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ON THE NUMBER OF COMPONENTS OF A GRAPH

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Abstract. Let $G := (V, E)$ be a simple graph; for $I \subseteq V$ we denote by $l(I)$ the number of components of $G[I]$, the subgraph of G induced by I. For $V_1, \dots, V_n$ finite subsets of V, we define a function $\beta(V_1, \dots, V_n)$ which is expressed in terms of $l\left(\bigcup_{i=1}^{n} V_i\right)$ and $l(V_i \cup V_j)$ for $i \leq j$. If $V_1, \dots, V_n$ are pairwise disjoint finite independent subsets of V, the number $\beta(V_1, \dots, V_n)$ can be computed in terms of the cyclomatic numbers of $G\left[\bigcup_{i=1}^{n} V_i\right]$ and $G[V_i \cup V_j]$ for $i \neq j$. In the general case, we prove that $\beta(V_1, \dots, V_n) \geq 0$ and characterize when $\beta(V_1, \dots, V_n) = 0$. This special case yields a formula expressing the length of members of an interval algebra [6] as well as extensions to pseudo-tree algebras. Other examples are given.

1. Presentation of the Main Result

1.1. **Main Result.** Let $G := (V, E)$ be a graph, where V is the vertex set and E is the edge set. We suppose that E is a subset of the set $[V]^2$ of unordered pairs of V. Let I be a subset of V, we denote by $G[I]$ the graph $(I, E \cap [I]^2)$ induced by G on I. We denote by $l(G[I])$, or $l_G(I)$, or more simply $l(I)$ if there is no ambiguity, the number of components of the graph $G[I]$. As much as possible, we abbreviate component of $G[I]$ by *component of* I. We assume that $l_G(\varnothing) = 0$.

Definition 1.1. *To an integer n and a family $(V_1, \dots, V_n)$ of finite subsets of V we associate a number, denoted $\beta_G(V_1, \dots, V_n)$, or $\beta(V_1, \dots, V_n)$ if there is no ambiguity, and defined as follows:*

$$\beta(V_1, \dots, V_n) := l\left(\bigcup_{i=1}^{n} V_i\right) - \sum_{1 \leq i < j \leq n} l(V_i \cup V_j) + (n-2) \sum_{i=1}^{n} l(V_i).$$

Notice that $\beta_G(V_1) = 0$. Notice also that $\beta(V_1, \dots, V_n, \varnothing) = \beta(V_1, \dots, V_n)$. So, in the sequel, when we calculate $\beta(V_1, \dots, V_n)$ we may suppose that each V_i is nonempty.

Let $n \leq m$ be two nonnegative integers, the set $\{k \in \mathbb{N} \mid n \leq k \leq m\}$ is denoted by $[n, m]$. For $n \in \mathbb{N} \setminus \{0\}$, the *successor function modulo* n denoted

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by $s_n : [0, n-1] \longrightarrow [0, n-1]$ is defined by $s_n(n-1) := 0$ and if $n \geq 2$ and $i \in [0, n-2]$ then $s_n(i) := i+1$.

A *path* is a sequence of pairwise distinct vertices of G, $v_0, \ldots, v_{k-1}$, such that, for each index $i \in [0, k-2]$, the pair $\{v_i, v_{i+1}\}$ is an edge; the *length* of this path is k. A *circuit* of G is a sequence $\sigma := v_0, \ldots, v_{k-1}$ of at least three vertices of G such that, for each index $i \in [0, k-1]$, $\{v_i, v_{s_k(i)}\}$ is an edge and such that all edges in $||\sigma|| := \{\{v_i, v_{s_k(i)}\} \mid i \in [0, k-1]\}$ are pairwise distinct; the set of edges $||\sigma||$ is called the *support* of σ. A path which is also a circuit is called a *cycle*.

Definition 1.2. *Let $\xi := (V_1, \ldots, V_n)$ be a family of subsets of V. A ξ-*labeled path *is a pair (π, i) such that π is a path whose vertices belong to V_i. Let (π, i) and (π', i') be two ξ-labeled paths with $\pi := v_0, \ldots, v_k$ and $\pi' := v'_0, \ldots, v'_{k'}$, we say that (π, i) is* joinable *to (π', i') if $i \neq i'$ and either $v_k = v'_0$ or $\{v_k, v'_0\}$ is an edge. A ξ-*path-cycle *is a sequence of ξ-labeled paths $(\pi_0, i_0), \ldots, (\pi_{k-1}, i_{k-1})$, $k \geq 3$, which satisfies two conditions:* (i) *for each index $l \in [0, k-1]$, (π_l, i_l) is joinable to $(\pi_{s_k(l)}, i_{s_k(l)})$ and* (ii) *if $l \neq l'$ and $i_l = i_{l'}$ then π_l and $\pi_{l'}$ belong to different components of V_{i_l}. This ξ-path-cycle is* 2-colored *if the set $\{i_0, \ldots, i_{k-1}\}$ has at most two elements.*

We notice that if $\xi := (V_1, \ldots, V_n)$ is a family of pairwise disjoint sets of vertices then every ξ-path-cycle $\Pi := (\pi_0, i_0), \ldots, (\pi_{k-1}, i_{k-1})$ induces an ordinary cycle $C := \pi_0, \ldots, \pi_{k-1}$ in G; such a ξ-path-cycle is 2-colored if and only if the vertices of the induced cycle belong to $V_i \cup V_j$ for some indexes i, j.

Recall that a partition $\kappa := (V_1, \ldots, V_n)$ of the set of vertices of a graph $G := (V, E)$ into independent subsets is said to be a *coloration of* G. A cycle $C := v_0, \ldots, v_{k-1}$ is 2-*colored by* κ if there are two distinct indexes i, j such that for each index $l \in [0, k-1]$, either $v_l \in V_i$ and $v_{s_k(l)} \in V_j$ or $v_l \in V_j$ and $v_{s_k(l)} \in V_i$. Let $\kappa := (V_1, \ldots, V_n)$ be a coloration of a graph G. A 2-colored κ-path-cycle induces a cycle which is 2-colored by κ and conversely.

We state next the main result of this paper:

Theorem 1.3. *Let $G := (V, E)$ be a graph and $\xi := (V_1, \ldots, V_n)$ be a family of finite subsets of V. Then*

(a) $\beta(V_1, \ldots, V_n) \geq 0$;
(b) $\beta(V_1, \ldots, V_n) = 0$ *if and only if every ξ-path-cycle of $G[V_1 \cup \cdots \cup V_n]$ is 2-colored.*

We give two proofs of Theorem 1.3. The first one (see Section 2) is algebraic. The second one (see Section 5) is purely combinatorial.

1.2. **Motivation.** These results originate in the study of Boolean algebras. Let C be a chain with a first element. The *interval algebra $B(C)$ of C* is the subalgebra of the power set $\mathfrak{P}(C)$ of C generated by the collection $I_h(C)$ of half-open intervals $[a, b[$ with $a \in C$, $b \in C \cup \{+\infty\}$ and $a < b$. To each element $x \in B(C)$ we associate an integer, the *length* of x denoted by

$l_{B(C)}(x)$, or $l(x)$ when there is no ambiguity, and defined as follows: $l(x) := 0$ if $x = \varnothing$, otherwise $l(x)$ is the unique integer n such that $x = \bigcup_{i<n}[a_{2i}, a_{2i+1}[$ and $a_0 < a_1 < \cdots < a_{2n-1} \in C \cup \{+\infty\}$. A formula involving lengths of unions of elements of an interval algebra $B(C)$ appeared in Pouzet and Rival [4]. In order to prove that countable lattices are retracts of products of chains, they proved that for every $x, y \in B(C)$:

$$l(x \cup y) + l(x \cap y) \leq l(x) + l(y). \tag{1.1}$$

Later, Bonnet and Si Kaddour [5], in order to prove that interval algebras on a scattered chain have a well-founded set of generators, proved that for every $x, y \in B(C)$:

$$l(x \cup y) + l(x \cap y) + l(x \setminus y) + l(y \setminus x) = l(x) + l(y) + l(x \Delta y). \tag{1.2}$$

Note that (1.2) implies (1.1). The proof of (1.2) needed a lengthy case analysis. The last author [6], gave an equivalent formulation of (1.2) by proving that for every pairwise disjoint elements x, y and z of $B(C)$:

$$l(x \cup y \cup z) = l(x \cup y) + l(x \cup z) + l(y \cup z) - l(x) - l(y) - l(z). \tag{1.3}$$

He also extended the above formula, proving that for every integer and every family $(x_i)_{1 \leq i \leq n}$ of pairwise disjoint elements of $B(C)$:

$$l\left(\bigcup_{i=1}^{n} x_i\right) = \sum_{1 \leq i < j \leq n} l(x_i \cup x_j) - (n-2)\sum_{i=1}^{n} l(x_i). \tag{1.4}$$

It was natural to ask if formula (1.4) holds in a more general situation. This was the motivation for this research. It led us to the Theorems 1.3 and 2.2.

1.3. **Application.** To conclude, let us explain how to derive formula (1.4) from Theorem 1.3. Let $G = (I_h(C), E)$ be the graph where $\{x, y\} \in E$ if and only if $x \cup y$ is an interval. Let $(x_i)_{1 \leq i \leq n}$ be a family of pairwise disjoint elements of $B(C)$. Each x_i is an union of $l(x_i)$ intervals x_{i_j}, where $1 \leq j \leq l(x_i)$, such that the union of x_{i_j} and $x_{i_{j'}}$ for $j \neq j'$ is not an interval of C (in particular x_{i_j} and $x_{i_{j'}}$ are disjoint). For each i set $V_i := \{x_{i_j} \mid 1 \leq j \leq l(x_i)\}$. The V_i's are pairwise disjoint (since the x_i's are pairwise disjoint) contain $l(x_i)$ vertices and are independent. Moreover, $G[V_1 \cup \cdots \cup V_n]$ contains no cycle at all. Indeed, if it contains a cycle S, let p be the leftmost half-open interval belonging to S. Let q, r be the neighbours of p in S. Since $p \cup q$ and $p \cup r$ are intervals, q and r must be on the right of p and hence they have the same minimum. This contradicts the fact that q and r are disjoint intervals. From (2) of Theorem 1.3 we have $\beta(V_1, \ldots, V_n) = 0$. Since a set of vertices $\{p_1, \ldots, p_l\}$ of $G[V_1 \cup \cdots \cup V_n]$ is connected if and only if $p_1 \cup \cdots \cup p_l$ is an interval, then $l_{G[V_1 \cup \cdots \cup V_n]}(U) = l_{B(C)}(\cup U)$ for every subset U of vertices; hence $\beta(V_1, \ldots, V_n) = 0$ translates to formula (1.4).

This approach generalizes. Consider a set $\mathcal{P}$ of subsets of some set X. Let $\mathcal{F}(\mathcal{P})$ be the set of finite unions of members of $\mathcal{P}$. The *length*, $l(u)$, of $u \in \mathcal{F}(\mathcal{P})$ is the least n such that there are $p_1, \ldots, p_n \in \mathcal{P}$ satisfying

$u = p_1 \cup \cdots \cup p_n$. In Section 6 we give a condition which ensures that formula (1.4) holds for all members of $\mathcal{F}(\mathcal{P})$. In Section 7 we give examples of such $\mathcal{P}$'s:

- The set of intervals of a chain. (Proposition 7.5).
- The set of connected sets of a forest. (Proposition 7.2).
- The set of truncated cones. (Proposition 7.9). This says that formula (1.4) holds for members of pseudo-tree algebras. These boolean algebras generalize interval algebras [1, 3].

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2. Algebraic study of β

In this section we prove Theorem 1.3, by algebraic means, for the case of families of disjoint and independent sets of vertices.

2.1. **Relation Between β and the Cyclomatic Number.** Following [2] we give the definition of the cycle space in a graph. Let G be a finite graph and $u_1, \ldots, u_p$ be a *fixed* enumeration of the edge set E of G. Let $\mathbb{F}_2 := \{0, 1\}$ be the two element field. To each subset C of E, associate the vector $[C] = (c_1, \ldots, c_p) \in \mathbb{F}_2^p$ with $c_i = 1$ if $u_i \in C$, and $c_i = 0$ otherwise. For each vector $D := (d_1, \ldots, d_p) \in \mathbb{F}_2^p$, its *support* is the set $||D|| := \{u_i \mid d_i = 1\}$. Note that, if C is a cycle of G, $||C||$ does not mean the support of a vector in $\mathbb{F}_2^p$ just defined (as a cycle is a sequence of vertices, not a vector in $\mathbb{F}_2^p$), but the support of a circuit defined just before Definition 1.2. The *cycle space*, $\mathcal{S}(G)$, is the $\mathbb{F}_2$-vector space generated by the family of vectors $[||C||]$ for C cycle of G. For each cycle C of G, since there is no ambiguity, we will use the notation $[C]$ instead of $[||C||]$.

A circuit which visits each edge of G is an *Eulerian tour* of G. A connected graph is *Eulerian* if it has a an Eulerian tour. As proved in [2], $\mathcal{S}(G)$ is characterized by the following:

Proposition 2.1. *Let $G := (V, E)$ be a finite graph such that $|E| = p$. A vector $D \in \mathbb{F}_2^p$ belongs to $\mathcal{S}(G)$ if and only if each component of the graph $(V, ||D||)$ is Eulerian.*

The dimension of $\mathcal{S}(G)$, denoted by $\nu(G)$, is called the *cyclomatic number* of G. We recall the equality—see for example [2] or Corollary 4.7—relating the numbers $v(G)$ of vertices, $e(G)$ of edges, $l(G)$ of components of a graph G and its cyclomatic number $\nu(G)$:

$$\nu(G) = e(G) - v(G) + l(G). \tag{2.1}$$

It turns out that the numbers β and ν are closely related:

Theorem 2.2. *Let $G := (V, E)$ be a finite graph and $\xi := (V_1, \ldots, V_n)$ be a coloration of G, then:*

(a) $\beta(V_1, \ldots, V_n) = \nu(G) - \sum_{1 \le i < j \le n} \nu\left(G[V_i \cup V_j]\right)$;

(b) $\beta(V_1,\ldots,V_n)=\nu(G)$ *if and only if* G *contains no cycle that is 2-colored by* ξ.

Proof. Item (a): By definition of β we have

$$\begin{aligned}\beta(V_1,\ldots,V_n) &= l\left(\bigcup_{i=1}^{n} V_i\right) - \sum_{1\leq i<j\leq n} l(V_i\cup V_j) + (n-2)\sum_{i=1}^{n} l(V_i)\\ &= l(V) + (n-2)\mid V\mid - \sum_{1\leq i<j\leq n} l(V_i\cup V_j).\end{aligned}$$

From equation (2.1) we have

$$\begin{aligned}&= \nu(G) - e(G) + \mid V\mid + (n-2)\mid V\mid\\ &\quad - \sum_{1\leq i<j\leq n}\left[\nu\left(G[V_i\cup V_j]\right) - e\left(G[V_i\cup V_j]\right) + \mid V_i\cup V_j\mid\right].\end{aligned}$$

Hence

$$\begin{aligned}&= \nu(G) - \sum_{1\leq i<j\leq n}\nu\left(G[V_i\cup V_j]\right)\\ &\quad + (n-1)\mid V\mid - \sum_{1\leq i<j\leq n}\mid V_i\cup V_j\mid\\ &\quad - \left(e(G) - \sum_{1\leq i<j\leq n} e\left(G[V_i\cup V_j]\right)\right).\end{aligned}$$

Since $(V_1,\ldots,V_n)$ is a coloration of G, we have

$$e(G) = \sum_{1\leq i<j\leq n} e\left(G[V_i\cup V_j]\right) \quad \text{and} \quad \sum_{1\leq i<j\leq n}\mid V_i\cup V_j\mid = (n-1)\mid V\mid.$$

Consequently:

$$\beta(V_1,\ldots,V_n) = \nu(G) - \sum_{1\leq i<j\leq n}\nu\left(G[V_i\cup V_j]\right).$$

Item (b): Applying item (a) we have

$$\begin{aligned}\beta(V_1,\ldots,V_n)=\nu(G) &\iff \sum_{1\leq i<j\leq n}\nu(G[V_i\cup V_j]) = 0\\ &\iff \nu(G[V_i\cup V_j]) = 0, \text{for all } i\neq j\\ &\iff \text{There is no cycle that is 2-colored by } \xi.\end{aligned}$$

□

We recall that a graph G is *triangulated* if each cycle C of length greater than 3 has a chord *i.e.* an edge joining two non-adjacent vertices of C. A graph G is *acyclic* or a *forest* if it has no cycle. An instance of Theorem 2.2 is:

Proposition 2.3. *Let $G = (V, E)$ be a finite graph and $(V_1, \dots, V_n)$ be a coloration of G. If G is acyclic then $\beta(V_1, \dots, V_n) = 0$. If G has no cycle of even length or is triangulated then $\beta(V_1, \dots, V_n) = \nu(G)$.*

2.1.1. *A derivation of formula (1.4) from Proposition 2.3.* Associate to the chain C the chain C' which includes for each element a of C three elements a_l, a_m and a_r. The order on C' being defined by: $a_l < a_m < a_r$ and $a_r < b_l$ for each $a < b$ in C. Let $G' = (I_h(C'), E')$ be the graph where $\{x', y'\} \in E'$ if and only if $x' \cap y' \neq \varnothing$. This graph as well as its induced subgraphs are called *interval graph.* It is well-known that interval graphs are triangulated.[1] Let $G = (I_h(C), E)$ be the graph defined in Subsection 1.3. For each interval $p := [a, b[$ of C we associate the interval $p' := [a_l, b_r[$ of C'. We remark that $[a, b[\cup [c, d[$ is an interval of C if and only if

$$[a_l, b_r[\, \cap\, [c_l, d_r[\, \neq \varnothing.$$

So, G is isomorphic to an induced subgraph of G'. Hence, denoting by U' the set of vertices in G' associated to a set of vertices U of G, we have $\beta_G(V_1, \dots, V_n) = \beta_{G'}(V'_1, \dots, V'_n)$ for any family $(V_1, \dots, V_n)$ of sets of vertices of G. Since G' is an interval graph we have by Proposition 2.3, $\beta_{G'}(V'_1, \dots, V'_n) = \nu(G'[V'_1 \cup \dots \cup V'_n])$ whenever the $V'_i s$ are pairwise disjoint independent sets. Moreover, in our case $G'[V'_1 \cup \dots \cup V'_n]$ has no cycle (see subsection 1.3), hence $\nu(G'[V'_1 \cup \dots \cup V'_n]) = 0$. Formula (1.4) follows.

2.2. **The Sign of β.** For a subset F of the vector space $\mathbb{F}_2^p$, we denote by $\langle F \rangle$ the vector subspace generated by F. Let $(V_1, \dots, V_n)$ be a coloration of G. Given $1 \le i < j \le n$, a family $B_{i,j} \subseteq \mathcal{S}(G)$ is a *cycle basis* of $G[V_i \cup V_j]$ if $B_{i,j}$ is a basis of $\langle \{[C] \in \mathcal{S}(G) \mid C \text{ is a cycle of } G[V_i \cup V_j]\ \} \rangle$. We notice that $\nu(G[V_i \cup V_j]) = |B_{i,j}|$.

Lemma 2.4. *Let $G = (V, E)$ be a finite graph and $\{V_1, \dots, V_n\}$ be a coloration of G. For each $\{i, j\} \subseteq [1, n]$ such that $i \neq j$, let $B_{i,j}$ be a cycle basis of $G[V_i \cup V_j]$. Then $B := \bigcup_{1 \le i < j \le n} B_{i,j}$ is linearly independent.*

Proof. For $\{k, l\} \neq \{i, j\}$, the set of edges of $G[V_i \cup V_j]$ is disjoint from the set of edges of $G[V_k \cup V_l]$. Hence, if $D := (d_1, \dots, d_{|E|}) \in \langle B_{i,j} \rangle$ and $1 \le q \le |E|$ such that $d_q = 1$ then for each vector $D' := (d'_1, \dots, d'_{|E|}) \in \langle B_{k,l} \rangle$ we have $d'_q = 0$. So, $D \notin \langle \bigcup \{B_{k,l} \mid 1 \le k < l \le n, \{k, l\} \neq \{i, j\}\} \rangle$. □

Theorem 2.5. *Let G be a finite graph and let $(V_1, \dots, V_n)$ be a coloration of G. Then $\beta(V_1, \dots, V_n) \ge 0$.*

Proof. Let $B_{i,j}$ be a cycle basis of $G[V_i \cup V_j]$. By Lemma 2.4, $\bigcup_{1 \le i < j \le n} B_{i,j}$ is linearly independent and, since the $B_{i,j}$'s are pairwise disjoint, we have:

$$\sum_{1 \le i < j \le n} \nu\left(G[V_i \cup V_j]\right) \le \nu(G).$$

The conclusion follows from item (a) of Theorem 2.2. □

[1] The converse of this theorem is false; see for example [7].

Theorem 2.6. *Let G be a finite graph and $\xi := (V_1, \ldots, V_n)$ be a coloration of G. Then $\beta(V_1, \ldots, V_n) = 0$ if and only if every cycle of G is 2-colored by ξ.*

Proof. Let $B_{i,j}$ be a cycle basis of $G[V_i \cup V_j]$ and let $B := \bigcup_{1 \le i < j \le n} B_{i,j}$.

Assume that $\beta(V_1, \ldots, V_n) = 0$: Then by item (a) of Theorem 2.2:

$$\nu(G) = \sum_{1 \le i < j \le n} \nu(G[V_i \cup V_j]). \tag{2.2}$$

By Lemma 2.4, the set of vectors B is linearly independent, and by (2.2) and the fact that $B_{i,j} \cap B_{k,l} = \varnothing$ for $\{i, j\} \ne \{k, l\}$, the set B is maximal, hence is a basis of $\mathcal{S}(G)$.

Since B is a basis, if C is a cycle of G, there is a family $\{D_{i,j} \in \langle B_{i,j} \rangle \mid 1 \le i < j \le n\}$ such that $[C] = \sum_{1 \le i < j \le n} D_{i,j}$. Since the sets of edges of $G[V_i \cup V_j]$ and of $G[V_k \cup V_l]$ are disjoint for $\{i, j\} \ne \{k, l\}$, we have $\| C \| = \bigcup_{1 \le i < j \le n} \| D_{i,j} \|$. This implies that $\|C\| = \|D_{i,j}\|$ for some i, j proving that C is 2-colored by ξ. Indeed, since $\{i, j\} \ne \{k, l\}$, $D_{i,j} \ne \varnothing$ and $D_{k,l} \ne \varnothing$ would imply, by Proposition 2.1, that $\|C\|$ contains the support of two cycles with no common edge.

Assume that every cycle of G is 2-colored by ξ: This means that for every cycle C there are $i \ne j$ such that C is a cycle of $G[V_i \cup V_j]$. Hence B is a generating family of $\mathcal{S}(G)$. By Lemma 2.4 we deduce that B is a basis of $\mathcal{S}(G)$. That is

$$\nu(G) = \sum_{1 \le i < j \le n} | B_{i,j} | = \sum_{1 \le i < j \le n} \nu(G[V_i \cup V_j]).$$

By item (a) of Theorem 2.2, $\beta(V_1, \ldots, V_n) = 0$.

□

3. The Connection Graph

In this section we prove that Theorem 1.3 can be derived from the fact that it holds for the case of families of disjoint and independent sets of vertices.

Given a graph $G := (V, E)$ and a family $(V_1, \ldots, V_n)$ of (nonempty) finite subsets of vertices of G, we show in Theorem 3.6 that $\beta(V_1, \ldots, V_n)$ is equal to $\beta(V'_1, \ldots, V'_n)$ for a family $(V'_1, \ldots, V'_n)$ of pairwise disjoint independent sets of vertices in $G(V_1, \ldots, V_n)$, the connection graph, we introduce below.

Definition 3.1. *The* connection graph *of G, denoted by $G(V_1, \ldots, V_n)$, is defined as follows:*

- *The vertices of $G(V_1, \ldots, V_n)$ are pairs (p, i) where p is a component of $G[V_i]$.*
- *The edges of $G(V_1, \ldots, V_n)$ are pairs of distinct vertices (p, i), (q, j) such that $p \cup q$ is connected.*

To each set q' of vertices of $G(V_1, \dots, V_n)$ we associate the set $\psi(q')$ of vertices of $G[V_1 \cup \dots \cup V_n]$ defined as follows:

$$\psi(q') := \bigcup \{p \mid (p, i) \in q' \text{ for some } i,\ 1 \le i \le n\}$$

Lemma 3.2. *A set q' of vertices of $G(V_1, \dots, V_n)$ is connected if and only if $\psi(q')$ is connected.*

Proof. Suppose that $\psi(q')$ is connected. Let (p, i) and (p', i') be elements of q'. Pick $v \in p$ and $v' \in p'$. Since $\psi(q')$ is connected and contains $p \cup p'$, there is a sequence $v_0, \dots, v_k$ of vertices of $\psi(q')$ such that $v_0 = v$, $v_k = v'$ and $\{v_j, v_{j+1}\} \in E$ for every j such that $0 \le j \le k - 1$. For each one of these j's choose (p_j, i_j) in q' such that $v_j \in p_j$, $(p_1, i_1) = (p, i)$ and $(p_k, i_k) = (p', i')$. Then there is a subsequence of $(p_1, i_1), \dots, (p_k, i_k)$ that is a path in q' containing (p_1, i_1) and (p_k, i_k). Hence q' is connected.

Conversely, suppose that q' is connected. Assume first that q' is finite. We prove by induction on the size $k := |q'|$ of q' that $\psi(q')$ is connected.

Assume $k = 1$: We have $q' = \{(p, i)\}$ and $\psi(q') = p$ which is connected.

Assume $k > 1$: Since q' is connected, there is some $(p, i) \in q'$ such that $q'' := q' \setminus \{(p, i)\}$ is connected. Since q' is connected there is some $(p', i') \in q'$, $(p', i') \neq (p, i)$ such that $p \cup p'$ is connected. Since $p' \subseteq \psi(q'')$ and by inductive hypothesis $\psi(q'')$ is connected, then $\psi(q') = p \cup \psi(q'')$ is connected.

If q' is infinite, let $\{v, v'\} \subseteq \psi(q')$. Take $\{(p, i), (p', i')\} \subseteq q'$ such that $v \in p$ and $v' \in p'$. There is a finite connected set q'' such that $\{(p, i), (p', i')\} \subseteq q'' \subseteq q'$. Hence, v and v' belong to $\psi(q'')$ which is connected. So, there is a path between v and v' which lies in $\psi(q'')$ and *a fortiori* in $\psi(q')$. □

Remark 3.3: If q is a component of $G[V_1 \cup \dots \cup V_n]$ and p is a component of $G[V_i]$ for some $i \in [1, n]$, then either $p \subseteq q$ or $p \cap q = \varnothing$.

If q' is a component of $G(V_1, \dots, V_n)$ and $(p, i) \notin q'$, then not only is $p \cap \psi(q') = \varnothing$, but p also does not contain a neighbour of a vertex in $\psi(q')$. (Otherwise, there is some $(p', i') \in q'$ such that $p \cap p' \neq \varnothing$; since q' is a component, $(p, i) \in q'$.)

Lemma 3.4. *A set q' of vertices of $G(V_1, \dots, V_n)$ is a component if and only if $\psi(q')$ is a component of $G[V_1 \cup \dots \cup V_n]$.*

Proof. We abbreviate $H := G[V_1 \cup \dots \cup V_n]$ and $H' := G(V_1, \dots, V_n)$. Assume that $\psi(q')$ is a component of H. According to Lemma 3.2, q' is connected. Let q'' be the component containing q'. Then $\psi(q') \subset \psi(q'')$. By Lemma 3.2, $\psi(q'')$ is connected. If $q'' \neq q'$, take $(p, i) \in q'' \setminus q'$. Then, by Remark 3.3, $p \subseteq \psi(q'') \setminus \psi(q')$. Thus $\psi(q') \subsetneq \psi(q'')$. This contradicts the fact that $\psi(q')$ is a component of H.

Conversely, assume that q' is a component of H' but that $\psi(q')$ is not a component of H. There is $v \in H \setminus \psi(q')$ such that v is connected to some $v' \in \psi(q')$. Take $(p, i), (p', i')$ such that $v \in p$ and $v' \in p'$. Hence,

$(p, i) \notin q'$ and $(p', i') \in q'$ are connected which contradicts the fact that q' is a component of H'. □

Moreover, we have:

Lemma 3.5. *Let $G := (V, E)$ be a graph and $(V_1, \ldots, V_n)$ be a family of nonempty finite subsets of V. For each i, let P_i be the set of components of $G[V_i]$. Then*

$$l_G(V_1 \cup \cdots \cup V_n) = l_{G(V_1,\ldots,V_n)}(P_1 \times \{1\} \cup \cdots \cup P_n \times \{n\}).$$

Proof. It suffices to prove that ψ induces a bijection ψ' from the set of components of $H' := G(V_1, \ldots, V_n)$ onto the set of components of $H := G[V_1 \cup \cdots \cup V_n]$.

ψ' **is one to one:** Suppose that $\psi'(q) = \psi'(q')$ and that $(p, i) \in q \setminus q'$. Since q' is a component, by Remark 3.3, $p \cap \psi'(q') = \varnothing$, but $p \subseteq \psi'(q)$, which contradicts the fact that $\psi'(q) = \psi'(q')$.

ψ' **is onto:** For any component π of H, by Remark 3.3 we have:

$$\pi = \bigcup \{p \subseteq \pi \mid \exists i \in [1, n] \text{ such that } p \text{ is a component of } G[V_i]\}.$$

Set

$$q' := \{(p, i) \mid p \subseteq \pi,\ p \text{ is a component of } G[V_i]\ (i \in [1, n])\}.$$

We have $\pi = \psi(q')$. By Lemma 3.4, q' is a component of H' and $\pi = \psi'(q')$.

□

A consequence of the previous lemma is the following result:

Theorem 3.6. *Let $G := (V, E)$ be a graph and $(V_1, \ldots, V_n)$ be a family of finite subsets of V. For each i, let P_i be the set of components of $G[V_i]$. Then:*

$$\beta_G(V_1, \ldots, V_n) = \beta_{G(V_1,\ldots,V_n)}(P_1 \times \{1\}, \ldots, P_n \times \{n\}).$$

Proof. We set $H' := G(V_1, \ldots, V_n)$. We notice first that for every subfamily $(V_{i_1}, \ldots, V_{i_k})$, the connection graph $G(V_{i_1}, \ldots, V_{i_k})$ is an induced subgraph of H'. Hence, if U is a set of vertices of $G(V_{i_1}, \ldots, V_{i_k})$ we have $l_{G(V_{i_1},\ldots,V_{i_k})}(U) = l_{H'}(U)$. Then, applying Lemma 3.5 to $G[V_1 \cup \cdots \cup V_n]$, $G[V_i \cup V_j]$ and $G[V_i]$ for all i, j's we have:

$$\begin{aligned} \beta_G(V_1, \ldots, V_n) &= l_G\left(\bigcup_{i=1}^{n} V_i\right) - \sum_{1 \le i < j \le n} l_G(V_i \cup V_j) + (n-2) \sum_{i=1}^{n} l_G(V_i) \\ &= l_{H'}\left(\bigcup_{i=1}^{n} P_i \times \{i\}\right) - \sum_{1 \le i < j \le n} l_{H'}(P_i \times \{i\} \cup P_j \times \{j\}) \\ &\quad + (n-2) \sum_{i=1}^{n} l_{H'}(P_i \times \{i\}), \end{aligned}$$

and so,

$$\beta_G(V_1,\dots,V_n) = \beta_{G(V_1,\dots,V_n)}(P_1 \times \{1\},\dots,P_n \times \{n\}).$$

□

Lemma 3.7. *Let $G := (V,E)$ be a graph and $\xi = (V_1,\dots,V_n)$ be a family of subsets of V. For each index i let P_i be the set of components of $G[V_i]$ and let $\xi' := (P_1 \times \{1\},\dots,P_n \times \{n\})$. The following statements are equivalent:*

(i) *Every ξ-path-cycle of $G[V_1 \cup \dots \cup V_n]$ is 2-colored.*
(ii) *Every cycle of the connection graph $G(V_1,\dots,V_n)$ is 2-colored by ξ'.*

Proof. Let $H := G[V_1 \cup \dots \cup V_n]$ and $H' := G(V_1,\dots,V_n)$.

(i) implies (ii): Let $C := (p_0,i_0),\dots,(p_{k-1},i_{k-1})$ be a cycle of H'. For each index $l \in [0,k-1]$, we have $i_l \neq i_{s_k(l)}$ and we construct a ξ-path-cycle $\Pi := (\pi_0,i_0),\dots,(\pi_{k-1},i_{k-1})$ such that the vertices of π_l belong to p_l. For this, for each index l we select a pair $\{x_l,y_l\} \subseteq p_l$ such that $\{y_l,x_{l+1}\}$ is an edge or $y_l = x_{l+1}$. For each index l we select a path $\pi_l := v_{l,0},\dots,v_{l,j_l} \subseteq p_l$ such that $x_l = v_{l,0}$ and $y_l = v_{l,j_l}$. Since the ξ-path-cycle Π is 2-colored then C is 2-colored by ξ'.

(ii) implies (i): Let $\Pi := (\pi_0,i_0),\dots,(\pi_{k-1},i_{k-1})$ be a ξ-path-cycle of H and, for each $l \in [0,k-1]$, let p_l be the component of V_{i_l} which contains π_l. Then $C := (p_0,i_0),\dots,(p_{k-1},i_{k-1})$ is a cycle of H'. Since C is 2-colored by ξ', Π is 2-colored.

□

Proof of Theorem 1.3. In Theorem 3.6, $(P_1 \times \{1\},\dots,P_n \times \{n\})$ is a family of pairwise disjoint independent sets of vertices of $G(V_1,\dots,V_n)$. Hence, item (a) is a consequence of Theorem 2.5; and item (b) is a consequence of Lemma 3.7 and Theorem 2.6. □

4. Recursive Properties of β and ν

In this section we give recursive definitions of β and ν. These definitions allow us to give an inductive proof of Theorem 2.2 and a derivation of equation (2.1).

Given a family $(V_1,\dots,V_n)$, where $n \geq 2$, of finite sets of vertices of a graph G, we will define $\beta(V_1,\dots,V_n)$ recursively on $(n,|V_n|)$ lexicographically ordered. We notice first that $\beta(V_1,V_2) = 0$ and $\beta(V_1,\dots,V_n,\varnothing) = \beta(V_1,\dots,V_n)$. Hence, in order to complete a recursive definition of β we require to evaluate $\beta(V_1,\dots,V_n \cup \{v\}) - \beta(V_1,\dots,V_n)$. For this we define the connection degree:

Definition 4.1. *Let $G := (V,E)$ be a graph. Given $u \in V$ and $U \subseteq V$, U finite. The set of* neighbors *of u in U is $N_G(U,u) := \{v \in U \mid \{u,v\} \in E\}$. We say that u is* connected *with U if $u \in U$ or $N_G(U,u) \neq \varnothing$. We denote by $d(U,u) := |N_G(U,u)|$. We denote by $K(U,u)$ or $K(U,\{u\})$ the set of components of U which are connected to u. The* connection degree *of the* *vertex u with the set of vertices U, denoted by $\kappa(U,u)$ or $\kappa(U,\{u\})$, is the cardinality of $K(U,u)$.*

Note that $\kappa(U,u) = 1$ if $u \in U$. The connection degree satisfies the following lemma which is easy to prove:

Lemma 4.2. *Let $G := (V,E)$ be a graph. Let u be a vertex and let U be a set of finite vertices, we have $l(U \cup \{u\}) = l(U) - \kappa(U,u) + 1$.*

Lemma 4.3. *Let $G := (V,E)$ be a graph. Let v be a vertex and let $(V_1,\dots,V_n)$, where $n \geq 2$, be a family of finite subsets of V, the following equation holds:*

$$\beta(V_1,\dots,V_n \cup \{v\}) - \beta(V_1,\dots,V_n) = \left(\sum_{i=1}^{n-1} \kappa(V_i \cup V_n, v)\right) - \kappa(V_1 \cup \cdots \cup V_n, v) - (n-2)\kappa(V_n, v).$$

Proof. By application of Definition 1.1 and Lemma 4.2 we have

$$\begin{aligned}\beta(V_1,\dots,V_n \cup \{v\}) - \beta(V_1,\dots,V_n) &= l(V_1 \cup \cdots \cup V_n \cup \{v\}) - l(V_1 \cup \cdots \cup V_n) \\ &\quad + (n-2)[l(V_n \cup \{v\}) - l(V_n)] \\ &\quad + \sum_{i=1}^{n-1} l(V_i \cup V_n) - l(V_i \cup V_n \cup \{v\}) \\ &= 1 - \kappa(V_1 \cup \cdots \cup V_n, v) \\ &\quad - (n-2)(\kappa(V_n, v) - 1) \\ &\quad + \sum_{i=1}^{n-1} (\kappa(V_i \cup V_n, v) - 1).\end{aligned}$$

□

Definition 4.4. *Let $G := (V,E)$ be a graph. Let $U \subseteq V$ and a vertex $u \notin U$. We define $\nu(U,u) := \nu(G[U \cup \{u\}]) - \nu(G[U])$.*

We will prove Theorem 2.2 using the recursive definition of β and a recursive property of the cyclomatic number given by the following lemma:

Lemma 4.5. *Let $G := (V,E)$ be a graph. Given a finite set of vertices U and a vertex $u \notin U$ then:*

$$\nu(G[U \cup \{u\}]) - \nu(G[U]) = d(U,u) - \kappa(U,u).$$

Proof. Let $U_1,\dots,U_k$ be the components of U. For each $i \in [1,k]$ let B_i and B'_i be subsets of $\mathcal{S}(G)$ such that B_i is a cycle basis of $G[U_i]$ and $B_i \cup B'_i$ is a cycle basis of $G[U_i \cup \{u\}]$. Then, $B := \bigcup_{i=1}^{i=k} B_i$ is a cycle basis of $G[U]$ and, if $B' := \bigcup_{i=1}^{i=k} B'_i$, then $B \cup B'$ is a cycle basis of $G[Ui \cup \{u\}]$. Therefore, $\nu(U,u) = \sum_{i=1}^{k} \nu(U_i,u)$.

In order to conclude, it is sufficient to show that $\nu(U_i, u) = d(U_i, u) - 1$ for each index i. In fact, if a vertex u is connected with a connected set of vertices W, such that $u \notin W$, then $\nu(W, u) = d(W, u) - 1$. Indeed, suppose that W is a connected subset of V. Let $v_1, \dots v_{d(W,u)}$ be an enumeration of the set of neighbors of u in W. For each $l \in [1, d(W, u) - 1]$, we choose a path $\pi_l := v_l, x_0, \dots, x_{j_l}, v_{l+1}$ in W. Thus, $C_l := u, v_l, x_0, \dots, x_{j_l}, v_{l+1}$ is a cycle. It is straightforward to show that if B is a cycle basis of $G[W]$ then $B \cup \{C_1, \dots, C_{d(W,u)-1}\}$ is a cycle basis of $G[W \cup \{u\}]$. □

Proposition 4.6. *Let $G = (V, E)$ be a graph. If $(V_1, \dots, V_{n-1}, V_n \cup \{v\})$ is a family of pairwise disjoint finite independent subsets of V, with $v \in V \setminus V_n$, then:*

$$\beta(V_1, \dots, V_{n-1}, V_n \cup \{v\}) - \beta(V_1, \dots, V_n) = \nu(V_1 \cup \dots \cup V_n, v) - \sum_{i=1}^{n-1} \nu(V_i \cup V_n, v).$$

Proof. Since $(V_1, \dots, V_{n-1}, V_{n-1}, V_n \cup \{v\})$ is a family of pairwise disjoint independent subsets of V, then:

$$\left(\sum_{i=1}^{n-1} d(V_i, v)\right) - d(V_1 \cup \dots \cup V_n, v) = 0.$$

Since $V_n \cup \{v\}$ is independent, $\kappa(V_n, v) = 0$. By lemmas 4.3 and 4.5, we have:

$$\begin{aligned}
\beta(V_1, \dots, V_n \cup \{v\}) &- \beta(V_1, \dots, V_n) \\
&= \left(\sum_{i=1}^{n-1} \kappa(V_i \cup V_n, v)\right) - \kappa(V_1 \cup \dots \cup V_n, v) \\
&= \sum_{i=1}^{n-1} [d(V_i \cup V_n, v) - \nu(V_i \cup V_n, v)] \\
&\qquad + \nu(V_1 \cup \dots \cup V_n, v) - d(V_1 \cup \dots \cup V_n, v) \\
&= \nu(V_1 \cup \dots \cup V_n, v) - \sum_{i=1}^{n-1} \nu(V_i \cup V_n, v) \\
&\qquad + \left(\sum_{i=1}^{n-1} d(V_i, v)\right) - d(V_1 \cup \dots \cup V_n, v) \\
&= \nu(V_1 \cup \dots \cup V_n, v) - \sum_{i=1}^{n-1} \nu(V_i \cup V_n, v).
\end{aligned}$$

□

Now, we are able to give an inductive proof of Theorem 2.2.

Proof of Theorem 2.2. We prove item (a) only, from which item (b) is a direct consequence. Let $G = (V, E)$ be a graph and $(V_1, \dots, V_n)$ be a coloration of G. We give a proof by induction on $(n, |V_n|)$ lexicographically ordered.

Initial step: If $n = 2$ then $G = G[V_1 \cup V_2]$, and $\nu(G) = \nu(G[V_1 \cup V_2])$, on the other hand $\beta(V_1, V_2) = 0$, so item (a) holds.

Inductive step: Suppose that $\{V_1, \dots, V_{n-1}, V_n \cup \{v\}\}$ is a family of pairwise disjoint independent subsets of V and that, by inductive hypothesis (I.H.), item (a) of Theorem 2.2 holds for the family $(V_1, \dots, V_n)$. Then by Proposition 4.6:

$$\begin{aligned}
&\beta(V_1, \dots, V_{n-1}, V_n \cup \{v\}) \\
&\quad = \beta(V_1, \dots, V_n) + \nu(V_1 \cup \dots \cup V_n, v) - \sum_{i=1}^{n-1} \nu(V_i \cup V_n, v) \\
&\quad \overset{I.H.}{=} \nu(G[V_1 \cup \dots \cup V_n]) - \sum_{i=1}^{n-1} \nu(G[V_i \cup V_n]) \\
&\qquad\quad - \sum_{1 \le i < j \le n-1} \nu(G[V_i \cup V_j]) \\
&\qquad\qquad + \nu(V_1 \cup \dots \cup V_n, v) - \sum_{i=1}^{n-1} \nu(V_i \cup V_n, v) \\
&\quad = \nu(G[V_1 \cup \dots \cup V_n \cup \{v\}]) - \sum_{i=1}^{n-1} \nu(G[V_i \cup V_n \cup \{v\}]) \\
&\qquad\quad - \sum_{1 \le i < j \le n-1} \nu(G[V_i \cup V_j]).
\end{aligned}$$

□

As a corollary of Theorem 2.2, and using the notations of Section 2, we have the classical result:

Corollary 4.7. *The cyclomatic number $\nu(G)$ of G, is given by the formula:*

$$\nu(G) = e(G) - v(G) + l(G).$$

Proof. Let $\{v_1, \dots, v_n\}$ be an enumeration of the vertices of G. For each index i, let $V_i := \{v_i\}$. Thus, $\xi := (V_1, \dots, V_n)$ is a coloration of G and we have:

$$\sum_{1 \le i < j \le v(G)} l(\{v_i, v_j\}) = v(G)^2 - v(G) - e(G).$$

Since each V_i is a singleton, there is no cycle that is 2-colored by ξ. Hence, by item (b) of Theorem 2.2 we have that $\beta(\{v_1\}, \dots, \{v_{v(G)}\}) = \nu(G)$. Thus

$$\begin{aligned}\nu(G) &= l(G) - \sum_{1\le i<j\le v(G)} l(\{v_i, v_j\}) + (v(G)-2)\sum_{i=1}^{v(G)} l(\{v_i\}) \\ &= l(G) - (v(G)^2 - v(G) - e(G)) + (v(G)-2)v(G) \\ &= l(G) - v(G) + e(G)\,.\end{aligned}$$

□

5. A Combinatorial Proof of Theorems 2.5 and 2.6

In this section we prove Theorems 2.5 and 2.6 by recursive and combinatorial means. For, we need to define an auxiliary function.

Definition 5.1. *Let G be a graph. The function δ is defined for any family of finite sets of vertices $(V_1, \dots, V_n)$, where $n \ge 2$, and any vertex v by:*

$$\delta(V_1, \dots, V_n, v) := \left(\sum_{i=1}^{n-1} \kappa(V_i \cup V_n, v)\right) - \kappa(V_1 \cup \dots \cup V_n, v).$$

Remark 5.2: Let $G = (V, E)$ be a graph. Let $(V_1, \dots, V_{n-1}, V_n)$ be a family of pairwise disjoint finite independent subsets of V. Let $V_n' := V_n \setminus \{v\}$ with $v \in V_n$. By hypothesis V_n is independent, hence $\kappa(V_n', v) = 0$. Thus, by Lemma 4.3 we have that

$$\begin{aligned}&\beta(V_1, \dots, V_{n-1}, V_n' \cup \{v\}) - \beta(V_1, \dots, V_{n-1}, V_n') \\ &\qquad = \left(\sum_{i=1}^{n-1} \kappa(V_i \cup V_n', v)\right) - \kappa(V_1 \cup \dots \cup V_{n-1} \cup V_n', v) \\ &\qquad = \delta(V_1, \dots, V_{n-1}, V_n', v).\end{aligned}$$

Lemma 5.3. *Let $G := (V, E)$ be a graph. For any family $(V_1, \dots, V_n)$, where $n \ge 2$, of finite sets of vertices and for any vertex v we have:*

(a) $\delta(V_1, \dots, V_n, v) \ge 0$.
(b) $\delta(V_1, \dots, V_n, v) = 0$ *if and only if for each $p \in K(V_1 \cup \dots \cup V_n, v)$ there is a unique pair (q, i) such that $i \in [1, n-1]$, $q \in K(V_i \cup V_n, v)$ and $q \subseteq p$.*

Proof. Let ϕ be the function which maps $(U', j) \in \bigcup_{i=1,\dots,n-1} K(V_i \cup V_n, v) \times \{i\}$ to $\phi((U', j)) \in K(V_1 \cup \dots \cup V_n, v)$ such that $U' \subseteq \phi((U', j))$.

This map is onto: let $U \in K(V_1 \cup \dots \cup V_n, v)$, and $u \in N_G(U, v)$; if $u \in V_i \cup V_n$ let $U' \in K(V_i \cup V_n, v)$ such that $u \in U'$, then $U = \phi((U', i))$. Thus, $\sum_{i=1}^{n-1} \kappa(V_i \cup V_n, v) \ge \kappa(V_1 \cup \dots \cup V_n, v)$. Hence $\delta(V_1, \dots, V_n, v) \ge 0$.

Moreover, $\delta(V_1, \dots, V_n, v) = 0$ if and only if ϕ is one-to-one. This amounts to the conclusion of (b). □

Corollary 5.4. *Let $G := (V, E)$ be a graph. Let $(V_1, \ldots, V_n)$ be a family of disjoint finite independent subsets of V. Let $(V_1', \ldots, V_n')$ be a family of subsets of V satisfying $V_i' \subseteq V_i$ for all $i \in [1, n]$. Then $\beta(V_1', \ldots, V_n') \leq \beta(V_1, \ldots, V_n)$.*

Proof. By induction on $\sum_{i=1}^{n-1} |V_i|$, using Remark 5.2 and Lemma 5.3 (a). □

Proof of Theorem 2.5. By lexicographic induction on $(n, |V_n|)$.

Initial step: For $n \in \{1, 2\}$ we have $\beta(V_1, V_n) = 0$.

Inductive step: Let $v \in V_n$ and $V_n' = V_n \setminus \{v\}$. Either $V_n' = \varnothing$ and then $(n-1, |V_{n-1}|) < (n, 0)$, or $V_n' \neq \varnothing$ and then $(n, |V_n'|) < (n, |V_n' \cup \{v\}|)$. In both cases we may apply the inductive hypothesis. Hence, $\beta(V_1, \ldots, V_{n-1}, V_n') \geq 0$. On the other hand, $\delta(V_1, \ldots, V_{n-1}, V_n', v) \geq 0$ by Lemma 5.3. We conclude, by Remark 5.2, that $\beta(V_1, \ldots, V_{n-1}, V_n' \cup \{v\}) \geq 0$.

□

Lemma 5.5. *Let $G = (V, E)$ be a graph and $\xi := (V_1, \ldots, V_{n-1}, V_n)$ be a family of pairwise disjoint finite independent subsets of V. Let $V_n' := V_n \setminus \{v\}$ with $v \in V_n$. If every cycle in $G[V_1 \cup \cdots \cup V_n]$ is 2-colored by ξ, then $\delta(V_1, \ldots, V_{n-1}, V_n', v) = 0$.*

Proof. By instantiating Lemma 5.3 with the family $(V_1, \ldots, V_{n-1}, V_n')$ and with the vertex v, we see that it suffices to prove that, for each $p \in K(V_1 \cup \cdots \cup V_{n-2} \cup V_{n-1} \cup V_n', v)$ there is a unique pair (q, i) such that $i \in [1, n-1]$, $q \in K(V_i \cup V_n', v)$ and $q \subseteq p$. We distinguish two cases, both of which will lead to a contradiction.

- If there are $i \neq j$ such that p contains a component of $V_i \cup V_n'$ connected with v and a component of $V_j \cup V_n'$ connected with v, then, there are $v_i \in (V_i \cup V_n') \cap p$ and $v_j \in (V_j \cup V_n') \cap p$ such that v is connected with v_i and v_j. Since $V_n = V_n' \cup \{v\}$ is independent, necessarily $v_i \in V_i$ and $v_j \in V_j$. Hence, there is a cycle in $G[V_1 \cup \cdots \cup V_n]$ containing v, v_i, v_j, this cycle is not 2-colored by ξ.
- If p contains two components, p_1 and p_2, of some $V_i \cup V_n'$ connected with v, then there are two vertices $v_1 \in V_i \cap p_1$ and $v_2 \in V_i \cap p_2$ both connected with v. But, since v_1 and v_2 belong to p there is in p a path $\pi := v_1, u_0, \ldots, u_{k-1}, v_2$ in $G[V_1 \cup \cdots \cup V_{n-1} \cup V_n']$. Since p_1 and p_2 are two different components of $V_i \cup V_n'$, the path π is not contained in $V_i \cup V_n'$. Necessarily π contains a vertex which belongs to V_j for some $j \notin \{i, n\}$. Then the cycle $v, v_1, u_0, \ldots, u_{k-1}, v_2$, in $G[V_1 \cup \cdots \cup V_n]$, is not 2-colored by ξ.

□

We denote by S_n the set of permutations of $[1, n]$.

Lemma 5.6. *Let $G = (V, E)$ be a graph and $\xi := (V_1, \dots, V_n)$ be a family of pairwise disjoint finite independent subsets of V. The following statements are equivalent:*

(i) *For every $\sigma \in S_n$ and every $v \in V_{\sigma(n)}$,*

$$\delta(V_{\sigma(1)}, \dots, V_{\sigma(n-1)}, V_{\sigma(n)} \setminus \{v\}, v) = 0.$$

(ii) *Every cycle in $G[V_1 \cup \dots \cup V_n]$ is 2-colored by ξ.*

Proof.

(i) implies (ii): Let C be a cycle in $G[V_1 \cup \dots \cup V_n]$ not 2-colored by ξ. Necessarily C contains a path v_1, v, v_2 such that $(v_1, v, v_2) \in V_{\sigma(1)} \times V_{\sigma(n)} \times V_{\sigma(2)}$ for some permutation $\sigma \in S_n$. Let $V'_{\sigma(n)} := V_{\sigma(n)} \setminus \{v\}$. Thus, v_1 and v_2 belong to the same component in $V_{\sigma(1)} \cup \dots \cup V_{\sigma(n-1)} \cup V'_{\sigma(n)}$, but v_1 belongs to a component of $V_{\sigma(1)} \cup V'_{\sigma(n)}$ which is different from the component of $V_{\sigma(2)} \cup V'_{\sigma(n)}$ containing v_2. So, by Lemma 5.3, $\delta(V_{\sigma(1)}, \dots, V_{\sigma(n-1)}, V_{\sigma(n)} \setminus \{v\}, v) > 0$.

(ii) implies (i): We instantiate Lemma 5.5 with $(V_{\sigma(1)}, \dots, V_{\sigma(n-1)}, V_{\sigma(n)})$ and with the vertex v.

□

Proof of Theorem 2.6. Assume that $\beta(V_1, \dots, V_n) = 0$. By symmetry of the function β with respect to its arguments, for each $\sigma \in S_n$ we have $\beta(V_{\sigma(1)}, \dots, V_{\sigma(n)}) = \beta(V_1, \dots, V_n) = 0$. Moreover, by Theorem 2.5 and Corollary 5.4, for each $v \in V_{\sigma(n)}$ we have $\beta(V_{\sigma(1)}, \dots, V_{\sigma(n-1)}, V_{\sigma(n)} \setminus \{v\}) = 0$; thus, by Remark 5.2 the equality $\delta(V_{\sigma(1)}, \dots, V_{\sigma(n-1)}, V_{\sigma(n)} \setminus \{v\}, v) = 0$. We conclude, by Lemma 5.6, that every cycle in $G[V_1 \cup \dots \cup V_n]$ is 2-colored by ξ.

Conversely, we prove by lexicographic induction on $(n, |V_n|)$ that, for each graph G and coloration $\xi := (V_1, \dots, V_n)$ of G, if all cycles of G are 2-colored by ξ then $\beta(V_1, \dots, V_n) = 0$.

Initial step: For $n \in \{1, 2\}$ we have $\beta(V_1, V_n) = 0$.

Inductive step: Let $v \in V_n$ and $V'_n := V_n \setminus \{v\}$. Then, we have all cycles of the graph $G' := G[V_1, \dots, V_{n-1}, V'_n]$ are 2-colored by the coloration $(V_1, \dots, V_{n-1}, V'_n)$. Either $V'_n = \varnothing$ and then $(n-1, |V_{n-1}|) < (n, 0)$, or $V'_n \neq \varnothing$ and then $(n, |V'_n|) < (n, |V'_n \cup \{v\}|)$. By inductive hypothesis, $\beta_{G'}(V_1, \dots, V_{n-1}, V'_n) = 0$. Therefore, we have $\beta_G(V_1, \dots, V_{n-1}, V'_n) = \beta_{G'}(V_1, \dots, V_{n-1}, V'_n) = 0$ since the graph G' is induced from the graph G. On the other hand, by Lemma 5.5, the equality $\delta(V_1, \dots, V_{n-1}, V'_n, v) = 0$ holds. We conclude, by Remark 5.2, that $\beta(V_1, \dots, V_{n-1}, V'_n \cup \{v\}) = 0$.

□

6. Representable Properties

In this section we give a characterization of a family of properties for which we can apply equation (1.4).

Let X be a set, we say that $\mathcal{P}$ is *a property* on subsets of X if $\mathcal{P} \subseteq \mathcal{P}(X)$. We set $\mathcal{P}^* = \mathcal{P} \setminus \{\varnothing\}$ and we define the graph $G_{\mathcal{P}} = (\mathcal{P}^*, E)$ by:

$$\{p_1, p_2\} \in E \iff (p_1 \cup p_2 \in \mathcal{P} \text{ and } p_1 \neq p_2) \ .$$

We denote by $\mathcal{F}(\mathcal{P})$ the set of finite unions of elements of $\mathcal{P}$. A nonempty element q of $\mathcal{P}$ is a $\mathcal{P}$*-component* of $A \subseteq X$ if $q \subseteq A$ and if for every $p \in \mathcal{P}$ such that $p \subseteq A$ and $p \cap q \neq \varnothing$ we have $p \subseteq q$.

Remark 6.1: Let $\mathcal{P}$ be a property on subsets of X. Let p and q be $\mathcal{P}$-components of $A \subseteq X$; then, either $p = q$ or $p \cap q = \varnothing$. Hence, for every $u \in \mathcal{F}(\mathcal{P}) \setminus \{\varnothing\}$, the set of $\mathcal{P}$-components of u, denoted by $K(u)$, satisfies that its elements are pairwise disjoint. The cardinality of $K(u)$ is called the *length* of u and is denoted by $l_{\mathcal{P}}(u)$ or $l(u)$. We assume that $K(\varnothing) = \varnothing$ and $l_{\mathcal{P}}(\varnothing) = 0$.

Definition 6.2. *Let X be a set, and $\mathcal{P}$ be a property on subsets of X. We say that $\mathcal{P}$ is* representable *by $G_{\mathcal{P}}$ (or $\mathcal{P}$ is representable) if it satisfies the following properties:*

(a) *For every $u \in \mathcal{F}(\mathcal{P}) \setminus \{\varnothing\}$, $K(u)$ is a partition of u.*
(b) *For every $p_1, \dots, p_k \in \mathcal{P}$, we have $p_1 \cup \cdots \cup p_k \in \mathcal{P}$ if and only if $G_{\mathcal{P}}[\{p_1, \dots, p_k\}]$ is connected.*

Example 6.3. *The two properties of representability are independent. Indeed,*

(b) does not imply (a): *Take*

$$X := \{a, b, c, d\} \text{ and } \mathcal{P} := \{\{a, b\}, \{c, d\}, \{b, c\}\}.$$

The set of vertices of $G_{\mathcal{P}}$ is independent, Property (b) is satisfied, but $K(\{a, b\} \cup \{b, c\}) = \varnothing$.

(a) does not imply (b): *Take $X := \{a, b, c\}$ and $\mathcal{P} = \{p_1, p_2, p_3, p_4\}$, with $p_1 = \{a\}$, $p_2 = \{b\}$, $p_3 = \{c\}$ and $p_4 = \{a, b, c\}$. For each $u \in \mathcal{F}(\mathcal{P}) \setminus \{p_4\}$, $K(u) = \{\{x\} \mid x \in u\}$; $K(p_4) = \{p_4\}$. Thus Property (a) holds. But Property (b) does not hold since $G_{\mathcal{P}}[\{p_1, p_2, p_3\}]$ is not connected although $p_1 \cup p_2 \cup p_3 \in \mathcal{P}$. Note that $l_{\mathcal{P}}(u) = l_{G_{\mathcal{P}}}(K(u))$ for all $u \in \mathcal{F}(\mathcal{P})$.*

Remark 6.4: Let $\mathcal{P}$ be a property. For all $u \in \mathcal{F}(\mathcal{P})$, the set $K(u)$ of $\mathcal{P}$-components of u is independent, this follows directly from Remark 6.1 and the definition of the edges in $G_{\mathcal{P}}$. Therefore, $l_{\mathcal{P}}(u) = l_{G_{\mathcal{P}}}(K(u))$.

Theorem 6.5. *Let $\mathcal{P}$ be a representable property. For all $u_1, \dots, u_n \in \mathcal{F}(\mathcal{P})$:*

$$l_{\mathcal{P}}\left(\bigcup_{i=1}^{n} u_i\right) = l_{G_{\mathcal{P}}}\left(\bigcup_{i=1}^{n} K(u_i)\right) \ .$$

Proof. For each $q \in K(u_1 \cup \cdots \cup u_n)$ we set

$$\phi(q) := \{q' \in K(u_1) \cup \cdots \cup K(u_n) \mid q' \subset q\}.$$

By Remark 6.4, it is sufficient to prove that ϕ is a bijection from $K(u_1 \cup \cdots \cup u_n)$ onto $K_{G_\mathcal{P}}(K(u_1) \cup \cdots \cup K(u_n))$ which denotes the set of components in $G_\mathcal{P}$ of $K(u_1) \cup \cdots \cup K(u_n)$. We prove the following statements.

$\cup\phi(q) = q$**:** It is clear that $\cup\phi(q) \subseteq q$. Let $x \in q$, there is i such that $x \in u_i$, let $q_{i,x} \in K(u_i)$ such that $x \in q_{i,x}$. Since $q_{i,x} \cap q \neq \varnothing$, $q_{i,x} \in \mathcal{P}$, $q_{i,x} \subseteq u_1 \cup \cdots \cup u_n$ and $q \in K(u_1 \cup \cdots \cup u_n)$ we have that $q_{i,x} \subseteq q$. Hence, $q_{i,x} \in \phi(q)$ which implies that $x \in \cup\phi(q)$, and thus $q \subseteq \cup\phi(q)$.

$\phi(q)$ **is connected:** Since $\cup\phi(q) = q \in \mathcal{P}$, by Property (b) of representability $\phi(q)$ is connected.

$\phi(q)$ **is a component in $G_\mathcal{P}$ of $K(u_1) \cup \cdots \cup K(u_n)$:** Let $g \subseteq K(u_1) \cup \cdots \cup K(u_n)$ be a connected set of $G_\mathcal{P}$ such that $g \cap \phi(q) \neq \varnothing$. Hence, $g \cup \phi(q)$ is $G_\mathcal{P}$-connected. By Property (b) of representability $(\cup g) \cup (\cup\phi(q)) \in \mathcal{P}$. So $(\cup g) \cup q \in \mathcal{P}$. But $\cup g \subseteq (\cup K(u_1)) \cup \cdots \cup (\cup K(u_n))$. By Property (a) of representability, for each i, $\cup K(u_i) = u_i$. Then $\cup g \subseteq u_1 \cup \cdots \cup u_n$. Since $q \in K(u_1 \cup \cdots \cup u_n)$ and $(\cup g) \cup q \in \mathcal{P}$, we have $(\cup g) \cup q \subseteq q$. Hence, $\cup g \subseteq q$ and so, for each $q' \in g$, $q' \subseteq q$ and $q' \in K(u_1) \cup \cdots \cup K(u_n)$. So, for each $q' \in g$, $q' \in \phi(q)$. Thus $g \subseteq \phi(q)$.

ϕ **is one to one:** Indeed if $\phi(q) = \phi(q')$ then $\cup(\phi(q)) = \cup(\phi(q'))$, hence $q = q'$.

ϕ **is onto:** Let $g \in K_{G_\mathcal{P}}(K(u_1) \cup \cdots \cup K(u_n))$. Pick $q' \in g$. Since $q' \in K(u_1) \cup \cdots \cup K(u_n)$, there is i such that $q' \in K(u_i)$. Let $q \in K(u_1 \cup \cdots \cup u_n)$ such that $q' \subseteq q$. Hence $q' \in \phi(q) \cap g$, so $\phi(q) \cap g \neq \varnothing$. But g and $\phi(q)$ are components, hence $\phi(q) = g$.

□

Proposition 6.6. *Let $\mathcal{P}$ be a representable property. For all $u_1, \ldots, u_n \in \mathcal{F}(\mathcal{P})$:*

$$l_\mathcal{P}\left(\bigcup_{i=1}^n u_i\right) = \sum_{1 \le i < j \le n} l_\mathcal{P}(u_i \cup u_j) - (n-2)\sum_{i=1}^n l_\mathcal{P}(u_i) + \beta(K(u_1), \ldots, K(u_n)).$$

Proof. By Definition 1.1 we have

$$l_{G_\mathcal{P}}\left(\bigcup_{i=1}^n K(u_i)\right) = \sum_{1 \le i < j \le n} l_{G_\mathcal{P}}(K(u_i) \cup K(u_j)) - (n-2)\sum_{i=1}^n l_{G_\mathcal{P}}(K(u_i)) + \gamma.$$

where $\gamma := \beta(K(u_1), \ldots, K(u_n))$. We conclude by Theorem 6.5. □

Definition 6.7. *Let X be a set. A property $\mathcal{P}$ on subsets of X is called a* connection property *if and only if:*

$$p_1 \in \mathcal{P}, p_2 \in \mathcal{P}, p_1 \cap p_2 \neq \varnothing \implies p_1 \cup p_2 \in \mathcal{P}.$$

The connection property is equivalent to Property (a) of representability:

Proposition 6.8. *A property $\mathcal{P}$ is a connection property if and only if for every $u \in \mathcal{F}(\mathcal{P}) \setminus \{\varnothing\}$, $K(u)$ is a partition of u.*

Proof. Let $\mathcal{P}$ be a connection property. Let $u \in \mathcal{F}(\mathcal{P}) \setminus \{\varnothing\}$. Let $M := \{q_1, \dots, q_n\} \subseteq \mathcal{P}$ of minimum cardinality such that $u = \cup M$. Let $p \in \mathcal{P}$ such that $p \subseteq u$. Necessarily, there is one and only one index i such that $p \cap q_i \neq \varnothing$, otherwise, there are $i \neq j$ such that $p \cap q_i \neq \varnothing$ and $p \cap q_j \neq \varnothing$. By the connection property $p \cup q_i \in \mathcal{P}$ and $p \cup q_j \in \mathcal{P}$. Hence, by the connection property again, $p \cup q_i \cup q_j \in \mathcal{P}$. Let $M' := (M \setminus \{q_i, q_j\}) \cup \{p \cup q_i \cup q_j\}$. Thus $u = \cup M'$ which contradicts the minimality of the size of M. Hence, M is a partition of u and each element of M is a $\mathcal{P}$-component of u. So, $M = K(u)$.

Conversely, suppose that for every $u \in \mathcal{F}(\mathcal{P}) \setminus \{\varnothing\}$, $K(u)$ is a partition of u. Then $K(u) \neq \varnothing$, let $p_1, p_2 \in \mathcal{P}$ such that $p_1 \cap p_2 \neq \varnothing$. Let $q \in K(p_1 \cup p_2)$. There is $i \in \{1, 2\}$ such that $p_i \cap q \neq \varnothing$, we can assume $i = 1$. Then $p_1 \subseteq q$. But $p_2 \cap p_1 \neq \varnothing$ implies that $p_2 \cap q \neq \varnothing$. So, $p_2 \subseteq q$ and thus $p_1 \cup p_2 = q \in \mathcal{P}$. □

Even though, as seen in Example 6.3, Property (a) of representability does not imply Property (b) of representability, we have:

Lemma 6.9. *Let $\mathcal{P}$ be a connection property. For every finite subset $q := \{p_1, \dots, p_k\}$ of $\mathcal{P}$, if $G_{\mathcal{P}}[\{p_1, \dots, p_k\}]$ is connected then $p_1 \cup \dots \cup p_k \in \mathcal{P}$.*

Proof. Suppose that $\mathcal{P}$ is a connection property. We prove the conclusion of the lemma by induction on k.

Initial step: If $k \in \{1, 2\}$, the property is true by the definition of $G_{\mathcal{P}}$.

Inductive step: We have $p_i \neq \varnothing$, for all i. Since $G_{\mathcal{P}}[\{p_1, \dots, p_k\}]$ has a spanning tree, we may suppose *w.l.o.g.* that, p_1 is connected in $G_{\mathcal{P}}[\{p_1, \dots, p_k\}]$ with $c := \{p_2, \dots, p_k\}$, where c is connected in $G_{\mathcal{P}}[\{p_1, \dots, p_k\}]$. Thus, by inductive hypothesis $u := p_2 \cup \dots \cup p_k \in \mathcal{P}$. Because p_1 is connected with c, we may suppose *w.l.o.g.* that p_1 is connected with p_2 and thus $p_1 \cup p_2 \in \mathcal{P}$. From $u \cap (p_1 \cup p_2) \neq \varnothing$, we have by the connection property that $p_1 \cup u \in \mathcal{P}$.

□

Proposition 6.10. *A property $\mathcal{P}$ is representable if and only if*

(a) *$\mathcal{P}$ is a connection property.*
(b) *If $p_1, \dots, p_k \in \mathcal{P}$ and $p_1 \cup \dots \cup p_k \in \mathcal{P}$ then $G_{\mathcal{P}}[\{p_1, \dots, p_k\}]$ is connected.*

Proof. By Proposition 6.8 and Lemma 6.9. □

Definition 6.11. *A property $\mathcal{P}$ is* weak-Helly *if there is no cycle of pairwise disjoint sets in $G_{\mathcal{P}}$.*

Lemma 6.12. *A representable property $\mathcal{P}$ is weak-Helly if and only if there is no triangle of pairwise disjoint sets in $G_{\mathcal{P}}$.*

Proof. The first implication is obvious by definition of weak-Helly property. Conversely if $p_1, p_2, p_3, \dots, p_k$ is a cycle of disjoint sets in $G_{\mathcal{P}}$, then by representability, $p_3 \cup \cdots \cup p_k \in \mathcal{P}$, $p_2 \cup p_3 \cup \cdots \cup p_k \in \mathcal{P}$ and $p_3 \cup \cdots \cup p_k \cup p_1 \in \mathcal{P}$. Then $p_1, p_2, p_3 \cup \cdots \cup p_k$ is a triangle in $G_{\mathcal{P}}$ of pairwise disjoint sets. □

Proposition 6.13. *Let $\mathcal{P}$ be a weak-Helly representable property and let $u_1, \dots, u_n \in \mathcal{F}(\mathcal{P})$ be pairwise disjoint. Then:*

$$l_{\mathcal{P}}\left(\bigcup_{i=1}^{n} u_i\right) = \sum_{1 \le i < j \le n} l_{\mathcal{P}}(u_i \cup u_j) - (n-2)\sum_{i=1}^{n} l_{\mathcal{P}}(u_i).$$

Proof. From Lemma 6.12, $G_{\mathcal{P}}[K(u_1) \cup \cdots \cup K(u_n)]$ is acyclic, hence we have by Theorem 2.6 that $\beta_{G_{\mathcal{P}}}(K(u_1), \dots, K(u_n)) = 0$. Then we get the equality by Proposition 6.6. □

7. Examples of Representable Properties

Example 7.1. *We give two examples of representable non weak-Helly properties. The proofs of representability for both examples are straightforward.*

(1) *The set $\mathcal{P}$ of connected sets of a topology on a set X is a representable property. Note that, in the general case, $\mathcal{P}$ is not weak-Helly; for instance if we consider the set $\mathcal{P}$ of connected sets of $\mathbb{R}^2$ and take a triangle ABC in $\mathbb{R}^2$ then the three segments $[AB[$, $[BC[$, and $[CA[$ are pairwise disjoint but $\{[AB[, [BC[, [CA[\}$ is a triangle in $G_{\mathcal{P}}$.*

(2) *The set of connected sets of vertices of a graph is a representable property. Clearly, for some graphs this property is not weak-Helly as soon as the graph contains a cycle.*

Proposition 7.2. *The set of connected sets of vertices of a forest is a weak-Helly representable property.*

Proof. By item (2) of Example 7.1, we have to prove only the Weak-Helly property. For contradiction, let C_1, C_2, C_3, be pairwise disjoint nonempty connected sets of vertices in a forest F, with $C_i \cup C_j$ connected in F for every $i, j \in \{1,2,3\}$. Let $c_1 \in C_1$ and $c_2 \in C_2$ such that $\{c_1, c_2\}$ is an edge of F. Similarly let $c_2' \in C_2$, $c_3 \in C_3$, $c_1' \in C_1$ and $c_3' \in C_3$ such that $\{c_2', c_3\}$ and $\{c_1', c_3'\}$ are edges of F. Let $x_0 := c_2, x_1, \dots, x_k := c_2'$ be a path in C_2; $y_0 := c_3, y_1, \dots, y_l := c_3'$ be a path in C_3; $z_0 := c_1', z_1, \dots, z_m := c_1$ be a path in C_1. Note that these paths are pairwise disjoint. Now, $x_0, x_1, \dots, x_k, y_0, y_1, \dots, y_l, z_0, z_1, \dots, z_m$ is a cycle of F, this contradicts the fact that F is a forest. □

An ordered set T is a *pseudo-tree* (resp. a *tree*) if for every $u \in T$, the set $\{t \in T : t \le u\}$ is a chain (resp. a well-ordered chain). Let O be an order. An *interval* of O is any subset I of O such that if $x, y \in I$, $z \in O$ and $x \le z \le y$ then $z \in I$.

Example 7.3. *The set of intervals of a tree is not necessary representable. Consider the tree* $T = \{r, a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3\}$ *where the only comparabilities are:* $r < a_1 < c_1 < b_1$, $r < b_2 < a_2 < c_2$ *and* $r < a_3 < b_3 < c_3$. *Put* $a := \{a_1, a_2, a_3\}$, $b := \{b_1, b_2, b_3\}$ *and* $c := \{c_1, c_2, c_3\}$. *Let* $\mathcal{P}$ *be the set of intervals of* T. *We have that* $a, b, c, a \cup b \cup c \in \mathcal{P}$, *but* $G_{\mathcal{P}}[\{a, b, c\}]$ *is not connected. Nevertheless, we show in Proposition 7.5 that the set of intervals of a chain is representable and weak-Helly.*

The notation $A < B$ (resp. $A \leq B$) for subsets of a chain means $a < b$ (resp. $a \leq b$) for all $a \in A$ and $b \in B$. The Boolean algebra consisting of finite unions of intervals of C is denoted by $\widehat{B}(C)$. Elements of $\widehat{B}(C)$ satisfy the following property:

Lemma 7.4. *For all elements* u *and* u' *of* $\widehat{B}(C)$, *if* $u \cup u'$ *is an interval, then there are intervals* $p \subseteq u$ *and* $p' \subseteq u'$ *such that* $p \cup p'$ *is an interval.*

Proof. Each $u \in \widehat{B}(C)$ is a finite union of maximal disjoint intervals, called the components of u. Let p (resp. p') be the rightmost component of u (resp. of u'). Then $p \cup p'$ is an interval. Otherwise, *w.l.o.g.* we may suppose that there is $x \in C$ such that $p < \{x\} < p'$ which implies that $u < \{x\} < p'$. Since $u \cup u'$ is an interval, there is a component p'' of u' such that $x \in p''$, in this case there is $y \in C \setminus u'$ such that $p'' < \{y\} < p'$; this contradicts the fact that $u \cup u'$ is an interval. □

For $a \in \widehat{B}(C) \setminus \{\varnothing\}$, define the *length* of a, denoted by $l(a)$, as the least integer n such that a is the union of n intervals. If a is the empty interval, we set $l(a) := 0$. In [6], it was proved that formula (1.3) holds for any family, $(x_i)_{1 \leq i \leq n}$, of pairwise disjoint elements of $\widehat{B}(C)$. Actually this result is a direct consequence of:

Proposition 7.5. *The set of intervals of a chain is a representable weak-Helly property.*

Proof. Let C be a chain and $\mathcal{P}$ be the set of intervals of C. The connection property is trivially satisfied.

Representability: By Proposition 6.10, it remains to prove that for all $p_1, \ldots, p_k \in \mathcal{P}$,

$$p_1 \cup \cdots \cup p_k \in \mathcal{P} \Longrightarrow G_{\mathcal{P}}[\{p_1, \ldots, p_k\}] \text{ is connected.}$$

For contradiction, assume that $C_1, \ldots, C_n$, where $n \geq 2$, are the components of $G_{\mathcal{P}}[\{p_1, \ldots, p_k\}]$. *W.l.o.g.* we assume that

$$\begin{aligned}
C_1 &= \{p_{r_0}, p_{r_0+1}, \ldots, p_{r_1}\}, \\
C_2 &= \{p_{r_1+1}, p_{r_1+2}, \ldots, p_{r_2}\}, \\
&\ \ \vdots \\
C_n &= \{p_{r_{n-1}+1}, p_{r_{n-1}+2}, \ldots, p_{r_n}\},
\end{aligned}$$

with $r_0 = 1$, $r_n = k$, $r_0 \leq r_1 < \cdots < r_n$. For each i, let q_i be the union of the elements of C_i. By Lemma 6.9, each q_i is an interval and for all $i \neq j$, $q_i \cap q_j = \varnothing$ and, by Lemma 7.4, $q_i \cup q_j$ is not an interval. Then, since C is a chain, *w.l.o.g.* we can assume that $q_1 < q_2 < \cdots < q_n$. Thus, by Lemma 7.4, $q_1 \cup \cdots \cup q_n$ is not an interval. But $p_1 \cup \cdots \cup p_k = q_1 \cup \cdots \cup q_n$, therefore $p_1 \cup \cdots \cup p_k$ is not an interval. This is a contradiction.

Weak-Helly: Let I_1, I_2, I_3 be pairwise disjoint nonempty intervals of C. *W.l.o.g.* we assume that $I_1 < I_2 < I_3$. Thus $I_1 \cup I_3$ is not an interval since $I_2 \neq \varnothing$.

□

Let T be a pseudo-tree, the *pseudo-tree algebra of* T is the subalgebra $B[T]$ of the power set $\mathcal{P}(T)$ generated by the family $\{b_t : t \in T\}$, where $b_t := \{u \in T : t \leq u\}$. For each $i \in T$ and I finite antichain of T above i (that means $i < i'$ for all $i' \in I$), $e_{i,I} := b_i \setminus \bigcup_{u \in I} b_u$. The set $e_{i,I}$ is called a *truncated cone*. Let $\mathcal{E}$ be the set of truncated cones: $\mathcal{E} := \{e_{i,I} \mid i \in T, I \text{ is a finite antichain in } T \text{ and } \{i\} < I\}$, then $B[T] = \mathcal{F}(\mathcal{E})$. For a basic exposition of this notion see [1] and [3]. If T is a chain with a first element, the pseudo-tree algebra $B[T]$ coincides with the interval algebra $B(T)$.

Lemma 7.6. *Let $e_{i_1,I_1} \in \mathcal{E}$ and $e_{i_2,I_2} \in \mathcal{E}$. The following statements are equivalent:*

(i) $e_{i_1,I_1} \cup e_{i_2,I_2} \in \mathcal{E}$.
(ii) *Either ($i_1 \leq i_2$ and $\forall j \in I_1,\ j \not< i_2$) or ($i_2 \leq i_1$ and $\forall j \in I_2,\ j \not< i_1$).*
(iii) *i_1 and i_2 are comparable and*

$$e_{i_1,I_1} \cup e_{i_2,I_2} = e_{\min\{i_1,i_2\},(I_1 \setminus e_{i_2,I_2}) \cup (I_2 \setminus e_{i_1,I_1})}.$$

Proof. Each one of statements (i), (ii) and (iii) implies that i_1 and i_2 are comparable; by symmetry, we may assume that $i_1 \leq i_2$.

(i) implies (ii): Suppose that $j \in I_1$ with $j < i_2$. Then, $i_1 < j < i_2$ and $j \notin (e_{i_1,I_1} \cup e_{i_2,I_2})$ which contradicts the fact that $e_{i_1,I_1} \cup e_{i_2,I_2} \in \mathcal{E}$.

(ii) implies (iii): We prove, in the first place, that $(I_1 \setminus e_{i_2,I_2}) \cup (I_2 \setminus e_{i_1,I_1})$ is an antichain. Let $j_1 \in I_1 \setminus e_{i_2,I_2}$ and $j_2 \in I_2 \setminus e_{i_1,I_1}$. Suppose that $j_1 < j_2$. Since $i_2 < j_2$, hence j_1 and i_2 are comparable. By (ii) we have $j_1 \not< i_2$. Hence, $i_2 \leq j_1$. Therefore, $j_1 \in e_{i_2,I_2}$. That contradicts the hypothesis $j_1 \in I_1 \setminus e_{i_2,I_2}$. Suppose that $j_2 < j_1$. Hence, $i_1 \leq i_2 < j_2 < j_1$. Therefore, $j_2 \in e_{i_1,I_1}$. That contradicts the hypothesis $j_2 \in (I_2 \setminus e_{i_1,I_1})$.

We prove, in the second place, that

$$e_{i_1,I_1} \cup e_{i_2,I_2} \subseteq e_{i_1,(I_1 \setminus e_{i_2,I_2}) \cup (I_2 \setminus e_{i_1,I_1})}.$$

Let $s \in e_{i_1,I_1} \cup e_{i_2,I_2}$.

Assume that $s \in e_{i_1,I_1}$**:** We have that $i_1 \leq s$ and, for all $j \in I_1$, $j \not\leq s$. It remains to show that for every $j_2 \in I_2 \setminus e_{i_1,I_1}$ we have $j_2 \not\leq s$. Let

$j_2 \in I_2 \setminus e_{i_1,I_1}$. Then $i_1 \leq i_2 < j_2$. Since $i_1 < j_2$ and $j_2 \notin e_{i_1,I_1}$, there is $j_i \in e_{i_1,I_1}$ such that $j_1 \leq j_2$. But $j_1 \not\leq s$, then $j_2 \not\leq s$.

Assume that $s \in e_{i_2,I_2}$**:** We have that $i_1 \leq i_2 \leq s$ and, for all $j \in I_2$, $j \not\leq s$. It remains to prove that for every $j_1 \in I_1 \setminus e_{i_2,I_2}$ we have $j_1 \not\leq s$. Let $j_1 \in I_1 \setminus e_{i_2,I_2}$. If $j_1 \in I_2$, since $s \in e_{i_2,I_2}$ then $j_1 \not\leq s$. If $j_1 \notin I_2$, we study two cases.

Case 1: $i_2 \leq j_1$: Then $j_1 \in b_{i_2}$. Since $j_1 \in I_1 \setminus e_{i_2,I_2}$, there is $j_2 \in I_2$ such that $j_2 < j_1$. Then $j_1 \not\leq s$.

Case 2: $i_2 \not\leq j_1$: By (ii), $j_1 \not< i_2$. Hence i_2 and j_1 are incomparable. Since $i_2 \leq s$ then $j_1 \not\leq s$.

We prove, at the last, that $e_{i_1,(I_1 \setminus e_{i_2,I_2}) \cup (I_2 \setminus e_{i_1,I_1})} \subseteq e_{i_1,I_1} \cup e_{i_2,I_2}$. Let $s \in e_{i_1,(I_1 \setminus e_{i_2,I_2}) \cup (I_2 \setminus e_{i_1,I_1})}$. Then $i_1 \leq s$. Assume that $s \notin e_{i_2,I_2}$, we have two cases.

Case 1: $i_2 \not\leq s$: By definition, $j \not\leq s$ for all $j \in I_1 \setminus e_{i_2,I_2}$. Moreover, $j \not\leq s$ for all $j \in I_1 \cap e_{i_2,I_2}$ since $i_2 \not\leq s$. From $s \in e_{i_1,(I_1 \setminus e_{i_2,I_2}) \cup (I_2 \setminus e_{i_1,I_1})}$, we have that $s \in e_{i_1,I_1}$.

Case 2: $i_2 \leq s$: Since $s \notin e_{i_2,I_2}$, there is $j_2 \in I_2$ such that $j_2 \leq s$. But $j \not\leq s$ for every $j \in I_2 \setminus e_{i_1,I_1}$. Therefore $j_2 \in e_{i_1,I_1}$. For contradiction suppose that $s \notin e_{i_1,I_1}$, then $j_1 \leq s$ for some $j_1 \in I_1$. Since $j_2 \leq s$ then j_1 and j_2 are comparable. Necessarily $j_2 < j_1$ since $j_2 \in e_{i_1,I_1}$. Hence $j_1 \in I_1 \setminus e_{i_2,I_2}$. This implies that $s \notin e_{i_1,(I_1 \setminus e_{i_2,I_2}) \cup (I_2 \setminus e_{i_1,I_1})}$. That contradicts our hypothesis.

(iii) implies (i): This is immediate.

□

Proposition 7.7. *The set of truncated cones of a pseudo-tree is a connection property.*

Proof. Let T be a pseudo-tree and $\mathcal{E}$ be the set of truncated cones of T. Let $e_{i_1,I_1}, e_{i_2,I_2} \in \mathcal{E}$ such that $e_{i_1,I_1} \cap e_{i_2,I_2} \neq \varnothing$. We may assume, *w.l.o.g.* that $i_1 \leq i_2$. Let $j_1 \in I_1$ and $s \in e_{i_1,I_1} \cap e_{i_2,I_2}$. Suppose that $j_1 < i_2$ then j_1 and s are comparable. We would have $s < j_1$ since $s \in e_{i_1,I_1}$, and $j_1 < s$ since $s \in e_{i_2,I_2}$. Hence, for all $j_1 \in I_1$, $j_1 \not< i_2$. We conclude by Lemma 7.6 that $e_{i_1,I_1} \cup e_{i_2,I_2} \in \mathcal{E}$. □

Lemma 7.8. *Let $e_{i_1,I_1}, e_{i_2,I_2} \in \mathcal{E}$. Then $e_{i_1,I_1} \cup e_{i_2,I_2} \in \mathcal{E}$ if and only if $e_{i_1,I_1} \cap e_{i_2,I_2} \neq \varnothing$ or $i_1 \in I_2$ or $i_2 \in I_1$.*

Proof. By symmetry we may suppose that $i_1 \leq i_2$.

If $e_{i_1,I_1} \cup e_{i_2,I_2} \in \mathcal{E}$ and $e_{i_1,I_1} \cap e_{i_2,I_2} = \varnothing$ then there is $j_1 \in I_1$ such that $j_1 \leq i_2$. By Lemma 7.6, $j \not< i_2$ for all $j \in I_1$. Hence $j_1 = i_2$, so $i_2 \in I_1$.

Conversely, if $e_{i_1,I_1} \cap e_{i_2,I_2} \neq \varnothing$ then, by Proposition 7.7, $e_{i_1,I_1} \cup e_{i_2,I_2} \in \mathcal{E}$. On the other hand, if $e_{i_1,I_1} \cap e_{i_2,I_2} = \varnothing$ and $i_2 \in I_1$, then $j \not< i_2$ for all $j \in I_1$ since I_1 is an antichain. Thus, Lemma 7.6, $e_{i_1,I_1} \cup e_{i_2,I_2} \in \mathcal{E}$. □

Proposition 7.9. *The set of truncated cones of a pseudo-tree is a weak-Helly representable property.*

Proof. Let T be a pseudo-tree and $\mathcal{E}$ be the set of truncated cones of T.

Representability: By Propositions 6.10 and 7.7, it remains to prove that for all $p_1, \dots, p_n \in \mathcal{E}$

$$p_1 \cup \dots \cup p_n \in \mathcal{E} \Longrightarrow G_{\mathcal{E}}[\{p_1, \dots, p_n\}] \text{ is connected.}$$

We shall prove the implication by induction on n. The property is trivially true for $n \in \{1, 2\}$. Assume that it is true for all $k < n$. Let $A := \{e_{i_1,I_1}, \dots, e_{i_n,I_n}\} \subseteq \mathcal{E}$ and let $u := \cup_{p \in A} p$. Suppose that $u \in \mathcal{E}$, we shall prove that $G_{\mathcal{E}}[A]$ is connected.

Since $u \in \mathcal{E}$, we may suppose, *w.l.o.g.* that $i_1 \leq i_k$ for all $k \in [1, n]$. If $i_1 = i_k$ for all $k \in [1, n]$ then, by Lemma 7.8, $G_{\mathcal{E}}[A]$ is connected. Assume that i_m is a maximal element in $\{i_1, \dots, i_n\}$ and that $i_1 < i_m$. Necessarily there is $l \in [1, n]$ such that $i_l < i_m$ and $e_{i_l,I_l} \cup e_{i_m,I_m} \in \mathcal{E}$, otherwise, by Lemma 7.6, for all $i_l < i_m$, there is $j_l \in I_l$ such that $j_l < i_m$. Note that the set consisting of such i_l's and j_l's is a chain. Let j_h be the maximum of such j_l's. Hence $j_h \notin u$ and $i_1 < j_h < i_m$, this contradicts the fact that $u \in \mathcal{E}$. Let $l \in [1, n]$ such that $i_l < i_m$ and $e_{i_l,I_l} \cup e_{i_m,I_m} \in \mathcal{E}$. Then by Lemma 7.6 we have $e_{i_l,I_l} \cup e_{i_m,I_m} = e_{i_l,(I_l \setminus e_{i_m,I_m}) \cup (I_m \setminus e_{i_l,I_l})}$. Let $A' := (A \setminus \{e_{i_l,I_l}, e_{i_m,I_m}\}) \cup \{e_{i_l,I_l} \cup e_{i_m,I_m}\}$. Since $u = \cup_{p \in A'} p \in \mathcal{E}$, we have by inductive hypothesis that $G_{\mathcal{E}}[A']$ is connected.

It remains to prove that for each $k \in [1, n] \setminus \{l, m\}$ if $e_{i_k,I_k} \cup (e_{i_l,I_l} \cup e_{i_m,I_m}) \in \mathcal{E}$ then $e_{i_k,I_k} \cup e_{i_l,I_l} \in \mathcal{E}$ or $e_{i_k,I_k} \cup e_{i_m,I_m} \in \mathcal{E}$. If $e_{i_k,I_k} \cap (e_{i_l,I_l} \cup e_{i_m,I_m}) \neq \varnothing$ then $e_{i_k,I_k} \cap e_{i_l,I_l} \neq \varnothing$ or $e_{i_k,I_k} \cap e_{i_m,I_m} \neq \varnothing$ and we conclude by Proposition 7.7. If $e_{i_k,I_k} \cap (e_{i_l,I_l} \cup e_{i_m,I_m}) = \varnothing$ then $e_{i_k,I_k} \cap e_{i_l,(I_l \setminus e_{i_m,I_m}) \cup (I_m \setminus e_{i_l,I_l})} = \varnothing$. Since $e_{i_k,I_k} \cup e_{i_l,(I_l \setminus e_{i_m,I_m}) \cup (I_m \setminus e_{i_l,I_l})} \in \mathcal{E}$, then by Lemma 7.6 we have two alternatives: either there is $j_k \in I_k$ such that $j_k = i_l$, which implies that $e_{i_k,I_k} \cup e_{i_l,I_l} \in \mathcal{E}$; or there is $j_r \in (I_l \setminus e_{i_m,I_m}) \cup (I_m \setminus e_{i_l,I_l})$ such that $j_r = i_k$, which implies that $e_{i_k,I_k} \cup e_{i_l,I_l} \in \mathcal{E}$ or $e_{i_k,I_k} \cup e_{i_m,I_m} \in \mathcal{E}$.

Weak-Helly: Let $e_{r,R}$, $e_{s,S}$ and $e_{t,T}$ be pairwise disjoint elements of $\mathcal{E}$. Assume that $e_{i,I} \cup e_{j,J} \in \mathcal{E}$ for every $i \neq j$. By Lemma 7.6, for all $i, j \in \{r, s, t\}$, i and j are comparable, so *w.l.o.g.* we can assume that $r < s < t$. Again by Lemma 7.8, we have that $t \in R$ since $r < t$, $e_{r,R} \cap e_{t,T} = \varnothing$ and $e_{r,R} \cup e_{t,T} \in \mathcal{E}$. Similarly we have that $s \in R$. But R is an antichain, so s and t are incomparable, thus $e_{s,S} \cup e_{t,T} \notin \mathcal{E}$. A contradiction.

□

By Propositions 6.13, 7.2, 7.5 and 7.9 we obtain the main application of representability:

Theorem 7.10. *Let $\mathcal{P}$ be either the set of intervals of a chain, or the family of connected sets of a forest, or the set of truncated cones of a pseudo-tree.*

Let $\{x_1, \ldots, x_n\}$ *be a family of pairwise disjoint elements of* $\mathcal{F}(\mathcal{P})$*, then:*

$$l_{\mathcal{P}}\left(\bigcup_{i=1}^{n} x_i\right) = \sum_{1\leq i<j\leq n} l_{\mathcal{P}}(x_i \cup x_j) - (n-2)\sum_{i=1}^{n} l_{\mathcal{P}}(x_i).$$

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References

[1] M. Bekkali, *On superatomic boolean algebras*, Ph.D. thesis, University of Colorado, 1991.
[2] R. Diestel, *Graph Theory*, second ed., Springer, 2000.
[3] S. Koppelberg and J. D. Monk, *Pseudo-trees and boolean algebras*, Order **8** (1992), 359–374.
[4] M. Pouzet and I. Rival, *Every countable lattice is a retract of a direct product of chains*, Algebra universalis **18** (1984), 295–307.
[5] H. Si Kaddour, *Ensembles de générateurs d'une algèbre de Boole*, Diplôme de doctorat, Université Claude Bernard Lyon 1, 1988.
[6] ———, *A note on the length of members of an interval algebra*, Algebra universalis **44** (2000), 195–198.
[7] D. B. West, *Introduction to Graph Theory*, Prentice-Hall, 1996.

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