Contributions to Discrete Mathematics

Volume 1, Number 1, Pages 57–67
ISSN 1715-0868

UNIQUELY CIRCULAR COLOURABLE AND UNIQUELY FRACTIONAL COLOURABLE GRAPHS OF LARGE GIRTH

SHUYUAN LIN AND XUDING ZHU

ABSTRACT. Given any rational numbers $r \geq r' > 2$ and an integer g, we prove that there is a graph G of girth at least g, which is uniquely circular r-colourable and uniquely fractional r'-colourable. Moreover, the graph G has maximum degree bounded by a number which depends on r and r' but does not depend on g.

1. INTRODUCTION

Suppose G is a graph with at least one edge and $r \geq 2$ is a rational number. A *circular r-colouring* of G is a mapping $f : V(G) \to [0, r)$ such that for any edge xy of G, $1 \leq |f(x) - f(y)| \leq r - 1$. We say G is *circular r-colourable* if there is a circular r-colouring of G. The *circular chromatic number* of G is defined as

$$\chi_c(G) = \inf\{r : G \text{ is circular } r\text{-colourable}\}.$$

It is known that for any graph G, $\chi(G) = \lceil \chi_c(G) \rceil$. Hence the circular chromatic number of a graph is a refinement of its chromatic number.

Suppose f is a circular r-colouring of G. Then for any $c \in [0, r)$ and for $\tau \in \{1, -1\}$, $g : V(G) \to [0, r)$ defined as $g(x) = [c + \tau f(x)]_r$ is also a circular r-colouring of G. (For a real number x and a positive real number r, we denote by $[x]_r$ the remainder of x dividing r, i.e., $[x]_r \in [0, r)$ is the unique number for which $x - [x]_r$ is a multiple of r.) If f and g are r-colourings of G such that $g(x) = [c + \tau f(x)]_r$ for some $c \in [0, r)$ and $\tau \in \{1, -1\}$, then we say f and g are *equivalent circular r-colourings* of G, written as $f \cong g$. It is obvious that '$\cong$' is an equivalence relation. A graph G is called uniquely circular r-colourable if up to equivalence, there is only one circular r-colouring of G. It is proved in [10] that for any rational $r \geq 2$, for any integer g, there is a graph G of girth at least g which is uniquely circular r-colourbale.

Received by the editors May 5, 2005, and in revised form, Nov. 30, 2005.

2000 *Mathematics Subject Classification.* 05C15.

Key words and phrases. Uniquely circular r-colourable graphs, uniquely fractional r'-colourable graphs, graph homomorphism, graph product.

This research is partially supported by the National Science Council under grant NSC92-2115-M-110-007.

Let $I(G)$ be the family of independent sets of G. A *fractional colouring* f of G is an assignment of nonnegative weights to independent sets of G, i.e., a mapping $f : I(G) \to R^{\geq 0}$, such that for each $x \in V(G)$, $\sum_{x \in I, I \in I} f(I) = 1$.

A fractional colouring f is called a *fractional r-colouring* of G if the sum $\sum_{I \in I} f(I)$ is equal to r. The *fractional chromatic number* of G, denoted by $\chi_f(G)$, is the least r such that G has a fractional r-colouring. We say that a graph G is *uniquely fractional r-colourable* if there is exactly one fractional r-colouring of G. I.e., there is a fractional r-colouring f of G and if f' is a fractional r-colouring of G, then $f(I) = f'(I)$ for all $I \in I(G)$. It is proved in [5] that for any rational $r \geq 2$, for any integer g, there is a uniquely fractional r-colourable graph of girth at least g.

In this paper, we consider unique circular colourability and unique fractional colourability simultaneously. It is known [12] that for any graph G, $\chi_f(G) \leq \chi_c(G)$. On the other hand, it is not difficult to show that for any rationals $2 < r' \leq r$, there is a graph G with $\chi_f(G) = r'$ and $\chi_c(G) = r$. In this paper, we prove that for any rationals $2 < r' \leq r$, for any integer g, there is a graph G of girth at least g such that G is uniquely fractional r'-colourable, and at the same time, uniquely circular r-colourable. In particular, $\chi_f(G) = r'$ and $\chi_c(G) = r$.

Both circular chromatic number and fractional chromatic number of a graph can be defined through graph homomorphisms. Suppose G and H are graphs. A *homomorphism* of G to H is a mapping $f : V(G) \to V(H)$ such that $\{f(x), f(y)\} \in E(H)$ whenever $\{x, y\} \in E(G)$. A homomorphism of G to H is also called an *H-colouring* of G. A graph G is said to be H-colourable if there exists a homomorphism of G to H. A graph G is said to be *uniquely H-colourable*, if there exists an H-colouring f of G such that f is an onto homomorphism and for any other H-colouring f' of G, f' is the composition $f \circ \sigma$ of f with an automorphism σ of H.

Note that a K_n-colouring of G is equivalent to an n-colouring of G, and unique n-colourability of G is equivalent to unique K_n-colourability of G. So the study of the chromatic number of a graph and unique colourability of a graph can be carried out in terms of graph homomorphisms. The same is true for the circular colouring.

For a pair of positive integers p, q such that $p \geq 2q$. Let $K_{\frac{p}{q}}$ be the graph which has vertices $\{0, \cdots, p-1\}$ and in which $\{i, j\}$ is an edge if and only if $q \leq |i - j| \leq p - q$. A $K_{\frac{p}{q}}$-colouring of a graph G is also called a (p, q)-colouring of G. It is known [12] and easy to see that for any graph G, $\chi_c(G) = \inf\{\frac{p}{q} : G \text{ is } K_{\frac{p}{q}}\text{-colourable }\}$. It is also easy to show that a graph G is uniquely $\frac{p}{q}$-colourable if and only if it is uniquely $K_{\frac{p}{q}}$-colourable.

The fractional chromatic number of a graph can be defined through graph homomorphisms to Kneser graphs. Suppose $n \geq 2k$ are positive integers. Let $[n] = \{0, 1, 2, \cdots, n-1\}$ and denote by $\binom{[n]}{k}$ the set of all k-subsets of $[n]$. The *Kneser graph* $K(n, k)$ has vertex set $V = \binom{[n]}{k}$ in which two vertices

A and B are adjacent if, when regarded as subsets of $[n]$, they do not intersect, i.e., $A \cap B = \emptyset$. A homomorphism f from a graph G to $K(n,k)$ is also called a *k-tuple n-colouring* of G. Such a homomorphism f assigns to each vertex x of G a set $f(x)$ of k colours, and if x and y are adjacent, then $f(x) \cap f(y) = \emptyset$, i.e., no colour is assigned to two adjacent vertices. It is known [9] that the fractional chromatic number of G is $\chi_f(G) = \min\{\frac{n}{k} : G$ is $K(n,k)$-colourable $\}$. However, unique fractional p/q-colourability is different from unique H-colourability for any graph H [5]. In particular, a uniquely $K(n,k)$-colourable graph G maybe not uniquely fractional n/k-colourable. This is due to the fact that a fractional n/k-colourable graph may not be $K(n,k)$-colourable. On the other hand, it is proved in [5] that if a graph G is uniquely $K(pt,qt)$-colourable for some integer t, and moreover, for any integer t', if G is $K(pt',qt')$-colourable, then G is uniquely $K(pt',qt')$-colourable, then G is uniquely fractional p/q-colourable.

The purpose of this paper is to construct, for any $2 < \frac{p'}{q'} \leq \frac{p}{q}$, for any integer g, a graph G of girth at least g such that (1): G is uniquely circular $\frac{p}{q}$-colourable, and (2): G is uniquely fractional $\frac{p'}{q'}$-colourable.

2. Main result and some preliminaries

The main result of this paper is the following theorem:

Theorem 1. *Given any two rational numbers $2 < r' \leq r$, for any integer g, there is a graph G of girth at least g such that G is uniquely circular r-colourable and uniquely fractional r'-colourable. Moreover, the graph G has maximum degree bounded by a number which depends on r and r' but does not depend on g.*

To prove Theorem 1, we shall first relax the condition on large girth and prove that for any $2 < r' \leq r$, there is a graph G' which is uniquely circular r-colourable, and also uniquely fractional r'-colourable. Assume $r = \frac{p}{q}$ and $r' = \frac{p'}{q'}$. If $\frac{p}{q} = \frac{p'}{q'}$, then $G' = K_{\frac{p}{q}}$ is uniquely circular r-colourable and uniquely fractional r-colourable. Assume $\frac{p}{q} > \frac{p'}{q'}$. The graph which is uniquely circular r-colourable, and also uniquely fractional r'-colourable is constructed through graph product. For graphs G and H, the categorical product $G \times H$ has vertex set $\{(x,y) : x \in V(G), y \in V(H)\}$. Two vertices (x,y) and (x',y') are adjacent in $G \times H$ if and only if x and x' are adjacent in G, y and y' are adjacent in G. We shall prove that if t is a large enough integer, then the categorical product graph $K(p't,q't) \times K_{\frac{p}{q}}$ is uniquely circular r-colourable and uniquely fractional r'-colourable. The following lemma is easy.

Lemma 2. *For any $2 < \frac{p'}{q'} < \frac{p}{q}$, if t is a large enough integer, then $K(p't,q't) \times K_{\frac{p}{q}}$ is uniquely circular $\frac{p}{q}$-colourable.*

Proof. Suppose H is a core graph, i.e., a graph which admits no homomorphism to any of its proper subgraphs. Let $C(H)$ be the graph whose vertices are all the mappings $f : V(H) \to V(H)$ which are not automorphisms, and whose edges are pairs $\{f, g\}$ such that for any $\{x, y\} \in E(H)$, $\{f(x), g(y)\} \in E(H)$. Since H is a core graph and no vertex f of $C(H)$ is an automorphism, the graph $C(H)$ is loopless. It is proved in [10] that if $\chi(G) > \chi(C(H))$ then $G \times H$ is uniquely H-colourable. As $\chi(K(p't, q't)) = (p' - 2q')t + 2$ [6], it follows that if $t > (\chi(C(K_{\frac{p}{q}})) - 2)/(p' - 2q')$, then $K(p't, q't) \times K_{\frac{p}{q}}$ is uniquely $K_{\frac{p}{q}}$-colourable, and hence uniquely circular $\frac{p}{q}$-colourable. □

Lemma 3. *For any* $2 < \frac{p'}{q'} < \frac{p}{q}$ *and for any integer* t, $K(p't, q't) \times K_{\frac{p}{q}}$ *is uniquely fractional* $\frac{p'}{q'}$*-colourable.*

Proof. For each $i \in \{0, 1, \cdots, p't - 1\}$, let $I_i = \{x \in V(K(p't, q't)) : i \in x\}$ (recall that each vertex of $K(p't, q't)$ is a $q't$-subset of $\{0, 1, \cdots, p't - 1\}$). Then I_i is a maximum independent set of $K(p't, q't)$ and $I_i \times V(K_{\frac{p}{q}})$ is an independent set of $K(p't, q't) \times K_{\frac{p}{q}}$. Let $f : I(K(p't, q't) \times K_{\frac{p}{q}}) \to [0, 1]$ be defined as $f(I_i \times V(K_{\frac{p}{q}})) = 1/q't$ for each $i \in \{0, 1, \cdots, p't - 1\}$ and $f(I) = 0$ for any other independent set I of $K(p't, q't) \times K_{\frac{p}{q}}$. Then f is a $\frac{p'}{q'}$-fractional colouring of $K(p't, q't) \times K_{\frac{p}{q}}$. We need to prove that, up to equivalence, f is the unique fractional $\frac{p'}{q'}$-colouring of $K(p't, q't) \times K_{\frac{p}{q}}$.

Lemma 4. *The independent sets* $I_i \times V(K_{\frac{p}{q}})$ *for* $i = 0, 1, \cdots, p't - 1$ *are the only maximum independent sets of* $K(p't, q't) \times K_{\frac{p}{q}}$.

We shall delay the proof of Lemma 4 for a little while. Now we use Lemma 4 to show that up to equivalence, f is the unique fractional $\frac{p'}{q'}$-colouring of $K(p't, q't) \times K_{\frac{p}{q}}$.

Assume g is a fractional $\frac{p'}{q'}$-coloring of $K(p't, q't) \times K_{\frac{p}{q}}$. We need to prove that for any independent set U of $K(p't, q't) \times K_{\frac{p}{q}}$,

$$g(U) = \begin{cases} 1/q't & \text{if } U = I_i \times V(K_{\frac{p}{q}}) \text{ for some } i \in \{0, 1, \cdots, p't - 1\} \\ 0 & \text{otherwise.} \end{cases}$$

It is well-known [9] that for any vertex transitive graph G, $\chi_f(G) = \frac{|V(G)|}{\alpha(G)}$ and for any optimal fractional colouring f of G, $f(I) = 0$ if I is not a maximum independent set. By Lemma 4, $I_i \times V(K_{\frac{p}{q}})$ for $i = 0, 1, \cdots, p't - 1$ are the only maximum independent sets. Therefore $g(I) = 0$ if $I \neq I_i \times V(K_{\frac{p}{q}})$ for some $i \in \{0, 1, \cdots, p't - 1\}$.

Assume there exists I_t such that $g(I_t \times V(K_{\frac{p}{q}})) \neq 1/q't$. Without loss of generality, assume $g(I_t \times V(K_{\frac{p}{q}})) > 1/q't$. Since $\sum_{i=0}^{p't-1} g(I_i \times V(K_{\frac{p}{q}})) = \frac{p'}{q'}$, there exist $I_{i_1} \times V(K_{\frac{p}{q}}), I_{i_2} \times V(K_{\frac{p}{q}}), \cdots, I_{i_{q't}} \times V(K_{\frac{p}{q}})$ such that $\sum_{t=1}^{q't} g(I_{i_t} \times V(K_{\frac{p}{q}})) < 1$. Let $x = \{i_1, \cdots, i_{q't}\} \in V(K(p't, q't))$. Since $I_{i_1} \times V(K_{\frac{p}{q}}), I_{i_2} \times V(K_{\frac{p}{q}}), \cdots, I_{i_{q't}} \times V(K_{\frac{p}{q}})$ are the only maximum independent sets containing (x, a) for any $a \in V(K_{\frac{p}{q}})$, it follows that $\sum_{(x,a)\in I} g(I) = \sum_{t=1}^{q't} g(I_{i_t} \times V(K_{\frac{p}{q}})) < 1$, in contrary to the assumption that g is a fractional colouring of $K(p't, q't) \times K_{\frac{p}{q}}$. Therefore,

$$g(U) = \begin{cases} 1/q't & \text{if } U = I_i \times V(K_{\frac{p}{q}}) \text{ for some } i \in \{0, 1, \cdots, p't - 1\} \\ 0 & \text{otherwise.} \end{cases}$$

i.e., $K(p't, q't) \times K_{\frac{p}{q}}$ is uniquely fractional $\frac{p'}{q'}$-colourable.

□

3. The proof of Lemma 4

Problems concerning independent sets of the categorical product of graphs have been studied in many papers. For example, Frankl [3] determined the maximum size of independent set of the categorical product of Kneser graphs. Ahlswede , Aydinian and Khachatrian [1] determined the size of the maximum independent set of the categorical product of certain generalized Kneser graphs. The size of the maximum independent set of the categorical product of a Kneser graph with a circular complete graph also follows from a result in [13] concerning the fractional chromatic number of such graphs. In Lemma 4, besides the size of a maximum independent set, we need to determine the structure of all maximum independent sets of the product of a Kneser graph with a circular complete graph. The proof given below is a refinement of the corresponding argument in [13].

Assume that U is a maximum independent set of $K(p't, q't) \times K_{\frac{p}{q}}$ and $U \neq I_i \times K_{\frac{p}{q}}$ for any $i \in \{0, 1, \cdots, p't - 1\}$.

For each vertex x of $K(p't, q't)$, let $U_x = \{y \in K_{\frac{p}{q}} : (x, y) \in U\}$.

Claim 1. *If $\{x, x'\} \in E(K(p't, q't))$ and $U_x \neq \emptyset, U_{x'} \neq \emptyset$, then $|U_x| + |U_{x'}| \leq 2q$.*

Proof. Assume $\{x, x'\} \in E(K(p't, q't))$ and $|U_x| + |U_{x'}| > 2q$. Since $U_x \neq \emptyset$ and $U_{x'} \neq \emptyset$, it is known [13] and easily to verify directly that there exist $a \in U_x$ and $b \in U_{x'}$ such that $\{a, b\} \in E(K_{\frac{p}{q}})$. Then $\{(x, a), (x', b)\} \in E(K(p't, q't) \times K_{\frac{p}{q}})$, in contrary to the assumption that U is an independent set of $K(p't, q't) \times K_{\frac{p}{q}}$. □

Claim 2. *For any vertex x of $K(p't, q't)$, either $|U_x| < 2q$ or $|U_x| = p$.*

Proof. Assume to the contrary that there exists $x \in V(K(p't, q't))$ such that $2q \le |U_x| < p$. By Claim 1, for all $y \in N(x)$, $U_y = \emptyset$. Therefore $U' = U \cup \{(x, a) : a \in K_{\frac{p}{q}} - U_x\}$ is an independent set of $K(p't, q't) \times K_{\frac{p}{q}}$. Since $|U_x| < p$, U' is strictly larger than U. This is in contrary to our assumption that U is a maximum independent set. □

Claim 3. *For any vertex x of $K(p't, q't)$, either $U_x = V(K_{\frac{p}{q}})$ or $U_x = \emptyset$.*

Proof. Let $Y = \{x \in V(K(p't, q't)) : U_x = V(K_{\frac{p}{q}})\}$. By Claim 1, for all $x \in N(Y)$, $U_x = \emptyset$. Let

$$U^* = U \cap (V(K(p't, q't)) - N[Y]) \times V(K_{\frac{p}{q}}).$$

Then U^* is an independent set of $(K(p't, q't) - N[Y]) \times K_{\frac{p}{q}}$. If $U^* = \emptyset$, then we are done. Assume $U^* \neq \emptyset$.

For each independent set Z of $K(p't, q't) - N[Y]$, $Z \cup Y$ is an independent set of $K(p't, q't)$, and hence has cardinality $|Z| + |Y| \le \binom{p't-1}{q't-1}$. Therefore $\alpha(K(p't, q't) - N[Y]) \le \binom{p't-1}{q't-1} - |Y|$. Since $\chi_f(K(p't, q't) - N[Y]) \le \chi_f(K(p't, q't)) = \frac{p'}{q'}$, it follows that

$$\begin{aligned} |V(K(p't, q't) - N[Y])| &\le \alpha(K(p't, q't) - N[Y])\chi_f(K(p't, q't) - N[Y]) \\ &\le (\binom{p't-1}{q't-1} - |Y|)\frac{p'}{q'}. \end{aligned}$$

Since $\frac{p'}{q'} < \frac{p}{q}$, this implies that

$$|V(K(p't, q't) - N[Y])|q + |Y|p < \binom{p't-1}{q't-1}p = |I_i \times V(K_{\frac{p}{q}})|. \quad (1)$$

Let $\kappa = \max\{|U_x| : x \in K(p't, q't) - N[Y]\}$. By Claim 2 and the definition of Y, we know that $\kappa < 2q$. If $\kappa \le q$, then by (1),

$$|U| \le |V(K(p't, q't) - N[Y])|q + |Y|p < |I_i \times V(K_{\frac{p}{q}})|.$$

This is in contrary to the assumption that U is a maximum independent set of $K(p't, q't) \times K_{\frac{p}{q}}$.

Thus we may assume that $q < \kappa < 2q$. For $s = q+1, q+2, \cdots, 2q-1$, let $Y_s = \{x \in V(K(p't, q't)) - N[Y] : |U_x| = s\}$.

Let $q + 1 \le s_0 < s_1 < \cdots < s_m < 2q$ be the integers such that either $Y_{s_i} \neq \emptyset$ or $Y_{2q-s_i} \neq \emptyset$.

$$\begin{aligned} \text{And let} \quad Z_{s_i} &= \{x \in V(K(p't, q't)) - N[Y] : |U_x| = 2q - s_i\} \\ Y_{s_i} &= \{x \in V(K(p't, q't)) - N[Y] : |U_x| = s_i\} \\ \text{and} \quad B &= \{x \in V(K(p't, q't)) - N[Y] : |U_x| = q\}. \end{aligned}$$

Then

$$|U| = |Y|p + |B|q + \sum_{i=0}^{m}(|Y_{s_i}| + |Z_{s_i}|)q - \sum_{i=0}^{m}(|Z_{s_i}| - |Y_{s_i}|)(s_i - q).$$

Now we need the following lemma which is slightly different from Lemma 4.5 of [13], but can be proved the same way.

Lemma 5. *Suppose $\alpha_0, \cdots, \alpha_m$ and $\beta_0, \cdots, \beta_m$ are positive real number such that $\frac{\beta_i}{\alpha_i} \geq \frac{\beta_{i+1}}{\alpha_{i+1}}$ for $i = 0, \cdots, m-1$. If $x_0, \cdots, x_m$ are real numbers satisfying $\sum_{j=0}^{i} \alpha_j x_j > 0$ for all $0 \leq i \leq m$, then $\sum_{j=0}^{i} \beta_j x_j > 0$ for all $0 \leq i \leq m$.*

Let $x_i = |Z_{s_i}| - |Y_{s_i}|$, $\beta_i = s_i - q$, $\alpha_i = 2q - s_i$. Then $\beta_i > 0$ and $\alpha_i > 0$ for all $i = 0, \cdots, m$ and

$$|U| = |Y|p + |B|q + \sum_{i=0}^{m}(|Y_{s_i}| + |Z_{s_i}|)q - \sum_{j=0}^{m} \beta_j x_j$$

If $\sum_{j=0}^{i} \alpha_j x_j > 0$ for all i, then by Lemma 5, $\sum_{j=0}^{i} \beta_j x_j > 0$. This implies that

$$\begin{aligned} |U| &= |Y|p + |B|q + \textstyle\sum_{i=0}^{m}(|Y_{s_i}| + |Z_{s_i}|)q - \sum_{j=0}^{m} \beta_j x_j \\ &< |Y|p + |B|q + \textstyle\sum_{i=0}^{m}(|Y_{s_i}| + |Z_{s_i}|)q \\ &\leq |Y|p + |V(K(p't, q't) - N[Y])|q < |I_i \times V(K_{\frac{p}{q}})|. \end{aligned}$$

This is in contrary to the assumption that U is a maximum independent set of $K(p't, q't) \times K_{\frac{p}{q}}$.

Thus we assume that $\sum_{j=0}^{i} \alpha_j x_j \leq 0$ for some $0 \leq i \leq m$. Let U' be the independent set of $K(p't, q't) \times K_{\frac{p}{q}}$ defined as

- $U'_x = V(K_{\frac{p}{q}})$ if $x \in Y_{s_j}$ for some $j \leq i$;
- $U'_x = \emptyset$ if $x \in Z_{s_j}$ for some $j \leq i$;
- $U'_x = U_x$ otherwise.

Then U' is an independent set of $K(p't, q't) \times K_{\frac{p}{q}}$, and

$$\begin{aligned} |U'| &= |U| - \sum_{j=0}^{i} |Z_{s_j}|(2q - s_j) + \sum_{j=0}^{i} |Y_{s_j}|(p - s_j) \\ &> |U| - \sum_{j=0}^{i} |Z_{s_j}|(2q - s_j) + \sum_{j=0}^{i} |Y_{s_j}|(2q - s_j) \\ &\geq |U|. \end{aligned}$$

This is again in contrary to the assumption that U is an maximum independent set of $K(p't, q't) \times K_{\frac{p}{q}}$.

It follows from Lemma 3 that $U = I \times V(K_{\frac{p}{q}})$ for some independent set I of $K(p't, q't)$. Since U is a maximum independent set of $K(p't, q't) \times K_{\frac{p}{q}}$, we conclude that I is a maximum independent set of $K(p't, q't)$ and hence $I =$

I_i for some $i \in \{0, 1, \cdots, p't - 1\}$, which is in contrary to our assumption. This completes the proof of Lemma 4.

□

4. THE PROOF OF THEOREM 1

For arbitrary core graphs H, uniquely H-colourable graphs of large girth have been studied in many papers. As observed before, unique circular p/q-colourability of a graph is equivalent to the unique $K_{p/q}$-colourability of the graph. However, unique fractional p'/q'-colourability is not equivalent to unique H-colourability for any graph H. As noted in [5], if t is large enough, then $K(p't, q't) \times K(p', q')$ is uniquely $K(p', q')$-colourable but not uniquely fractional p'/q'-colourable. For this reason, the existing results concerning uniquely H-colourable graphs of large girth cannot be applied directly to obtain Theorem 1. Nevertheless, the proof of Theorem 1 below is parallel to the existing probabilistic proofs concerning uniquely H-colourable graphs of large girth.

Suppose F is an n vertex graph with vertices $0, 1, \cdots, n-1$. Given a positive integer m, we denote by $F[m] = F[\overline{K_m}]$ the lexicographic product of F and $\overline{K_m}$. In other words, for each vertex v of F, let $v[m]$ be a set of cardinality m. Then $F[m]$ has vertex set $\cup_{v \in V(F)} v[m]$ such that $x \in v[m]$ is adjacent to $x' \in v'[m]$ if and only if $\{v, v'\}$ is an edge of F.

It is proved in [9, 5] that for any integer g, there exists an integer m, such that $F[m]$ has a spanning subgraph G of girth at least g for which the following hold:

(1) $V(G) = W_0 \cup W_1 \cup \cdots \cup W_{n-1}$ where $W_i = i[m]$ for each $i \in V(F)$.
(2) For any edge $\{v, v'\}$ of F, for any $X \subseteq v[m]$, $Y \subseteq v'[m]$, if $|X| \geq m/40n$ and $|Y| \geq m/40n$, then there is an edge (in G) between X and Y.
(3) For any edge $\{v, v'\}$ of F, for any $X \subseteq v[m]$, $Y \subseteq v'[m]$ with $n \leq |X| = n|Y| \leq \frac{m}{40}$, there are less than $|Y|n^{10}/2$ edges between X and Y.
(4) For any edge $\{v, v'\}$ of F, for any vertex $x \in v[m]$, x has at least $n^{10}/2$ neighbours in $v'[m]$.
(5) Each vertex of G has degree at most $5|V(F)|^{13}$.

If $\frac{p'}{q'} = \frac{p}{q}$, then let $F = K_{\frac{p}{q}}$. If $2 < \frac{p'}{q'} < \frac{p}{q}$, then let $F = K(p't, q't) \times K_{\frac{p}{q}}$, where t is large enough so that F is uniquely circular $\frac{p}{q}$-colourable. To prove Theorem 1, we shall prove that the spanning subgraph G of $F[m]$ with properties (1) and (4) listed above is uniquely circular $\frac{p}{q}$-colourable and also uniquely fractional $\frac{p'}{q'}$-colourable. Property (5) implies that the maximum degree of G is bounded by a number which does not depends on g (but depends on p/q and p'/q'). As unique circular $\frac{p}{q}$-colourability is equivalent to unique $K_{p/q}$-colourability, the following lemma is a special case of Theorem 4 in [4].

Lemma 6. *Suppose G is a spanning subgraph of $F[m]$ with properties (1)-(4) listed above. Then G is uniquely circular $\frac{p}{q}$-colourable.*

Lemma 7. *Suppose G is a spanning subgraph of $F[m]$ with properties (1)-(4) listed above. Then G is uniquely fractional $\frac{p'}{q'}$-colourable.*

Proof. Since $G \subseteq F[m]$ and F is fractional $\frac{p'}{q'}$-colourable, it follows that G is fractional $\frac{p'}{q'}$-colourable. To prove that G is uniquely fractional $\frac{p'}{q'}$-colourable, it suffices to show that each maximum independent set of G is of the form $I[m]$ for a maximum independent set of I of F.

Let α_F and α_G be the size of the maximum independent set of F and G, respectively.

Since G is a spanning subgraph of $F[m]$, we have $\alpha_G \geq \alpha_F m$. Assume $J \in I(G)$, $|J| = \alpha_G \geq \alpha_F m$. Let v be a vertex of F, we denote by $\varphi(v)$ the size of $v[m] \cap J$, i.e., $\varphi(v) = |J \cap v[m]|$. Then, there exists an order of $V(F)$, $\{v_1, v_2, \cdots, v_n\}$, such that $\varphi(v_1) \geq \varphi(v_2) \geq \cdots \varphi(v_n) \geq 0$. Since $\sum_{i=1}^{n} \phi(v_i) \geq \alpha_F m$, we have $\varphi(v_1) \geq \frac{\alpha_F m}{n}$, $\varphi(v_2) \geq \frac{\alpha_F m - m}{n}, \cdots, \varphi(v_{\alpha_F}) \geq \frac{\alpha_F m - (\alpha_F - 1)m}{n - (\alpha_F - 1)} \geq \frac{m}{n}$.

Let $I = \{v_1, v_2, \cdots, v_{\alpha_F}\}$. First we show that I is an independent set of F. If not, then there exists $v_i, v_j \in I$ such that $\{v_i, v_j\} \in E(F)$. Since $v_i[m] \cap J$ has size $\varphi(v_i) \geq \frac{m}{n}$ and $v_j[m] \cap J$ has size $\varphi(v_j) \geq \frac{m}{n}$, there are subsets U of $v_i[m] \cap J$ and W of $v_j[m] \cap J$ such that $|U| = |W| = \lceil \frac{m}{40n} \rceil$. However, by Property (2), there exists an edge between U and W, contrary to the assumption that J is an independent set of G.

Next we show that $\varphi(x_{\alpha+1}) = 0$. Assume to the contrary that if $\varphi(x_{\alpha+1}) \neq 0$, i.e., $v_{\alpha+1}[m] \cap J \neq \emptyset$. Since $I \cup v_{\alpha+1}$ is not independent set of F, there exists a $v_i \in I$ such that $\{v_i, v_{\alpha+1}\}$ is an edge of F. By Property (3), each vertex in $J \cap v_{\alpha+1}[m]$ has at least $n^{10}/2$ neighbours in $v_i[m]$. As $J \cap v_{\alpha+1}[m] \neq \emptyset$ and J is independent in G, it follows that $|v_i[m] - J| \geq n^{10}/2$. Let $W = v_i[m] - J$ and let $\beta = |W|$. Let $\ell = \varphi(v_{\alpha+1})$. Since $\varphi(v_{\alpha+1}) \geq \varphi(v_j)$ for $j = \alpha + 1, \cdots, n$, it follows that $\beta \leq \ell \cdot (n - \alpha) \leq \ell \cdot n$. So $\ell \geq \beta/n$. Let $U \subseteq v_{\alpha+1}[m] \cap J$ be a subset of size β/n. Since each vertex of U has at least $\frac{t^{10}}{2}$ neighbours in $v_i[m] - J = W$, we conclude that there are at least $\frac{t^{10}}{2}|U|$ edges between U and W. This is in contrary to Property (3). Therefore $\varphi(v_{\alpha+1}) = 0$, i.e., if J is a maximum independent set of G, then $J = I[m]$ for some maximum independent set I of F. And we have $I_i \times K_{\frac{p}{q}}$ for $i = 0, 1, \cdots, p't - 1$ are the only maximum independent set of F with size $\binom{p't-1}{q't-1}p$. Therefore, $J = (I_i \times K_{\frac{p}{q}})[m]$ for some $i = 0, 1, \cdots, p't - 1$.

Since G is a spanning subgraph of $F[m]$, G is fractional $\frac{p'}{q'}$-colourable. As $|V(F)| = \binom{p't}{q't}p$, we have $|V(G)| = m\binom{p't}{q't}p$, $\alpha_G = \alpha_F m = \binom{p't-1}{q't-1}pm$, so $\chi_f(G) \geq \frac{|V(G)|}{\alpha_G} = \frac{p'}{q'}$. Thus we know that $\chi_f(G) = \frac{p'}{q'}$. Let J_i be the maximum independent set of G such that $J_i = (I_i \times K_{\frac{p}{q}})[m]$ for $i = 0, 1, \cdots, p't-1$.

Let $f : I(G) \to [0,1]$ such that

$$f(U) = \begin{cases} 1/q't & \text{if } U = J_i \text{ for some } i \in \{0, 1, \cdots, p't-1\}, \\ 0 & \text{otherwise.} \end{cases}$$

then we know that f is a proper fractional $\frac{p'}{q'}$-colouring of G.

Next we want to show that for any fractional $\frac{p'}{q'}$-g of G, $g(I) = f(I)$ for any independent set I of G. As $\chi_f(G) = \frac{|V(G)|}{\alpha_G}$, for any optimal fractional colouring g of G, $g(I) = 0$ if I is not a maximum independent set. As J_i for $i = 0, 1, \cdots, p't-1$ are the only maximum independent sets of G, we have $g(I) = 0$ if $I \neq J_i$ for some $i \in \{0, 1, \cdots, p't-1\}$. It remains to show that for any fractional $\frac{p'}{q'}$-colouring g of G.

$$g(U) = \begin{cases} 1/q't & \text{if } U = J_i \text{ for some } i \in \{0, 1, \cdots, p't-1\}, \\ 0 & \text{otherwise.} \end{cases}$$

This part is similar to the proof of Lemma 3 and omitted. □

REFERENCES

[1] R. Ahlswede, H. Aydinian and L.H. Khachatrian, The intersection theorem for direct products, *Europ. J. Combinatorics*, **19** (1998), 649–661.

[2] T. Emden-Weibert, S. Hougardy and B. Kreuter, Uniquely colourable graphs and the hardness of colouring graphs of large girth, Combinatorics, *Probability and Computing* **7** (1998), 375–386.

[3] P. Frankl, An Erdős-Ko-Rado Theorem for direct products, *Europ. J. Combinatorics* **17** (1996), 727–730.

[4] H. Hajiabolhassan and X. Zhu, Sparse H-colourable graphs of bounded maximum degree, *Graphs and Combinatorics* **20** (2004) 65–71

[5] S. Lin and X. Zhu, Uniquely fractional r-colourable graphs of bounded maximum, degree and large girth, March 2004.

[6] L. Lovasz, Kneser's conjecture, chromatic number and homotopy, *J. Comb. Theory, Ser. A* **25** (1978), 319-324.

[7] J. Nešetřil and X. Zhu, On Sparse Graphs with Given Colorings and Homomorphisms, *J. Comb. Theory Series B* **90** (2004), 161–172.

[8] Z. Pan and X. Zhu, Graphs of large girth with prescribed circular coloring, *Graphs and Combinatorics* **21** (2005), 119–129.

[9] E. R. Scheinerman, D. H. Ullman, *Fractional Graph Theory*, Wiley-Interscience Series in Discrete Mathematics and Optimization (John Wiley & Son, New York, 1997).

[10] X. Zhu, Uniquely H-colorable graphs with large girth, *Journal of Graph Theory* **41** (1996), 33–41.

[11] X. Zhu, Construction of Uniquely H-colorable Graphs, *Journal of Graph Theory* **30** (1999), 1–6.

[12] X. Zhu, Circular chromatic number, a survey, *Taiwanese Journal of Mathematics* **4** (2000), 643–660.

[13] X. Zhu, The Fractional chromatic number of the direct product of graphs, *Glasgow Math. J.* **44** (2002), 103–105.

DEPARTMENT OF APPLIED MATHEMATICS, NATIONAL SUN YAT-SEN UNIVERSITY, KAOHSIUNG 80424, TAIWAN
E-mail address: zhu@math.nsysu.edu.tw

DEPARTMENT OF APPLIED MATHEMATICS, NATIONAL SUN YAT-SEN UNIVERSITY, KAOHSIUNG 80424, TAIWAN, AND NATIONAL CENTER FOR THEORETICAL SCIENCES, ROC
E-mail address: zhu@math.nsysu.edu.tw