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CONVOLUTION OVER LIE AND JORDAN ALGEBRAS

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ABSTRACT. Given a ternary relation C on a set U and an algebra A, we present a construction of a convolution algebra $A(U,C)$ of $U = (U,C)$ over A. This generalises both matrix algebras and algebras obtained from convolution of monoids. To any class of algebras corresponds a class of convolution structures. Our study cases are the classes of commutative, associative, Lie, and Jordan algebras. In each of these classes we give conditions on (U,C) under which $A(U,C)$ is in the same class as A. It turns out that in some situations these conditions are even necessary.

1. INTRODUCTION

The notion of convolution algebra over monoïds is not new, it appears for instance in [10]. The present construction is a generalisation to ternary relations. A more general construction of *convolution algebras of types over algebras* is recently given in [11, 12]. Our work on convolution structures and algebras is motivated and inspired by Algebraic Logic. Roughly speaking, vector spaces play the role of Boolean algebras, algebras the role of Boolean algebras with operator, bases of vector spaces the role of atom sets of Boolean algebras, and convolution structures the role of atom structures. Furthermore, as matrices and groups are simply atom sets of the most standard relation algebras and the universes of matrix algebras and group algebras are functions from matrices and groups respectively to algebras, it is natural to consider functions from atom sets of relations algebras to algebras. The duality between Boolean algebras with operators and their atom structures provides us with an interplay between Boolean algebras with operator and logic like the classical one between Boolean algebras and propositional logic. We mention at this point that in our subsequent paper [2] we have proved a structure theorem for algebras saying that any algebra with a basis is isomorphic to the (generalised) convolution algebra of this basis over the ring of scalars. This fact reveals that convolution algebras are not as specialised as they might seem at first glance and at the same time

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it prepares the ground for an interplay between Algebras and convolution structures.

We consider an algebra A together with a convolution structure (U, C) with universe U equipped with a ternary relation C. The collection of functions from U into A yields a new algebra $A(U, C)$ with naturally defined operations. For instance, two functions are multiplied by a convolution that is defined by the ternary relation C. If the ternary relation is the multiplication of a group G, we obtain the group algebra AG. If $U = \{1, \ldots, n\} \times \{1, \ldots, n\}$ and C is the usual composition of pairs, we obtain the matrix algebra $M_n(A)$.

In§2 we introduce the notion of a convolution structure together with basic notions such as homomorphisms, substructures, products, and disjoint unions. We also give some elementary properties of convolution structures which will be needed for the paper. In §3 we present convolution algebras and state some properties of such algebras. In particular, necessary conditions on A and (U, C) are provided for the algebra $A(U, C)$ to be simple. In §4 with any class $\mathcal{C}$ of algebras we associate a class $\mathfrak{Con}(\mathcal{C})$ of convolution structures as follows:

$$\mathfrak{Con}(\mathcal{C}) = \{(U, C) : \ A(U, C) \in \mathcal{C} \text{ provided } A \in \mathcal{C}\}.$$

There are given results on homomorphic images, products, and disjoint unions of elements of $\mathfrak{Con}(\mathcal{C})$. In §5 and §6 we investigate convolutions over the classes of commutative and associative algebras respectively. As to noncommutative algebras we study the case of Lie algebras in §7. Finally, in §8 we deal with convolutions over the class of Jordan algebras: an important class of nonassociative algebras.

2. CONVOLUTION STRUCTURES

Definition 2.1. A *convolution structure* $U = (U, C)$ consists of a non-empty set U together with a ternary relation C on U such that for any $x, y \in U$ the *set* $\{z \in U : \ (x, y, z) \in C\}$ *of outputs of* (x, y) is finite. The set U is *the universe of* (U, C) and the relation C is *the composition of* (U, C). If the composition C is empty, we say that (U, C) is *trivial*.

The sets of outputs are required to be finite in order to define properly the multiplication in convolution algebras, see Def. 3.2 below.

Examples 2.2. (1) Structures with a binary operation: Any structure U with a binary operation "$\cdot$" can be made into a convolution structure in a natural way by defining a ternary relation C as follows:

$$C = \{(x, y, z) \in U \times U \times U : \ x \cdot y = z\}.$$

The convolution structure obtained from a group G in this way will be called *the group convolution structure of* G. Clearly a ternary relation on a set comes from a binary operation iff the sets of outputs are singletons.

(2) Matrix convolution structures: Let V be a non-empty set. We can construct the V-matrix convolution structure (or the n-matrix convolution structure if V is a finite set with n elements) as follows. The universe is taken as $U = V \times V$ and C is taken as the familiar pair composition:

$$C = \{\big((i,j),(j,k),(i,k)\big) : \ i,j,k \in V\}.$$

(3) Products: The product of an indexed family $\{(U_i, C_i) : \ i \in I\}$ of convolution structures is $\prod_{i\in I}(U_i, C_i) = (\prod_{i\in I} U_i, C)$ where $\prod_{i\in I} U_i$ is the set theoretical Cartesian product of $\{U_i : \ i \in I\}$ and C is defined coordinatewisely:

$$\big((x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I}\big) \in C \text{ iff } (x_i, y_i, z_i) \in C_i \ \forall i \in I.$$

Clearly $(\prod_{i\in I} U_i, C)$ is a convolution structure if its factors are.

(3) Disjoint unions: The disjoint union of an indexed family $\{(U_i, C_i) : i \in I\}$ of pairwise disjoint convolution structures is

$$\biguplus_{i\in I}(U_i, C_i) = (\cup_{i\in I} U_i, \cup_{i\in I} C_i).$$

Note that $(x,y,z) \in \cup_{i\in I} C_i$ iff there is exactly one index $i \in I$ such that $(x,y,z) \in C_i$. This structure is easily seen to be a convolution structure.

We now introduce the notion of homomorphism of convolution structures.

Definition 2.3. Let $U_1 = (U_1, C_1)$ and $U_2 = (U_2, C_2)$ be two convolution structures and let $h : U_1 \to U_2$ be a mapping. We say that h is a *homomorphism* provided the following condition is satisfied:

$$(x,y,z) \in C_1 \quad \to \quad \big(h(x), h(y), h(z)\big) \in C_2. \tag{2.1}$$

We say that h is an *embedding* if it is an injective homomorphism which satisfies the following stronger version of (2.1):

$$(x,y,z) \in C_1 \quad \leftrightarrow \quad \big(h(x), h(y), h(z)\big) \in C_2. \tag{2.2}$$

If h is a surjective embedding, then it is called an *isomorphism*. We say that (U_1, C_1) and (U_2, C_2) are *isomorphic*, in symbols $(U_1, C_1) \equiv (U_2, C_2)$, if there exists an isomorphism from U_1 to U_2. Clearly, being isomorphic is an equivalence relation over the class of convolution structures. Furthermore, if h is a homomorphism which is injective, surjective, or bijective, then so is the induced mapping $h' : C_1 \to C_2$, $(x_1, y_1, z_1) \mapsto \big(h(x_1), h(y_1), h(z_1)\big)$. Finally, we say that U_1 is a *convolution substructure of* U_2 provided U_1 is a subset of U_2 and the inclusion mapping is an embedding.

One more stronger notion of a homomorphism which is important for our purposes is given in the following definition.

Definition 2.4. A homomorphism $h : (U_1, C_1) \to (U_2, C_2)$ is called a *strong homomorphism* if the following condition is satisfied:

$$\begin{aligned}(x_2, y_2, h(z_1)) \in C_2 \quad \to \quad & \exists!(x_1, y_1) \in U_1 \times U_1 : \\ & h(x_1) = x_2 \wedge h(y_1) = y_2 \wedge (x_1, y_1, z_1) \in C_1.\end{aligned} \tag{2.3}$$

Note that "$\exists!(x_1, y_1)$" means the uniqueness of the pair (x_1, y_1). We say that h is a *strong isomorphism* provided h is a bijective strong homomorphism.

Examples 2.5. **(1)** If $\{(U_i, C_i) : i \in I\}$ is an indexed family of convolution structures, then the *projection* $p_i : \prod_{i \in I} U_i \to U_i$ is a homomorphism which is surjective since no U_i is empty. Furthermore it is easily seen that p_j is strong iff for any $j \in I$, the composition C_j has exactly one element. If all the factors of the product $\prod_{i \in I}(U_i, C_i)$ are equal to some fixed convolution structure (U, C) (i.e. $U_i = U$ and $C_i = C$ for all i), then the diagonal mapping $U \to \prod_{i \in I} U$ is an embedding.

(2) Let $U_1 = \{a_1, b_1, c_1\}$, $C_1 = \{(a_1, b_1, c_1)\}$ and let $U_2 = \{a_2, b_2\}$, $C_2 = \{(a_2, a_2, a_2)\}$. Then the constant mapping $h : U_1 \to U_2$ with value a_2 is a strong homomorphism which is neither injective nor surjective.

The following proposition states that the notions of isomorphism and strong isomorphism coincide.

Proposition 2.6. *A homomorphism $h : (U_1, C_1) \to (U_2, C_2)$ is an isomorphism iff it is a strong isomorphism.*

Proof. The proof is routine and follows straightforwardly from the definitions. □

The following lemma gives conditions on a convolution structure under which any strong homomorphism into this structure must be a surjective one.

Lemma 2.7. *Let $\mathcal{U}_1 = (U_1, C_1)$ and $\mathcal{U}_2 = (U_2, C_2)$ be two convolution structures and let $h : \mathcal{U}_1 \to \mathcal{U}_2$ be any strong homomorphism. If $\mathcal{U}_2$ satisfies one of the following two formulae:*

(S1) $\forall x_2, z_2 \in U_2\ \exists u_2, v_2, w_2 \in U_2 : \ (u_2, v_2, z_2) \in C_2 \wedge (x_2, w_2, v_2) \in C_2$

(S2) $\forall x_2, z_2 \in U_2\ \exists u_2, v_2, w_2 \in U_2 : \ (u_2, v_2, z_2) \in C_2 \wedge (x_2, v_2, w_2) \in C_2,$

then h is surjective.

Proof. Assume that $\mathcal{U}_2$ satisfies (S1). Let $x_2 \in U_2$ be arbitrary, let $z_1 \in U_1$, and let $z_2 \in U_2$ be the h-image of z_1. Then there are $u_2, v_2, w_2 \in U_2$ such that

$$(u_2, v_2, z_2) \in C_2 \wedge (x_2, w_2, v_2) \in C_2.$$

As h is a strong homomorphism and $z_2 = h(z_1)$ there is $v_1 \in U_1$ such that $v_2 = h(v_1)$. Applying the same argument to $(x_2, w_2, v_2) \in C_2$ we find that x_2 admits an h-inverse image. If $\mathcal{U}_2$ satisfies (S2) we proceed similarly. □

We have the following corollary for group and matrix convolution structures, refer to Examples 2.2(1,2).

Corollary 2.8. *The following statements are valid:*
(1) Any strong homomorphism into a group convolution structure is surjective.
(2) Any strong homomorphism into a matrix convolution structure is surjective.

Proof. As to part (1), it is trivially checked that any group satisfies both (S1) and (S2) in Lemma 2.7. As to part (2) let V be a non-empty set, let $(V \times V, C)$ be the V-matrix convolution structure, and let (i, j) and (k, l) arbitrary from $V \times V$. Then evidently

$$\big((i,k),(k,j),(i,j)\big) \in C \wedge \big((k,l),(l,j),(k,j)\big) \in C.$$

This implies that $V \times V$ satisfies (S1). Hence by virtue of Lemma 2.7 the result follows. □

3. CONVOLUTION ALGEBRAS

We start this section by recalling the definition of an algebra.

Definition 3.1. Let A be a vector space over a commutative ring R. We say that A is an *algebra over* R if it is equipped with a bilinear *multiplication* $(x, y) \mapsto xy$. Bilinearity of multiplication means the right and left distributive laws $z(x + y) = zx + zy$ and $(x + y)z = xz + yz$, and the scalar condition $\alpha(xy) = (\alpha x)y = x(\alpha y)$. The algebra A is:

- *trivial* if $xy = 0$ for all $x, y \in A$.
- *commutative* if $xy = yx$ for all $x, y \in A$.
- *associative* if $(xy)z = x(yz)$ for all $x, y, z \in A$.

We now give a construction of convolution algebras.

Definition 3.2. Let A be an algebra and let $U = (U, C)$ be a convolution structure. Let

$$A(U, C) = \{f \in A^U : f(u) \neq 0 \text{ for only finitely many } u \in U\}.$$

Note that if U is finite, then $A(U, C)$ is the set A^U of all functions from U to A.

- The sum in $A(U, C)$ is pointwise and the zero-element is the constant function with value 0.
- If $f \in A(U, C)$ and $\alpha \in R$, then we define
$$\alpha f : U \to A,\ u \mapsto \alpha\big(f(u)\big).$$
- If $f, g \in A(U, C)$, then their multiplication is 0 if (U, C) is trivial (that is $C = \emptyset$), and otherwise
$$(f \cdot g)(z) = \sum_{x,y \in U} \{f(x)g(y) :\ (x, y, z) \in C\}.$$

Notice that $A(U, C)$ is closed under multiplication since the sets of outputs are assumed to be finite, see Def. 2.1. We shall see in Thm. 3.3 below that $A(U, C)$ is an algebra. We refer to $A(U, C)$ as the *convolution algebra of* (U, C) *over* A. This extends the terminology used in the literature when U

is a monoid, see [10]. In particular, if $(U,C) = (G,C)$ is a group convolution structure, then $A(G,C)$ is the group algebra AG and if (U,C) is the n-matrix convolution structure, then $A(U,C)$ is the n-matrix algebra $M_n(A)$. We shall often write $[xy : z]$ to mean $(x,y,z) \in C$. If $u_1,\ldots,u_n \in U$ and $a_1,\ldots,a_n \in A$, we shall use $\chi^{u_1,\ldots,u_n}_{a_1,\ldots,a_n}$ to denote the element of $A(U,C)$ which maps u_i to a_i and maps any other element of U to 0. Clearly $f \in A(U,C)$ iff f is of the form $\chi^{u_1,\ldots,u_n}_{a_1,\ldots,a_n}$.

Theorem 3.3. *If A is an algebra over R and U is a convolution structure, then $A(U,C)$ is an algebra over R.*

Proof. Routine verifications. □

As to trivial algebras we have:

Proposition 3.4. *The algebra $A(U,C)$ is trivial iff U is trivial or A is trivial.*

Proof. The implication from right to left is evident. For the converse assume that both A and U are non-trivial. Then there are $a,b \in A$ such that $ab \neq 0$ and there are $u,v,w \in U$ such that $(u,v,w) \in C$. It follows that

$$(\chi^u_a \cdot \chi^v_b)(w) = \chi^u_a(u)\chi^v_b(v) = ab \neq 0.$$

Hence $A(U,C)$ is not trivial. □

Definition 3.5. Let $U = (U,C)$ be a convolution structure, let A_1 and A_2 be algebras, and let $\varphi : A_1 \to A_2$ be a mapping. Then we define the following natural mapping:

$$\varphi(U,C) : A_1(U,C) \to A_2(U,C),\ f \mapsto \varphi f \text{ (composite function).}$$

Proposition 3.6. *The following statements are true:*

(1) If $\varphi : A_1 \to A_2$ is a homomorphism of algebras, then so is $\varphi(U,C)$.
(2) If φ is a (proper) embedding, then so is $\varphi(U,C)$.
(3) If φ is an isomorphism, then so is $\varphi(U,C)$.
(4) In the category of algebras, (U,C) can be seen as a covariant functor.

Proof. Parts (1) and (4) are directly verified. It is also easy to prove that if φ is an embedding then so is $\varphi(U,C)$. Assume now that φ is a proper embedding. Then there is $x_2 \in A_2$ with no φ-inverse image. Then for any $u \in U$ it is true that $\chi^u_{x_2}$ has no $\varphi(U,C)$-inverse image. Hence part (2) follows. As to part (3) we only need to show that $\varphi(U,C)$ is surjective if φ is an isomorphism. Let $g \in A_2(U,C)$, say $g = \chi^{u_1,\ldots,u_n}_{a_1,\ldots,a_n}$. For each $i = 1,\ldots,n$ let $x_i \in A_1$ be the unique element such that $\varphi(x_i) = g(u_i)$. Then $\varphi(U,C)(\chi^{u_1,\ldots,u_n}_{x_1,\ldots,x_n}) = g$ and so $\varphi(U,C)$ is surjective. □

Definition 3.7. Let A be an algebra, let (U_1,C_1) and (U_2,C_2) be two convolution structures, and let $\phi : U_1 \to U_2$ be a mapping. Then we define a natural mapping $A(\phi)$ of algebras as follows:

$$A(\phi) : A(U_2,C_2) \to A(U_1,C_1),\ f \mapsto f\phi \text{ (composite function).}$$

In the following result strong homomorphism is required.

Proposition 3.8. *Let A be an algebra, let (U_1, C_1) and (U_2, C_2) be two convolution structures, and let $\phi : U_1 \to U_2$ be a strong homomorphism. Then the following statements are true:*

(1) The mapping $A(\phi)$ is an algebra homomorphism.
(2) If $A \neq \{0\}$, then $A(\phi)$ is an algebra embedding iff ϕ is surjective.
(3) If ϕ is an isomorphism, then $A(\phi)$ is an algebra isomorphism.
(4) The algebra A can be seen as a contravariant functor from the category of convolution structures with strong homomorphisms as morphisms to the category of algebras with algebra homomorphisms as morphisms.

Proof. (1) It easy to check that $A(\phi)$ is a linear transformation. To show that it preserves products let $f, g \in A(U_2, C_2)$ and let $w_1 \in U_1$. Then we find:

$$\begin{aligned} A(\phi)(f \cdot g)(w_1) &= (f \cdot g)\phi(w_1) \\ &= \sum_{u_2, v_2} \{ f(u_2) g(v_2) : (u_2, v_2, \phi(w_1)) \in C_2 \} \\ &= \sum_{u_1, v_1} \{ f\big(\phi(u_1)\big) g\big(\phi(v_1)\big) : (u_1, v_1, w_1) \in C_1 \} \\ &= \big(A(\phi)(f) \cdot A(\phi)(g) \big)(w_1), \end{aligned}$$

where the third equality follows as ϕ is a strong homomorphism.

(2) Suppose that ϕ is a surjective strong homomorphism. Then by part (1) $A(\phi)$ is a homomorphism. To show that $A(\phi)$ is injective let $f \in A(U, C)$ such that $A(\phi)(f) = 0$. Let $u_2 \in U_2$ and let $u_1 \in U_1$ be the ϕ-inverse image of u_2. Then we have:

$$f(u_2) = f\big(\phi(u_1)\big) = A(\phi)(f)(u_1) = 0.$$

This shows that $f = 0$ and $A(\phi)$ is injective. Conversely suppose that ϕ is not surjective and let u_2 be an element of U_2 which has no ϕ-inverse image. Then for any $0 \neq x \in A$ we have $\chi_x^{u_2} \neq 0$ but $A(\phi)(\chi_x^{u_2}) = \chi_x^{u_2}\phi = 0$. Hence $A(\phi)$ is not injective.

(3) Suppose that ϕ is a convolution isomorphism. Then by statements (1) and (2), it suffices to show that $A(\phi)$ is surjective. Let $g \in A(U_1, C_1)$, say $g = \chi_{a_1,\ldots,a_n}^{u_1,\ldots,u_n}$ and let $v_1, \ldots, v_n \in U_2$ be the ϕ-images of $u_1, \ldots, u_n$ respectively. Then $A(\phi)(\chi_{a_1,\ldots,a_n}^{v_1,\ldots,v_n}) = g$ and so $A(\phi)$ is surjective.

(4) follows by routine verifications. □

Note that the implication from right to left in Prop. 3.8(2) is true even for $A = \{0\}$. The following result states that an algebra has an "exponential" behaviour when applied to a convolution structure.

Proposition 3.9. *Let A, A_1, A_2 be algebras and let (U, C), (U_1, C_1), and (U_2, C_2) be convolution structures. Then each one of the following natural mappings is an algebra isomorphism.*

(1) $F : (A_1 \times A_2)(U, C) \to A_1(U, C) \times A_2(U, C)$, $F(f)_i(u) = f(u)_i$ for $i = 1, 2$.

(2) $H : [A(U_1, C_1)](U_2, C_2) \to A[(U_1, C_1) \times (U_2, C_2)]$, $H(f)(u_1, u_2) = f(u_2)(u_1)$.

(3) $G : A[(U_1, C_1) \uplus (U_2, C_2)] \to A(U_1, C_1) \times A(U_2, C_2)$, $G(f)_i(u_i) = f(u_i)$ *for* $i = 1, 2$ *and* $u_i \in U_i$.

Proof. (1) It is immediately seen that F is an injective homomorphism of vector spaces. To show F is surjective let $(g_1, g_2) \in A_1(U, C) \times A_2(U, C)$. Then the function $f \in (A_1 \times A_2)(U, C)$ defined by $f(u)_i = g_i(u)$ is evidently an F-inverse image of g. It remains to check that F respects the multiplication. Let $f, g \in (A_1 \times A_2)(U, C)$ and let $w \in U$. Then for $i = 1, 2$ we have:

$$F(f \cdot g)_i(w) = (f \cdot g)(w)_i = \Big(\sum_{\substack{u,v \\ [uv:w]}} f(u)g(v) \Big)_i =$$

$$\sum_{\substack{u,v \\ [uv:w]}} f(u)_i g(v)_i = \sum_{\substack{u,v \\ [uv:w]}} F(f)_i(u) F(g)_i(v) = [F(f)_i \cdot F(g)_i](w).$$

(2) We only show that H respects the multiplication as it is easily verified that H is an isomorphism of vector spaces. Let $f, g \in [A(U_1, C_1)](U_2, C_2)$ and let $(w_1, w_2) \in U_1 \times U_2$. Then we find:

$$\begin{aligned} H(f \cdot g)(w_1, w_2) &= [(f \cdot g)(w_2)](w_1) \\ &= \sum_{\substack{u_2, v_2 \\ [u_2 v_2 : w_2]}} \big(f(u_2) \cdot g(v_2)\big)(w_1) \\ &= \sum_{\substack{(u_1,u_2),(v_1,v_2) \\ [u_1 v_1 : w_1] \wedge [u_2 v_2 : w_2]}} f(u_2)(u_1) g(v_2)(v_1) \\ &= \big(H(f) \cdot H(g)\big)(w_1, w_2). \end{aligned}$$

(3) This part follows by similar reasoning as parts (1,2) combined with the definition of disjoint unions. □

Remark 3.10. Prop. 3.9 can be extended to arbitrary indexed families of algebras and convolution structures.

We close this section with a result on simple algebras.

Definition 3.11. A subalgebra B of an algebra A is an *ideal* of A, in symbols, $B \rhd A$, if $xy \in B$ and $yx \in B$ for all $x \in B$ and all $y \in A$. Clearly A and $\{0\}$ are ideals. An ideal which is different from A and $\{0\}$ is said to be *proper*. An algebra is *simple* provided it is non-trivial and it has no proper ideals.

In the following result a necessary condition on (U, C) is given so that the class of simple algebras is closed under convolution over (U, C).

Proposition 3.12. *Let A be an algebra and let $U = (U, C)$ be a convolution structure. If $A(U, C)$ is simple, then A is simple, U is not a disjoint union of convolution structures, and any strong homomorphism into (U, C) is surjective.*

Proof. As a simple algebra is non-trivial we have that both A (as an algebra) and (U, C) (as a convolution structure) are non-trivial by Prop. 3.4. If A has a proper ideal B, then $B(U, C)$ is a proper subalgebra of $A(U, C)$ by Prop. 3.6(2). Furthermore if $f \in B(U, C)$ and $g \in A(U, C)$, then it is easy to check that both fg and gf are in $B(U, C)$. Then $B(U, C)$ is a proper ideal of $A(U, C)$, and so $A(U, C)$ is not simple. By virtue of Prop. 3.9(3) and Remark 3.10, it is evident that U is not a disjoint union whenever $A(U, C)$ is simple. Finally assume that there is a strong homomorphism $\phi : (U_1, C_1) \to (U, C)$ which is not surjective. By virtue of Prop. 3.8(2) the mapping $A(\phi) : A(U, C) \to A(U_1, C_1)$ defined in Def. 3.7 is a homomorphism which is not injective. This means that Ker $A(\phi) \neq \{0\}$. But we also have Ker $A(\phi) \neq A(U, C)$. Indeed, for any $u_1 \in U_1$ and any $0 \neq x \in A$ we have $A(\phi)(\chi_x^{\phi(u_1)})(u_1) = x \neq 0$. It follows that Ker $A(\phi)$ is a proper ideal of $A(U, C)$ and that $A(U, C)$ is not simple. $\square$

It is known that an n-matrix algebra over a simple algebra is simple too. Then any strong homomorphism into the n-matrix convolution structure is surjective by Prop. 3.12. This fact agrees with Cor. 2.8.

Problem 1. Which convolution structures respect the class of simple algebras?

4. CONVOLUTION CLASSES

In this section we associate a class of convolution structures with any non-empty class of algebras that is closed under isomorphisms in a natural manner. First note that if A is any algebra and $(U, C) = (\{x\}, \{(x, x, x)\})$, then the algebras A and $A(U, C)$ are isomorphic.

Definition 4.1. Let $\mathcal{C}$ be a non-empty class of algebras which is closed under isomorphisms. Then we define a class of convolution structures as follows:

$$\mathfrak{Con}(\mathcal{C}) = \{(U, C) : \text{ if } A \in \mathcal{C} \text{ then } A(U, C) \in \mathcal{C}\}.$$

We refer to $\mathfrak{Con}(\mathcal{C})$ as *the convolution class corresponding to* $\mathcal{C}$. Observe that by the previous note if a class of algebras is non-empty and is closed under isomorphisms, then its corresponding convolution class in non-empty. For instance, the convolution class corresponding to the class of trivial algebras is the class of all convolution structures by Prop. 3.4.

We have the following universal-algebraic results for convolution classes.

Proposition 4.2. *Let $\mathcal{C}$ be a non-empty class algebras which is closed under isomorphisms. Then the following statements are valid.*

(1) The class $\mathfrak{Con}(\mathcal{C})$ is closed under finite products.
(2) If $\mathcal{C}$ is closed under subalgebras, then $\mathfrak{Con}(\mathcal{C})$ is closed under strong homomorphic images.
(3) If $\mathcal{C}$ is closed under products, then $\mathfrak{Con}(\mathcal{C})$ is closed under disjoint unions.

Proof. (1) This part is obtained by induction combined with Prop. 3.9(2).

(2) Let $A \in \mathcal{C}$, let $(U_2, C_2) \in \mathfrak{Con}(\mathcal{C})$, and let $\phi : (U_2, C_2) \to (U_1, C_1)$ be a strong homomorphism which is surjective. Then by Prop. 3.8 the mapping $A(\phi) : A(U_1, C_1) \to A(U_2, C_2)$ is an embedding. Hence $A(U_1, C_1) \in \mathcal{C}$ as $\mathcal{C}$ is closed under subalgebras.

As to part (3), let $A \in \mathcal{C}$ and let $\{(U_i, C_i) : i \in I\}$ be an indexed family of pairwise disjoint elements of $\mathfrak{Con}(\mathcal{C})$. Then by Prop. 3.9(3) and Remark 3.10 the algebra $A[\biguplus_{i\in I}(U_i, C_i)]$ is isomorphic to $\prod_{i\in I} A(U_i, C_i)$. As $\mathcal{C}$ is closed under products we have that $\biguplus_{i\in I}(U_i, C_i) \in \mathfrak{Con}(\mathcal{C})$. □

5. COMMUTATIVE ALGEBRAS

Just as for algebras, the notion of being commutative is natural for convolution structures.

Definition 5.1. We say that a convolution structure $U = (U, C)$ is *commutative* if the following condition is satisfied:

$$\text{(CCom) } \forall x, y, z \in U\ [xy : z] \to [yx : z].$$

For instance, the group convolution structure over an abelian group is commutative. However, the n-matrix convolution structure is not commutative unless $n = 1$. We now see that convolution over commutative structures respects the class of commutative algebras.

Proposition 5.2. *Let A be an algebra and $U = (U, C)$ be a convolution structure. Then the following statements are true:*

(1) If A and U are commutative, then so is $A(U, C)$.

(2) If A is a non-trivial commutative algebra, then $A(U, C)$ is commutative iff U is commutative.

Proof. (1) This part follows directly by the definitions. The implication from right to left in part (2) is a special case of part (1). To show the other implication, assume that U is not commutative and let $u, v, w \in U$ such that $(u, v, w) \in C$ but $(v, u, w) \notin C$. As A is not trivial there are $a, b \in A$ such that $ab \neq 0$. Then we find:

$$(\chi^u_a \cdot \chi^v_b)(w) = \chi^u_a(u)\chi^v_b(v) = ab \neq 0 = (\chi^v_b \cdot \chi^u_a)(w).$$

So, $A(U, C)$ is not commutative. □

An immediate consequence of Prop. 5.2(2) is the known fact that the variety of commutative algebras is not closed under matrix forming for $n > 1$.

Corollary 5.3. *If $\mathcal{C}$ is the class of commutative algebras and $\mathfrak{C}$ is the class of commutative convolution structures, then $\mathfrak{C} \subseteq \mathfrak{Con}(\mathcal{C})$.*

Proof. This is nothing else but Prop. 5.2(1). □

Corollary 5.4. *If $\mathcal{C}'$ is the class of non-trivial commutative algebras and $\mathfrak{C}'$ is the class of non-trivial commutative convolution structures, then $\mathfrak{C}' = \mathfrak{Con}(\mathcal{C}')$.*

Proof. Let $\mathcal{U} = (U, C) \in \mathfrak{C}'$ and let $A \in \mathcal{C}'$. Then $A(U, C)$ is commutative by Prop. 5.2(1). But $A(U, C)$ is non-trivial by Prop. 3.4. It follows that $\mathcal{U} \in \mathfrak{Con}(\mathcal{C}')$. Now let $(U, C) \in \mathfrak{Con}(\mathcal{C}')$ and let $A \in \mathcal{C}'$. On the one hand, $\mathcal{U}$ is commutative by Prop. 5.2(2) as $A(U, C)$ is. On the other hand, $\mathcal{U}$ is non-trivial by Prop. 3.4 as $A(U, C)$ is. Hence $(U, C) \in \mathfrak{C}'$. $\square$

Problem 2. Let (U_1, C_1) and (U_2, C_2) be two finite commutative convolution structures such that the algebras $\mathbb{Z}(U_1, C_1)$ and $\mathbb{Z}(U_2, C_2)$ are isomorphic. Is it true that (U_1, C_1) and (U_2, C_2) are isomorphic?

G. Higman showed that the answer is positive if $\mathcal{U}_1$ and $\mathcal{U}_2$ are finite abelian group convolution structures. See [4]. In general, isomorphism of convolution algebras does not imply isomorphism of their convolution structures as the following example shows. If A is a trivial algebra (i.e. $ab = 0\ \forall a, b \in A$) and U is a non-empty set, then for any two convolution structures (U, C_1) and (U, C_2) on U we have an isomorphism between $A(U, C_1)$ and $A(U, C_2)$ given by the identity mapping.

6. ASSOCIATIVE ALGEBRAS

The notion of being an associative convolution structure is natural too.

Definition 6.1. We say that a convolution structure $\mathcal{U} = (U, C)$ is *associative* if the following two implications hold:

(CAss1) $\forall v, w, x, y, z \in U\ [vw : x] \wedge [xy : z] \ \rightarrow\ \exists u\ [wy : u] \wedge [vu : z]$,
(CAss2) $\forall u, v, w, y, z \in U\ [wy : u] \wedge [vu : z] \ \rightarrow\ \exists x\ [vw : x] \wedge [xy : z]$.

For group convolution structures, (CAss1) and (CAss2) simply mean that $(vw)y = v(wy)$. As this is one of the group axioms we see that any group convolution structure is associative. Moreover, matrix convolution structures are associative too.

Proposition 6.2. *The following statements are valid:*

1. *If A and $\mathcal{U}$ are associative, then so is $A(U, C)$.*
2. *If A is associative with a ring of scalars having characteristic* 0 *and there are $a, b, c \in A$ such that $abc \neq 0$, then $A(U, C)$ is associative iff $\mathcal{U}$ is associative.*

Proof. (1) The result is clear if A or U is trivial. Otherwise, let $f, g, h \in A(U, C)$ and let $z \in U$. Then we have:

$$\begin{aligned}
\big((f\cdot g)\cdot h\big)(z) &= \sum_{x,y}\{(f\cdot g)(x)h(y) :\ [xy:z]\}\\
&= \sum_{v,w,x,y}\{\big(f(v)g(w)\big)h(y) :\ [vw:x]\wedge[xy:z]\}\\
&= \sum_{v,w,x,y}\{f(v)\big(g(w)h(y)\big) :\ [vw:x]\wedge[xy:z]\}\\
&= \sum_{v,w,u,y}\{f(v)\big(g(w)h(y)\big) :\ [wy:u]\wedge[vu:z]\}\\
&= f\cdot(g\cdot h)(z),
\end{aligned}$$

where the third equality follows by the associativity of A and the fourth one follows from Def. 6.1.

(2) The implication from right to left is a consequence of the previous part. Conversely suppose that U is not associative. If (CAss1) fails in U, then there are $v, w, x, y, z \in U$ such that

$$[vw:x]\wedge[xy:z]\wedge\forall u\in U:\ (\neg[wy:u]\vee\neg[vu:z].$$

Then on the one hand,

$$\begin{aligned}
\big((\chi^v_a\cdot\chi^w_b)\cdot\chi^y_c\big)(z) &= \sum_{x}\{\chi^v_a(v)\chi^w_b(w)\chi^y_c(y) :\ [vw:x]\wedge[xy:z]\}\\
&= \sum_{x}\{abc :\ [vw:x]\wedge[xy:z]\}\neq 0.
\end{aligned}$$

On the other hand,

$$\big(\chi^v_a\cdot(\chi^w_b\cdot\chi^y_c)\big)(z) = \sum_{u}\{\chi^v_a(v)\chi^w_b(w)\chi^y_c(y) :\ [wy:u]\wedge[vu:z]\} = 0.$$

Hence $A(U, C)$ is not associative. Similar reasoning applies if U does not satisfy (CAss2). □

Direct consequences of Prop. 6.2(1) are the known facts that the variety of associative algebras is closed under matrix forming as well as under convolutions of monoids.

Corollary 6.3. *If $\mathcal{A}$ is the class of associative algebras and $\mathfrak{A}$ is the class of associative convolution structures, then $\mathfrak{A}\subseteq\mathfrak{Con}(\mathcal{A})$.*

Proof. This is nothing else than Prop. 6.2(1). □

Corollary 6.4. *Let R be a commutative ring with characteristic 0, let $\mathcal{A}'$ be the class of associative algebras over R satisfying the formula*

$$\exists a, b, c :\ abc\neq 0,$$

and let $\mathfrak{A}'$ be the class of associative convolution structures satisfying the formula:

$$\exists v, w, x, y, z\ \big(([vw:x]\wedge[xy:z])\ \vee\ ([wy:x]\wedge[vx:z])\big).$$

Then we have: $\mathfrak{A}'=\mathfrak{Con}(\mathcal{A}')$.

Proof. Let $U = (U, C) \in \mathfrak{A}'$ and let $A \in \mathcal{A}'$. Then $A(U, C)$ is associative by Prop. 6.2(1). Moreover for $a, b, c \in A$ such that $abc \neq 0$ and $v, w, x, y, z \in U$ such that $[vw : x] \wedge [xy : z]$ we have

$$(\chi_a^v \cdot \chi_b^w \cdot \chi_c^y)(z) = \sum_{\substack{t \in U \\ [vw:t] \wedge [ty:z]}} abc \neq 0.$$

If $[wy : x] \wedge [vx : z]$, we can find $t \in U$ such that $[vw : t] \wedge [ty : z]$ and therefore we again find that $(\chi_a^v \cdot \chi_b^w \cdot \chi_c^y)(z) \neq 0$. Consequently $A(U, C) \in \mathcal{A}'$ and so $U \in \mathfrak{Con}(\mathcal{A}')$. Now let $U \in \mathfrak{Con}(\mathcal{A}')$. Then U is associative by virtue of Prop. 6.2(2). Moreover (U, C) satisfies the required condition since for any $A \in \mathcal{A}'$ there are $a, b, c \in A$ such that $abc \neq 0$. Then $U \in \mathfrak{A}'$ and the equality of the two classes follows. $\square$

7. LIE ALGEBRAS

Definition 7.1. An algebra L over a commutative ring R with multiplication $(x, y) \mapsto [x, y]$ is called a *Lie algebra over R* if the following axioms are satisfied:

(L1) $[x, x] = 0 \; \forall x \in L$.
(L2) (Jacobi identity) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0 \; \forall x, y, z \in L$.

The multiplication $(x, y) \mapsto [x, y]$ is called the *bracket operation*. A trivial Lie algebra (i.e. $[x, y] = 0 \; \forall x, y$) is called *abelian*. Finally, we call L *3-nilpotent* if $[[x, y], z] = 0 \; \forall x, y, z \in L$.

Note that (L1) and (L2) imply that the bracket operation is *anticommutative* (i.e. $[x, y] = -[y, x]$ for all $x, y \in L$). We refer to [6, 7] for a survey on Lie algebras.

Example 7.2. Any associative algebra A gives rise to a Lie algebra A^- whose bracket operation is defined by

$$[x, y] = xy - yx.$$

It is known that any Lie algebra over a field is a subalgebra of a Lie algebra of the form A^-. See [7, p. 159-162].

We now introduce the corresponding Lie formulae for convolution structures.

Definition 7.3. We say that a convolution structure $U = (U, C)$ is *Lie* if the following two conditions are satisfied:

(LC1) U is commutative (i.e. $[xy : z] \to [yx : z]$).
(LC2) $[vw : x] \wedge [xy : z] \to \exists u \; [wy : u] \wedge [uv : z]$.

Examples 7.4. (1) Abelian groups: Any abelian group can be made into a Lie convolution structure in the usual way.

(2) Non-empty sets: If $a \in U$ then $(U, \{(a,a,a)\})$ is Lie. If a and b are two distinct elements of U, then U equipped with each of the following relations is Lie:

$$C_1 = \{(a,a,a),(a,a,b)\},\ C_2 = \{(a,a,a),(b,b,b),(a,b,a),(b,a,a)\},$$
$$C_3 = \{(a,a,a),(b,b,b),(a,b,a),(b,a,a),(b,a,b),(a,b,b)\}.$$

Solving the following problem may be useful to classify convolution algebras over Lie structures.

Problem 3. Given a finite set U, classify the Lie convolution structures on U up to isomorphism.

We have the following characterisation for Lie convolution structures:

Proposition 7.5. *A convolution structure is Lie iff it is commutative and associative.*

Proof. The proof follows straightforwardly from Defs. 5.1, 6.1, and 7.3. □

The main result of this section says among others that convolution with respect to Lie convolution structures respects the class of Lie algebras:

Theorem 7.6. *Let L be a Lie algebra over R and let $U = (U, C)$ be a convolution structure. Then we have:*

(1) If U is Lie, then so is $L(U, C)$.

(2) If U is non-trivial and L is non-abelian 3-nilpotent, then $L(U, C)$ is Lie iff U satisfies the commutative law (LC1).

(3) If there are $a, b, c \in L$ such that $n[[a,b],c] + m[[c,a],b] = 0$ for no positive integers m and n, then $L(U, C)$ is a Lie algebra iff U is Lie.

Proof. (1) Suppose that U is Lie. Then by (L1) and (LC1) we find that $[f, f] = 0$ for any $f \in L(U, C)$, showing that $L(U, C)$ satisfies (L1). As to (L2) let $f, g, h \in L(U, C)$ and let $z \in U$. Then we have:

$$\begin{aligned} [[f,g],h]\,(z) &= \sum_{x,y}\{[[f,g](x),h(y)] :\ [xy:z]\} \\ &= \sum_{v,w,x,y}\{[[f(v),g(w)],h(y)] :\ [vw:x] \wedge [xy:z]\}, \end{aligned}$$

$$\begin{aligned} [[g,h],f](z) &= \sum_{u,v}\{[[g,h](u),f(v)] :\ [uv:z]\} \\ &= \sum_{w,y,u,v}\{[[g(w),h(y)],f(v)] :\ [wy:u] \wedge [uv:z]\}, \end{aligned}$$

and

$$\begin{aligned} [[h,f],g]\,(z) &= \sum_{t,w}\{[[h,f](t),g(w)] :\ [tw:z]\} \\ &= \sum_{y,v,t,w}\{[[h(y),f(v)],g(w)] :\ [yv:t] \wedge [tw:z]\}. \end{aligned}$$

Now by (LC2) and Prop. 7.5, $[[f(v),g(w)],h(y)]$ occurs as a summand in $[[f,g],h](z)$ iff $[[g(w),h(y)],f(v)]$ occurs as a summand in $[[g,h],f](z)$ iff $[[h(y),f(v)],g(w)]$ occurs as a summand in $[[h,f],g](z)$. Then by (L2) applied to the algebra L we find that

$$[[f,g],h](z)+[[g,h],f](z)+[[h,f],g](z)=0.$$

Hence $L(U,C)$ satisfies (L2) too and it is therefore a Lie algebra.

(2) Assume that U satisfies (LC1). On the one hand, evidently $L(U,C)$ is 3-nilpotent as L is. Then $L(U,C)$ satisfies (L2). On the other hand, (L1) and (LC1) yield $[f,f]=0$ for all $f\in L(U,C)$. Then (L1) follows. Conversely, assume that U does not satisfy the condition (LC1). Then there are $u,v,w\in U$ such that $(u,v,w)\in C$ but $(v,u,w)\notin C$. Since L is non-abelian there are $a,b\in L$ such that $[a,b]\neq 0$. Then we have:

$$[\chi^{u,v}_{a,b},\chi^{u,v}_{a,b}](w)=[\chi^{u,v}_{a,b}(u),\chi^{u,v}_{a,b}(v)]=[a,b]\neq 0.$$

It follows that $L(U,C)$ is not a Lie algebra.

(3) The implication from right to left follows from part (1). Conversely, assume that U is not Lie. Then U is non-trivial. Suppose that U does not satisfy (LC1). Then there are $u,v,w\in U$ such that $(u,v,w)\in C$ but $(v,u,w)\notin C$. The assumption on a,b and c yields that $[a,b]\neq 0$ or $[c,a]\neq 0$. Without loss of generality, assume that $[a,b]\neq 0$. Then

$$[\chi^{u,v}_{a,b},\chi^{u,v}_{a,b}](w)=[\chi^{u,v}_{a,b}(u),\chi^{u,v}_{a,b}(v)]=[a,b]\neq 0.$$

So, $L(U,C)$ is not Lie. Suppose now that U satisfies (LC1) but not (LC2). Then there are $v,w,x,y,z\in U$ such that

$$[vw:x]\wedge[xy:z]\wedge\forall u\in U:\ (\neg[wy:u]\vee\neg[uv:z]).$$

Let n be the number of elements $t\in U$ such that $[vw:t]\wedge[ty:z]$ and let m be the number of elements $t\in U$ such that $[yv:t]\wedge[tw:z]$. Then on the one hand,

$$[[\chi^v_a,\chi^w_b],\chi^y_c](z)=\sum_t\{[[\chi^v_a,\chi^w_b](t),\chi^y_c(y)]:\ [ty:z]\}=$$

$$\sum_t\{[[\chi^v_a(v),\chi^w_b(w)],\chi^y_c(y)]:\ [vw:t]\wedge[ty:z]\}=n[[a,b],c],$$

and

$$[[\chi^y_c,\chi^v_a],\chi^w_b](z)=\sum_t\{[[\chi^y_c,\chi^v_a](t),\chi^w_b(w)]:\ [tw:z]\}=$$

$$\sum_t\{[[\chi^y_c(y),\chi^v_a(v)],\chi^w_b(w)]:\ [yv:t]\wedge[tw:z]\}=m[[c,a],b].$$

On the other hand,

$$[[\chi^w_b,\chi^y_c],\chi^v_a](z)=\sum_u\{[[\chi^w_b,\chi^y_c](u),\chi^v_a(v)]:\ [uv:z]\}=$$

$$\sum_u\{[[\chi^w_b(w),\chi^y_c(y)],\chi^v_a(v)]:\ [wy:u]\wedge[uv:z]\}=0.$$

As $n[[a,b],c] + m[[c,a],b] \neq 0$ we have

$$[[\chi_a^v, \chi_b^w], \chi_c^y] + [[\chi_c^y, \chi_a^v], \chi_b^w] + [[\chi_b^w, \chi_c^y], \chi_a^v] \neq 0,$$

showing that $L(U,C)$ is not a Lie algebra. □

An important consequence of Thm. 7.6(1) is that if L is a Lie algebra and G is an abelian group then the group algebra LG is a Lie algebra. Note also that the consequence of Thm. 7.6(3) holds if in particular there are $a,b,c \in L$ such that $[[a,b],c]$ and $[[c,a],b]$ are linearly independent.

Corollary 7.7. *If $\mathcal{L}$ is the class of Lie algebras and $\mathfrak{L}$ is the class of Lie convolution structures, then $\mathfrak{L} \subseteq \mathfrak{Con}(\mathcal{L})$.*

Proof. This is Thm. 7.6(1). □

Corollary 7.8. *If $\mathcal{L}'$ is the class of non-abelian 3-nilpotent Lie algebras and $\mathfrak{L}'$ is the class of non-trivial commutative convolution structures, then $\mathfrak{L}' = \mathfrak{Con}(\mathcal{L}')$.*

Proof. Let $U = (U,C) \in \mathfrak{L}'$ and let $L \in \mathcal{L}'$. Then by Thm. 7.6(1) $L(U,C)$ is Lie. Moreover for $a,b \in L$ such that $[a,b] \neq [b,a]$ and $(u,v,w) \in C$ we find:

$$[\chi_a^u, \chi_b^v] \neq [\chi_b^v, \chi_a^u],$$

showing that $L(U,C)$ is non-abelian. Furthermore $L(U,C)$ is clearly 3-nilpotent as L is. Whence $L(U,C) \in \mathcal{L}'$, hence $(U,C) \in \mathfrak{Con}(\mathcal{L}')$. Assume now that $(U,C) \in \mathfrak{Con}(\mathcal{L}')$. Then evidently (U,C) is non-trivial. Moreover (U,C) is commutative by Thm. 7.6(2) as $L(U,C)$ is Lie for any $L \in \mathcal{L}'$. Hence $(U,C) \in \mathfrak{L}'$. □

Corollary 7.9. *Let $\mathcal{L}''$ be the class of Lie algebras satisfying the following condition:*

$$\exists a,b,c:\ m[[a,b],c] + n[[c,a],b] = 0 \text{ for no positive integers } m \text{ and } n$$

and let $\mathfrak{L}''$ be the class of Lie convolution structures satisfying the following condition:

$$\exists v,w,y,z,t:\ [vw:t] \wedge [ty:z].$$

Then we have $\mathfrak{L}'' = \mathfrak{Con}(\mathcal{L}'')$.

Proof. Suppose that $(U,C) \in \mathfrak{L}''$ and that $L \in \mathfrak{Con}(\mathcal{L}'')$. Then by Thm. 7.6(1) $L(U,C)$ is Lie. Furthermore for $a,b,c \in L$ such that $m[[a,b],c] + n[[c,a],b] = 0$ for no positive integers m and n and for $v,w,y,z,t \in U$ such that $[vw : t] \wedge [ty : z]$ we find:

$$p[[\chi_a^v, \chi_b^w], \chi_c^y] + q[[\chi_c^y, \chi_a^v], \chi_b^w] = 0 \text{ for no positive integers } p \text{ and } q.$$

This implies that $L(U,C) \in \mathcal{L}''$ and so $(U,C) \in \mathfrak{Con}(\mathcal{L}'')$. Assume now that $(U,C) \in \mathfrak{Con}(\mathcal{L}'')$. This means that for any $L \in \mathcal{L}''$ we have $L(U,C) \in \mathcal{L}''$. On the one hand, (U,C) is Lie by Thm. 7.6(3). On the other hand, there are v,w,y,t,z such that $[vw:t] \wedge [ty:z]$ since otherwise $[[f,g],h] + [[h,f],g] = 0$ for any $f,g,h \in L(U,C)$. Consequently $(U,C) \in \mathfrak{L}''$. □

We end this section by a result on Lie algebras of the form A^-, refer to Example 7.2.

Proposition 7.10. *If A is an associative algebra and (U, C) is a Lie convolution structure, then $A(U, C)^- = A^-(U, C)$.*

Proof. Note that $A(U, C)$ is a associative by Props.7.5 and 6.2(1). Then $A(U, C)^-$ is well-defined. Note also that $A^-(U, C)$ is a Lie algebra by Thm. 7.6(1). To show the equality, let $f, g \in A(U, C)$ and let $z \in U$. To avoid confusion we let $[f \cdot g]$ be the multiplication in $A^-(U, C)$ and keep using the bracket notation $[\,,]$ in both A^- and $A(U, C)^-$. Then we have the following equalities:

$$[f, g](z) = (f \cdot g - g \cdot f)(z) = \sum_{\substack{x,y \\ [xy:z]}} \big(f(x)g(y) - g(x)f(y)\big) =$$

$$\sum_{\substack{x,y \\ [xy:z]}} f(x)g(y) - \sum_{\substack{x,y \\ [xy:z]}} g(x)f(y) = \sum_{\substack{x,y \\ [xy:z]}} f(x)g(y) - \sum_{\substack{x,y \\ [xy:z]}} g(y)f(x) =$$

$$\sum_{\substack{x,y \\ [xy:z]}} \big(f(x)g(y) - g(y)f(x)\big) = \sum_{\substack{x,y \\ [xy:z]}} [f(x), g(y)] = [f \cdot g](z),$$

where the fourth equality follows by commutativity of (U, C). □

8. Jordan algebras

An important class of nonassociative algebras is the class of Jordan algebras. Basic references on Jordan algebras are [8, 9].

Definition 8.1. An algebra J with multiplication $(x, y) \mapsto x \bullet y$ is a *Jordan algebra* if the following axioms are satisfied:

(J1) (Commutative law) $x \bullet y = y \bullet x$ for all $x, y \in J$.

(J2) (Jordan identity) $(x \bullet x) \bullet (y \bullet x) = \big((x \bullet x) \bullet y\big) \bullet x$ for all $x, y \in J$.

Example 8.2. Just as for Lie algebras, any associative algebra A gives rise to a Lie algebra A^+ whose multiplication is defined by

$$x \bullet y = xy + yx.$$

A Jordan algebra which is isomorphic to a subalgebra of an algebra of the form A^+ is called *special*. It is known that the class of special Jordan algebras is a proper subclass of the class of Jordan algebras. See for instance [8, p.11 Thm. 2]. A Jordan algebra which is not special is said to be *exceptional*. It is however still unknown whether the class of special Jordan algebras is finitely based over the class of Jordan algebras.

As to convolution structures we have the following definition.

Definition 8.3. We say that a convolution structure $U = (U, C)$ is *Jordan* provided it satisfies the following conditions:

(JC1) U is commutative.

(JC2) $[st : x] \wedge [vw : y] \wedge [xy : z] \rightarrow s = t = v \wedge [sy : z] \wedge$
$\exists u [ss : u] \wedge [uw : y]$.

(JC3) $[st : v] \wedge [vw : y] \wedge [xy : z] \rightarrow s = t = x \wedge [sw : y] \wedge$
$\exists u [ss : u] \wedge [uy : z]$.

Remark 8.4. We note that the axioms (JC1), (JC2), and (JC3) are independent. Indeed:

(1) If a and b are distinct elements in U and $C = \{(a, b, b)\}$, then trivially (U, C) is a noncommutative structure which satisfies both (JC2) and (JC3).
(2) For the same U as in (1) and $C = \{(a, b, b), (b, a, b)\}$ we have that (U, C) is a model for (JC1) and (JC2) but not for (JC3). The latter statement follows since $[ab : b] \wedge [ba : b] \wedge [ab : b]$ but $a \neq b$.
(3) If U has three distinct elements a, b, c and $C = \{(a, a, b), (b, b, c)\}$, then (U, C) satisfies (JC1) and (JC3) but not (JC2). Indeed we have $[aa : b] \wedge [aa : b] \wedge [bb : c]$ but $(a, b, c) \notin C$.

It is also worth to mention that the axioms (JC1)-(JC3) do not entail associativity. To see this let $U = \{a, b, c, d\}$ and let $C = \{(a, a, b), (b, c, d), (c, b, d)\}$. Then (U, C) is a Jordan structure which is not associative.

We now collect some elementary consequences of the axioms (JC1-JC3).

Proposition 8.5. *Let $U = (U, C)$ be a Jordan convolution structure and $a, b, c \in U$. Then the following statements are valid.*

(1) $[ab : b] \rightarrow a = b$.
(2) $[aa : a] \wedge [ba : c] \rightarrow a = b$.
(3) $[aa : a] \wedge [bb : a] \rightarrow a = b$.
(4) $[aa : b] \wedge [bb : a] \rightarrow a = b$.

Proof. (1) If $[ab : b]$, then we must have $[ba : b]$ by (JC1). Then applying (JC3) to $[ab : b] \wedge [ba : b] \wedge [ab : b]$, we find that $a = b$.

(2) Application of (JC3) for $s = t = v = w = y = a$, $x = b$, and $z = c$ yields that $a = b$.

(3) Let $s = t = x = y = z = a$ and $v = w = b$ and apply (JC2) to derive $a = b$.

(4) Let $s = t = v = x = y = a$ and $v = w = z = b$ and apply (JC3) to find $[ab : a]$. Then (JC1) and part (1) yield $a = b$. □

Example 8.6. Suppose that $U = \{a, b\}$ where $a \neq b$. Then there are exactly four non-isomorphic Jordan convolution structures on U. The corresponding ternary relations are:

$$C_1 = \{(a, a, a)\},\ C_2 = \{(a, a, b)\},\ C_3 = \{(a, a, a), (b, b, b)\}, \text{ and}$$
$$C_4 = \{(a, a, a), (a, a, b)\}.$$

Problem 4. Classify the Jordan convolution structures on a finite set U up to isomorphism.

We now give the main theorem of this section.

Theorem 8.7. *If J is a Jordan algebra and $U = (U, C)$ is a Jordan convolution structure, then $J(U, C)$ is a Jordan algebra.*

Proof. If J and U are commutative, then so is $A(U, C)$ by Prop. 5.2(1). To show the Jordan identity (J2) let $f, g \in A(U, C)$ and let $z \in U$. On the one hand, we have

$$(f \bullet f) \bullet (f \bullet g)(z) = \sum_{x,y} \{(f \bullet f)(x) \bullet (f \bullet g)(y) :\ [xy : z]\} =$$

$$\sum_{s,t,x,v,w,y} \{\big(f(s) \bullet f(t)\big) \bullet \big(f(v) \bullet g(w)\big) :\ [st : x] \wedge [vw : y] \wedge [xy : z]\} =$$

$$\sum_{s,w,x,y} \{\big(f(s) \bullet f(s)\big) \bullet \big(f(s) \bullet g(w)\big) :\ [ss : x] \wedge [sw : y] \wedge [xy : z]\} =$$

$$\sum_{s,w,x,y} \{f(s) \bullet \Big(\big(f(s) \bullet f(s)\big) g(w)\Big) :\ [ss : x] \wedge [sw : y] \wedge [xy : z]\},$$

where the third equality follows from (JC2-JC2) and the fourth one follows since J is Jordan. On the other hand, we have

$$f \bullet \big((f \bullet f) \bullet g\big)(z) = \sum_{x,y} \{f(x) \bullet \big((f \bullet f) \bullet g\big)(y) :\ [xy : z]\} =$$

$$\sum_{v,w,x,y} \{f(x) \bullet \big((f \bullet f)(v) \bullet g(w)\big) :\ [vw : y] \wedge [xy : z]\} =$$

$$\sum_{s,t,v,w,x,y} \{f(x) \bullet \Big(\big(f(s) \bullet f(t)\big) \bullet g(w)\Big) :\ [st : v] \wedge [vw : y] \wedge [xy : z]\} =$$

$$\sum_{s,v,w,y} \{f(s) \bullet \Big(\big(f(s) \bullet f(s)\big) \bullet g(w)\Big) :\ [ss : v] \wedge [vw : y] \wedge [sy : z]\} =$$

$$\sum_{s,w,x,y} \{f(s) \bullet \Big(\big(f(s) \bullet f(s)\big) \bullet g(w)\Big) :\ [ss : x] \wedge [sw : y] \wedge [xy : z]\},$$

where the fourth equality follows from (JC3) and the fifth equality follows by virtue of (JC2-JC3). Hence we have:

$$(f \bullet f) \bullet (g \bullet f) = f \bullet \big((f \bullet f) \bullet g\big) = \big((f \bullet f) \bullet g\big) \bullet f.$$

□

We were not able to find conditions under which a convolution structure is necessary Jordan whenever a convolution algebra of this structure is a Jordan algebra. As to special Jordan algebras, (see Example 8.2), we have the following facts. The proof of the following result is omitted as it is similar to the proof of Prop. 7.10.

Proposition 8.8. *If A is an associative algebra and (U, C) is an associative Jordan convolution structure, then we have that $A(U, C)^+ = A^+(U, C)$.*

Corollary 8.9. *If J is a special Jordan algebra and (U, C) is an associative Jordan convolution structure, then $J(U, C)$ is a special Jordan algebra.*

Proof. Let A be an associative algebra such that J is (isomorphic to) a subalgebra of A^+. Then $J(U, C)$ is (isomorphic to) a subalgebra of $A^+(U, C)$ by Prop. 3.6(2). But $A^+(U, C) = A(U, C)^+$ by virtue of Prop. 8.8. Hence $J(U, C)$ is special. □

The following three problems are suggested by Cor. 8.9.

Problems 5. Let J be a Jordan algebra and let (U, C) be a Jordan convolution structure.

(1) Assume that J is special and (U, C) is not associative. Is $J(U, C)$ special?
(2) Assume that $J(U, C)$ is exceptional and (U, C) is associative. Can we conclude something about $J(U, C)$?
(3) The same question arises if J is exceptional and (U, C) is not associative.

9. CONCLUDING REMARKS

9.1. **Connections with algebraic logic.** Convolution algebras with units and involutions can be obtained too. To this end, one needs structures of type $U = (U, C, r, I)$ where $C \subseteq U \times U \times U$, r is a unary function on U called *reverse*, and I is a subset of U called *the set of identities*. Such structures are referred to as *atom structures* or *frames* and are very important in algebraic logic and modal logic in particular. Atom structures give rise to so-called *Boolean algebras with operators* and dually, the atom collection of a complete atomic Boolean algebra with operators can be made into an atom structure. This duality principle led to the study of Boolean algebras "at the frame level", refer to [5, 1]. In [2] it is shown that a similar duality exist between algebras and their bases. So, can algebras be studied at the "basis level" too?

It turns out that if an atom structure (U, C, r, I) satisfies the following conditions:

- the reduct (U, C) is a convolution structure,
- the set I of identities is finite,
- if $u, v \in U$, then $u = v$ iff there is a unique $e \in I$ such that $(u, e, v) \in C$,

then convolution of U respects the class of algebras with units. If in addition it satisfies the following condition:

- $(x, y, z) \in C$ iff $(r(x), z, y)$ iff $(z, r(y), x)$,

then convolution of this structure respects the class of algebras with involution. Atom structures of relation algebras satisfy each one of these conditions, see [5].

9.2. **A new variant of the isomorphism problem.** In [4], G. Higman asked whether the isomorphism of two group convolution algebras $\mathbb{Z}G$ and $\mathbb{Z}H$ for finite G and H implies the isomorphism of the groups G and H. This problem is referred to as *the isomorphism problem for integral group rings*.

The question remained open until M. Hertweck recently gave a counterexample showing that the answer is negative. Refer to [3]. The following new variant of the isomorphism problem for integral group rings is suggested to us by an anonymous referee in connection with an application for a research grant.

Problem 6. Let $\mathfrak{C}$ be a class of convolution structures and let $\mathcal{A}$ be a class of algebras. To which extent is a convolution structure in $\mathfrak{C}$ determined by its convolution algebras constructed by means of algebras from $\mathcal{A}$?

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REFERENCES

[1] P. Blackburn, M. de Rijke, and Y. Venema, *Modal logic*, Cambridge Tracts in Theoretical Computer Science, vol. 53, Cambridge University Press, 2001, xxii+554 pp.
[2] M. El Bachraoui, *Generalised convolution algebras*, Al Akhawayn University Ifrane, Morocco (2005), In preparation.
[3] M. Hertweck, *A counterexample to the isomorphism problem for integral group rings*, Ann. of Math. (2) **154** (2001), no. 1, 115–138.
[4] G. Higman, *The units of group-rings*, Proc. London Math. Soc. (2) **46** (1940), 231–248.
[5] R. Hirsch and I. Hodkinson, *Relation algebras by games*, Studies in Logic and the Foundations of Mathematics, vol. 147, North-Holland, 2002, xviii+691 pp.
[6] J.E Humphreys, *Introduction to lie algebras and representation theory*, Graduate Texts in Mathematics, vol. 9, Springer Verlag New York, 1978, xii+173 pp.
[7] N. Jacobson, *Lie algebras*, Wiley (Interscience) New York, 1962.
[8] ———, *Structure and representations of jordan algebras*, Amer. Math. Soc. Providence, 1968, x+453.
[9] K. McCrimmon, *A taste of jordan algebras*, Springer Verlag, New York, 2004, xxvi+562 pp.
[10] R. Pierce, *Associative algebras*, *Graduate Texts in Mathematics*, Springer-Verlag New York-Berlin, 1982, xii+436 pp.
[11] S. Veldsman, *Radicals of convolution rings*, Bull. Acad. Stiinte Republic. Mold. Mat. **44** (2004), no. 1, 98–115.
[12] ———, *Convolution rings*, Algebra Coll. **13** (2006), no. 2, 211–238.

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