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S-SPACES FROM FREE EXTENSIONS

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Dedicated to the centenary of the birth of Ferenc Kárteszi (1907–1989).

Abstract. We prove that there exist S-spaces containing an arbitrary number of non-isomorphic affine planes of any admissible order. The proof is obtained by constructing some new S-spaces in two different ways. In one case we obtain S-spaces of finite order containing an infinite number of points, while in the other case we obtain S-spaces of infinite order.

1. Introduction

The study of S-spaces begun in the early 60's when E. Sperner [10] introduced certain incidence structures similar to ordinary affine spaces, but with some weaker properties regarding the classical Desargues theorem and the concept of dimension. Some fairly recent results on S-spaces are in [4, 7, 8, 9], while a good account on the basic properties of S–spaces can be found in [2].

A generalised affine space (briefly, an S-space) is an incidence structure $\mathfrak{S}$ of "points" and "lines", together with a binary relation between lines which is called "parallelism", satisfying the following axioms:

(1) Any two points are incident with exactly one line;
(2) All the lines are incident with the same number of points;
(3) The parallelism is an equivalence relation;
(4) Given a line ℓ and a point x, there exists exactly one line ℓ' in $\mathfrak{S}$ which is incident with x and parallel to ℓ.

Using Axioms (3) and (4) we find that if two lines ℓ_1 and ℓ_2 are parallel, then either $\ell_1 = \ell_2$ or $\ell_1 \cap \ell_2 = \varnothing$.

Ordinary affine spaces provide the first examples of S-spaces, while an S-space $\mathfrak{S}$ which is not an ordinary affine space is called a "proper" S-space. Further, if the number of points of Axiom (2) is finite, say n, then $\mathfrak{S}$ is called a finite S-space of order n.

It is well known that the only subspaces of dimension 2 contained in an ordinary affine space are Desarguesian affine planes, while this is not true

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in general when a proper S-space is considered. For a proper S-space $\mathfrak{S}$, the following questions arise:

- How many non-isomorphic affine planes are contained in $\mathfrak{S}$ as subspaces?
- What are the maximum and the minimum number of non-isomorphic affine planes through a point?

This problem was originally posed in a more general setting by Barlotti [2] who defined the regularity parameters of an S-space $\mathfrak{S}$, that is, the minimum number m_r and the maximum number M_r of ordinary affine spaces of dimension r through a point of $\mathfrak{S}$.

In this paper we construct finite S-spaces containing k non-isomorphic affine planes of given order n for any $k < \delta + 1$, with δ denoting the number of isomorphism classes of affine planes of order n, and show that for such an S-space $m_2 \geq k$ holds with an arbitrarily large number of non-isomorphic affine planes through each point.

2. Preliminaries

From Axiom (4) it follows that through every point of a finite S-space $\mathfrak{S}$ of order n there pass the same number of lines. Let $b(x)$ be the number of lines through each point x of an S-space $\mathfrak{S}$. The "dimension" of $\mathfrak{S}$ is given by one of the following:

- if $b(x) = \infty$ for any $x \in \mathfrak{S}$, then $\mathfrak{S}$ has infinite dimension;
- if there is a positive integer r such that
$$b(x) = \frac{n^r - 1}{n - 1}$$
for a fixed $n \in \mathbb{N}$ and any $x \in \mathfrak{S}$, then $\mathfrak{S}$ has regular dimension r;
- if none of the above cases occurs, then $\mathfrak{S}$ has no regular dimension.

S-spaces with no regular dimension actually exist, see [5, 6], while S-spaces of regular dimension 2 are always ordinary affine planes, see [1, Theorem 1.2.1] for instance.

In the remainder of this section we recall an inductive method for constructing S-spaces due to A. Barlotti [2].

Let $\mathcal{S} = (P, L)$ be a near linear space with set of points P and set of lines L (see [3]), such that no line contains more than s points for a certain positive integer s. Set

$$\{ A_j = (P_j, L_j) \mid j = 1, 2, 3, \dots \},$$

where A_j is an incidence structure of "points" and "lines" with point set P_j and line set L_j defined as follows:

(1) $A_0 = \mathcal{S}$;
(2) A_{h+1} is obtained from A_h as follows:
  (a) let $\mathcal{F}$ be a family of subsets of P_h such that:
    (i) no such subset contains two points on a line of L_h;

(ii) every two points of P_h belong to exactly one subset of $\mathcal{F}$;
(iii) every subset of $\mathcal{F}$ contains k points, with $1 < k \le s$.

If A_h is not an S-space in its own right, then there exists a set $\mathcal{F}$ as above: One example is provided by the set of pairs of points which are not joined by a line of L_h. Once we found such a set $\mathcal{F}$, we consider its subsets as new "lines" that will be added to those of L_h to obtain an incidence structure $A_h^{(1)} = (P_h^{(1)}, L_h^{(1)})$, with $L_h^{(1)} = L_h \cup \mathcal{F}$. Then we extend in a natural way the existing parallelism to these new lines by considering each of them parallel to itself. Doing so, we introduce some new classes of parallelism, each consisting of a single line.

(b) Add to any line of $L_h^{(1)}$ containing $k < s$ points $s-k$ new points. This yields a new incidence structure $A_h^{(2)} = (P_h^{(2)}, L_h^{(2)})$.
(c) Choose some subsets of $P_h^{(2)}$ such that no two points in each of them are on a line or on two parallel lines of $L_h^{(2)}$. Contract each of these subsets to one point, in order to obtain a new incidence structure $A_h^{(3)} = (P_h^{(3)}, L_h^{(3)})$.
(d) Let ℓ_1 and ℓ_2 be two lines such that no line parallel to one of them meets a line parallel to the other. If such pairs of lines exist, then define a new parallelism class containing ℓ_1, ℓ_2, all the lines parallel to ℓ_1 and all the lines parallel to ℓ_2. This yields new incidence structure $A_h^{(4)} = (P_h^{(4)}, L_h^{(4)})$.
(e) Let $\ell \in L_h^{(4)}$. For every point $x \in P_h^{(4)}$ not contained in a line parallel to ℓ add a new line ℓ_x (initially containing the point x only) to the parallelism class of ℓ. The incidence structure so constructed is A_{h+1}.

An incidence structure A_t, $t \in \mathbb{N}$, obtained from a near linear space $\mathcal{S}$ as above is called an extension of order t of $\mathcal{S}$. Such an extension is called a free extesion if every subset of 2a contains exactly two points and neither the contraction of 2c, nor the modification of 2d are performed.

Theorem 2.1 (Barlotti)**.** *The incidence structure*

$$\mathfrak{S} = \lim_{h \to \infty} A_h$$

is an S-space.

Using free extensions, Barlotti was able to construct a class of S-spaces with regularity parameter $M_2 = 0$, that is, S-spaces containing no affine planes.

3. The required S-spaces

Let δ denote the number of non-isomorphic affine planes of a certain order n.

Theorem 3.1. *For every positive integer $k < \delta + 1$, let $\{\pi_0, \pi_1, \ldots, \pi_{k-1}\}$ be a set of non-isomorphic affine planes of order n. Then there exists an S-space $\mathfrak{S}$ of order n containing all the π_j as subspaces. Furthermore, $\mathfrak{S}$ has regularity parameter $m_2 \geq k$.*

Proof. For a prime power n, let $A_0 = (P, L)$ be a near linear space whose longest line contains at most $k \leq n$ points, but containing some lines of size less than n. For $h > 0$ let A_h be a free extension of A_0, and $A_h^{(2)} = (P_h^{(2)}, L_h^{(2)})$ the incidence structure obtained after performing 2b on A_h. Let j be an integer with $1 \leq j \leq k$. If $h \equiv j \pmod{k}$, then for each point $x \in P_h^{(2)}$ not contained in any affine plane isomorphic to π_j add a set B of $n^2 - 1$ new points in such a way that $\{x\} \cup B$ yields an affine plane isomorphic to π_j. Denote the resulting incidence structure by A_{h+1}.

After m such extensions of A_h, with $1 < m < k$, we end up with an incidence structure A_{h+m} containing points which are in no affine plane isomorphic to π_j; however, these points can be included in such affine planes extending A_{h+m} again. The incidence structure

$$\mathfrak{N} = \lim_{h \to \infty} A_h$$

is a finite S-space of order q satisfying all the required conditions. The existence of the parallelism is granted by the fact that a free extesion includes 2e. □

Note that the S-space $\mathfrak{N}$ arising from Theorem 3.1 is a finite S-space of order n containing an infinite number of points. Now we are going to construct S-spaces of infinite order instead, in order to prove the following result.

Theorem 3.2. *There exist S-spaces satisfying $M_2 = m_2 = \infty$.*

Proof. As in the proof of Theorem 3.1, we start off with a near linear space A_0 whose lines have length at most s. For every $h \geq 0$, obtain a free extension of A_h by adding $s + h - r$ points to each line of $L_h^{(1)}$ containing $r \leq s$ points, and denote by $B_{h+1} = (P'_{h+1}, L'_{h+1})$ the resulting incidence structure. If h is an integer such that no affine plane of order $s + h$ exists, then put $B_{h+1} = A_{h+1}$ and go on; otherwise, for every point $x \in P'_{h+1}$ add $(s + h)^2 - 1$ more points in such a way that these points together with x constitute an affine plane of order $s + h$. The resulting S-space

$$\mathfrak{M} = \lim_{h \to \infty} A_h$$

has infinite order, and contains finite affine planes of any admissible order. Further, the condition $M_2 = m_2 = \infty$ is an obvious consequence of the construction. □

We remark that all the S-spaces arising from both Theorems 3.1 and 3.2 have infinite dimension.

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