Contributions to Discrete Mathematics

Volume 3, Number 1, Pages 6–13
ISSN 1715-0868

PARTITIONING THE FLAGS OF PG$(2,q)$ INTO STRONG REPRESENTATIVE SYSTEMS

CSABA MENGYÁN

Dedicated to the centenary of the birth of Ferenc Kárteszi (1907–1989).

Abstract. In this paper we show a natural extension of the idea used by Illés, Szőnyi and Wettl which proved that the flags of PG$(2,q)$ can be partitioned into $(q-1)\sqrt{q}+3q$ strong representative systems for q an odd square. From a generalization of the Buekenhout construction of unitals their idea can be applied for any non-prime q^h to yield that $q^{2h-1}+2q^h$ strong representative systems partition the flags of PG$(2,q^h)$. In this way we also give a solution to a question of Gyárfás about the strong chromatic index of the bipartite graph corresponding to PG$(2,q)$, for q not a prime.

1. Introduction

A *blocking set* in a projective plane is a set of points which intersects every line. A point P of a blocking set B is called *essential*, if $B \setminus \{P\}$ is not a blocking set. Geometrically, a point is essential if there is a tangent line at P, that is, a line intersecting B just at P. A blocking set is said to be *minimal*, when no proper subset of it is a blocking set, or, equivalently if each point of the blocking set is essential. Note that a minimal blocking set in a projective plane is either a line, or does not contain a line.

A *flag* of PG$(2,q)$ is an incident point-line pair (P,r). A set of flags $B = \{(P_1,r_1),...,(P_k,r_k)\}$ is a *strong representative system* if $P_i \in r_j \iff i=j$. B is *maximal* if it is maximal subject to inclusion.

It is easy to see that the idea of a strong representative system is a generalization of the notion of a minimal blocking set: any minimal blocking set can be represented as a strong representative system by taking the set of points together with one of their tangents. We note that the representation is unique if there is exactly one tangent at each point of the minimal blocking set.

The reason to introduce both ideas here is that we are going to use strong representative systems arising from minimal blocking sets to partition the flags of PG$(2,q)$.

A trivial way to partition the flags is given in the following lemma.

2000 *Mathematics Subject Classification.* 51E21.

Key words and phrases. Minimal blocking set, strong representative system, partitioning the flags, Buekenhout construction, strong chromatic index.

Lemma 1.1 (Trivial estimate)**.** *The flags of* PG$(2, q)$ *can be partitioned into* $q^2 + 2q$ *strong representative systems.*

Proof. We work in AG$(2, q) \subset$ PG$(2, q)$. Consider the set of flags (P_i, r_i), $i = 1, ..., q$, where the P_i-s are on the same vertical line $l(\neq l_\infty)$, and r_i is a non-vertical line through P_i, as a strong representative system. As the r_i-s run through every non-vertical line through their corresponding P_i-s, we get q disjoint strong representative systems. Repeating the procedure above for the remaining $q - 1$ vertical lines ($\neq l_\infty$) we can partition almost all flags into q^2 strong representative systems.

To finish the proof, we have to add strong representative systems partitioning the flags (P, r), where r is a vertical line or the line at infinity and the flags (P, r), where P is an infinite point and r is any line through P. To this aim we define two types of strong representative systems:

(1) (P_i, r_i), $i = 1, ..., q$, where the P_i-s are on the same horizontal line l, and r_i is the vertical line through P_i;
(2) (P_i, r_i), $i = 1, ..., q$, where the r_i-s are non-vertical lines through the same point P, and P_i is the infinite point of r_i.

If one lets l run through horizontal lines in the first case, and P run through points of a fixed vertical line, then these yield an additional $2q$ number of strong representative systems. In this way we have partitioned all flags except for the flag (Y, l_∞), where Y denotes the infinite point of the vertical lines, but this can be added to any of the aforementioned q^2 sets. □

The purpose of the present paper is to improve on this trivial estimate. The method we follow uses large minimal blocking sets.

In [4] the maximal size of a strong representative system was shown to be $q\sqrt{q}+1$, which is also the maximal size of minimal blocking sets (Bruen-Thas upper bound [1]). From this it follows that at least roughly $q\sqrt{q}$ strong representative systems are needed to partition all flags, as the number of flags is approximately q^3. Illés, Szőnyi and Wettl proved that this is indeed the case for q an odd square.

Theorem 1.2. *The flags of* PG$(2, q)$*,* q *an odd square, can be partitioned into* $(q - 1)\sqrt{q} + 3q$ *strong representative systems* [4]*.*

To prove Theorem 1.2, Illés, Szőnyi and Wettl used unitals as large minimal blocking sets arising from the parabola construction [7]. Their method involved partitioning the affine plane with these minimal blocking sets, and mapping such a minimal blocking set into another in a way that permuted the tangents in the affine points. Finally, the uncovered flags were covered with strong representative systems resembling the ones the trivial estimate described.

It is straightforward to show that the same method of proof applied in [4] can be used to verify Theorem 1.2 for q an even square also. As the result obtained in this paper is more general than this, we simply mention that the only necessary change that has to be made in that proof is that instead of the parabola construction from [7], Hermitian-curves of the form $x^{\sqrt{q}+1} + y^{\sqrt{q}}z + z^{\sqrt{q}}y = cz^{\sqrt{q}+1}$, $c \in \mathrm{GF}(\sqrt{q})$ should be considered.

In the present paper we investigate the more general case when q^h is not a prime. For this, we consider the generalized Buekenhout construction from [8] which produces minimal blocking sets of size $q^{h+1}+1$ in $\mathrm{PG}(2, q^h)$. Hoping that we can repeat the trick of partitioning the affine plane with copies of the affine part of this blocking set, from this size (and since the number of flags is approximately q^{3h}) it is natural to expect that something of order $q^{2h-1} + E(q)$ holds in $\mathrm{PG}(2, q^h)$ with $E(q) \ll q^{2h-1}$. In Theorem 3.5 we prove $E(q) \leq 2q^h$, showing that $q^{2h-1} + 2q^h$ strong representative systems suffice to partition the flags of $\mathrm{PG}(2, q^h)$. This gives a factor of q improvement on the trivial estimate.

In graph theoretic terminology we can thus answer a question of Gyárfás [2] on the strong chromatic index of the point-line incidence graph of $\mathrm{PG}(2, q)$, for q not a prime. For this we recall that a *strong colour class* in a graph G is a set of independent edges with the extra property that this set of edges is an induced subgraph of G, i.e. there are no edges in G joining two end-points of different edges in this strong colour class. Consider now the point-line incidence graph G of $\mathrm{PG}(2, q)$ (that is, the points and lines are the two colour classes and the edges are the flags of $\mathrm{PG}(2, q)$). In G a strong colour class is a strong representative system. The *strong chromatic index* of a graph G is the minimum number of colours in an edge-colouring with the property that the edges having the same colour form a strong colour class. Geometrically, for G this chromatic index is the minimum number of strong representative systems covering the flags of $\mathrm{PG}(2, q)$, the very question we have set out to solve.

2. The generalized Buekenhout construction

In this section we present a particular case of the generalized Buekenhout construction given in [8]. We will need some properties of the blocking set constructed (and the high dimensional structure behind it) which are only included implicitly in their work.

Consider the André, Bruck-Bose representation of the plane $\mathrm{PG}(2, q^h)$. This arises from a suitable $(h-1)$-spread of the hyperplane at infinity in $\mathrm{PG}(2h, q)$. The affine lines of the plane are h-dimensional subspaces containing the $(h-1)$-spaces of the $(h-1)$-spread. The ideal points correspond to the elements of the spread. The affine points of $\mathrm{PG}(2, q^h)$ are the affine points of $\mathrm{PG}(2h, q)$. The idea behind all blocking set constructions starting with this representation is that a point set intersecting every h-space in the underlying $\mathrm{PG}(2h, q)$ yields a blocking set in the plane $\mathrm{PG}(2, q^h)$.

Construction 2.1. *For the generalized Buekenhout construction consider the André, Bruck-Bose representation of the plane* $\mathrm{PG}(2, q^h)$*. Let* S *be the* $(h-1)$*-spread of the hyperplane* H *at infinity of* $\mathrm{PG}(2h, q)$*, defining the plane* $\mathrm{PG}(2, q^h)$*. Let* O *be an ovoid of a* 3*-space* π *and embed* π *in an* $(h+2)$*-space*

M. Construct a cone B in M with base O and vertex V, an $(h-2)$-space disjoint from π. Embed now M into PG$(2h,q)$ *in such a way that an element ρ of the spread S is generated by the vertex V and exactly one point T of the ovoid O; the hyperplane H is otherwise disjont from B.*

In a series of statements we prove that B, considered as a point set of PG$(2,q^h)$ is a minimal blocking set, and also derive some combinatorial properties.

Remark 2.2. Consider a plane α in M disjoint from V. $\alpha' := \langle V,\alpha\rangle \cap \pi$ is a plane in π. It is easy to see that $\langle \alpha, V\rangle = \langle \alpha', V\rangle$. A point of α' and V generate a space meeting α in one point. This gives a one-to-one correspondence between points of α and points of α'. Hence α meets B in either 1 or $q+1$ points.

The next lemma analyzes the embedding of M into PG$(2h,q)$. Denote by M' the infinite part of M (that is, $M\cap H$). In the following two lemmas whenever we talk about an h-space through an element of S, we mean an h-space not contained in H.

Lemma 2.3.

- (i) *M' contains one element of S and meets the other members of the spread in a line;*
- (ii) *a 1 co-dimensional subspace U of M' meeting ρ in precisely V contains q^{h-2} of the lines in (i) and meets the other members of the spread in a point;*
- (iii) *an h-space through ρ is either contained in M or their affine part is disjoint;*
- (iv) *an h-space through a spread element different from ρ meets M in a plane;*

Proof.

- (i) We know that M', which is an $h+1$ space, contains a spread element, namely ρ. The other members of the spread meet M' in at least a line (by a dimension argument), and since they are disjoint from ρ, the intersections must be lines (there is no room in M' for ρ and a disjoint 2-space).
- (ii) Note that U either meets the spread elements in a line or in a point. U is partitioned by these points, lines and V. All in all we have q^h+1 objects in the partition. A little counting shows that the number of lines is q^{h-2}.
- (iii) Note that through a spread element h-spaces arise by taking the space generated by an affine point and the spread element.
- (iv) By dimensions, an h-space meets M in at least a plane. On the other hand, if the h-space is through a spread element, then it has to be disjoint from ρ, hence the intersection cannot be bigger (again by dimensions).

□

Combining the remark and the previous lemma, we have the following.

Lemma 2.4.

(i) *An h-space through ρ meets the affine part of M in either* 0 *or* q^{h-1} *points. The latter arises when the h-space is generated by ρ and a point of O different from T.*

(ii) *An h-space not through ρ meets the affine part of M in either* 1 *or* $q+1$ *points.*

(iii) *Take an affine point P of O and denote by α_P the tangent plane of O at P (within π). Then h-spaces through spread elements that are tangents at an affine point of $\langle V, P\rangle$ meet M within $\langle V, \alpha_P\rangle$.*

Proof.

(i) All h-spaces through ρ within M can be generated by ρ and a point of π different from T. If this point is in O, then the affine part contains $1 + (q-1)\,|V| = q^{h-1}$ points of B. Otherwise the intersection in the affine part is empty.

(ii) If an h-space is not through ρ, then by Lemma 2.3 it meets M in a plane, and by Remark 2.2 the intersection is 1 or $q+1$.

(iii) Suppose h is an h-space meeting B only in the point Q. By Lemma 2.3 h meets M in a plane α_Q, and by Remark 2.2 $\langle V, \alpha_Q\rangle = \langle V, \alpha_P\rangle$, hence $\alpha_Q \subseteq \langle V, \alpha_P\rangle$.

$\square$

Theorem 2.5. *Considering B as a point set of* $\mathrm{PG}(2, q^h)$ *we find a minimal blocking set B' with the following properties:*

(i) *The size of B' is $q^{h+1} + 1$.*

(ii) *B' has a unique infinite point Y. There are q^2 lines through Y meeting B' in $q^{h-1}+1$ points, the rest of the lines through Y are tangents.*

(iii) *Lines not through Y are either tangents or $(q+1)$-secants.*

(iv) *Through an affine point of B' there are q^{h-2} tangents, $q^h - q^{h-2}$ $(q+1)$-secants and one $(q^{h-1}+1)$-secant.*

(v) *If P' and P'' are affine points of B' on the same $(q^{h-1}+1)$-secant, then infinite points of tangents through P' are the same as infinite points of tangents through P''.*

Proof. The spread element ρ (generated by V and T) becomes a point Y in $\mathrm{PG}(2, q^h)$, the only ideal point of B'. Lines through Y correspond to h-spaces through ρ, so (ii) follows from Lemma 2.4 (i).

For (iii) note that a line in $\mathrm{PG}(2, q^h)$ not through Y corresponds to an h-space through a spread element different from ρ, so we can apply Lemma 2.4 (ii).

(i) and (iv) follow from (ii) and (iii) by simple counting.

For (v) let $\langle P, V\rangle$ correspond to the $(q^{h-1}+1)$-secant with a $P \in O$ and choose a $Q \in \langle P, V\rangle$. By Remark 2.2 and Lemma 2.4 a tangent h-space through Q (corresponding to a tangent line of B') meets M in a plane α_Q within $\langle V, \alpha_P\rangle$. This plane meets M' in a line. This line is contained in the infinite part of $\langle V, \alpha_P\rangle$, a 1 co-dimensional subspace of M'. Hence the spread element within the tangent h-space in question is one of the q^{h-2} spread elements mentioned in Lemma 2.3 (ii). But by the just proved (iv) there are exactly q^{h-2} tangents through any point of $\langle P, V\rangle$ (that is, through any affine point of the $(q^{h-1}+1)$-secant in question). Hence the infinite points of tangents through any affine point of the $(q^{h-1}+1)$-secant in question are exactly the points corresponding to spread elements meeting $\langle \alpha, V\rangle$ in a line (and not in a point). $\square$

Remark 2.6. If $l_1, l_2, ..., l_{q^2}$ denote the $(q^{h-1}+1)$-secants and I_i, $i = 1, ..., q^2$, the infinite points of tangents through points on l_i, then these I_i-s partition the infinite points different from Y into sets of cardinality q^{h-2}.

Proof. This is a direct consequence of (v) of the previous lemma. $\square$

3. The main result

In this section we use copies of the blocking set constructed in the previous section to partition the flags of PG$(2, q^h)$ into strong representative systems.

Lemma 3.1. *Let $f(X, Y)$ denote a homogeneous irreducible quadratic polynomial over* GF(q) *and consider the following affine equation:*

$$Z = f(X, Y) + aX + bY + c.$$

As a, b and c run through all elements of GF(q)*, we find q^3 elliptic quadrics with the following properties:*

(i) $(0, 0, 1, 0)$ *is the unique infinite point of all the quadrics;*
(ii) *every affine point is on q^2 quadrics;*
(iii) *for any incident (P, α) pair with P an affine point and α a plane not through $(0, 0, 1, 0)$, there is exactly one quadric through P for which α is a tangent plane (at P).*

Proof. It is easy to check that the q^3 elliptic quadrics have $(0, 0, 1, 0)$ as the only point at infinity. (See [3] for the fact that these are elliptic quadrics.)

For (ii) note that after fixing x, y and z, the number of solutions for $z - xa - yb - f(x, y) = c$, is q^2.

For (iii) recall that the tangent plane of the quadric $Z = f(X, Y) + aX + bY + c$ at the point $(x, y, z, 1)$ is the plane through the point and orthogonal to the vector $(f'_X(x, y) + a, f'_Y(x, y) + b, -1, a + b + 2c - z)$ (see [3]). It is easy to see that for fixed x, y and z this vector uniquely determines a, b and c. As all quadrics contain $(0, 0, 1, 0)$, no tangent plane in question can pass through this point. $\square$

Lemma 3.2. *Suppose that in the generalized Buekenhout construction we fix everything apart from O and let O run through all ovoids from Lemma 3.1 with T corresponding to the point* $(0, 0, 1, 0)$*. Then we find q^3 minimal* *blocking sets of* $\mathrm{PG}(2, q^h)$ *with the same unique infinite point that cover* q^2 *times the affine points of* M*.*

Proof. As $\langle \pi, V \rangle = M$, and $\pi \setminus H$ was covered q^2 times this is also true for the affine part of M. (Any point in $\langle P, V \rangle$, $P \in \pi \setminus H$, is covered as many times as P.) $\square$

Lemma 3.3. *Pick two minimal blocking sets from Lemma 3.2 sharing the point* $P \in M \setminus H$*. Then the tangents to* P *will meet the line at infinity in disjoint pointsets (both of size* q^{h-2}*) for the two cases.*

Proof. $\langle V, P \rangle$ meets π in a point P', this should be a common point of O_1 and O_2, the two ovoid bases for the two minimal blocking sets. From Lemma 3.1 we know that the tangent planes α_1 and α_2 at P' have to be different. On the other hand, by Lemma 2.4 (iii), an h-space corresponding to a common tangent through P would meet M in a plane (disjoint from V) within $\langle \alpha_1, V \rangle \cap \langle \alpha_2, V \rangle = \langle \alpha_1 \cap \alpha_2, V \rangle$. This is not possible by dimensions. $\square$

Corollary 3.4. *Let* U *denote the points of* $\mathrm{PG}(2, q^h)$ *(considered in the André, Bruck-Bose representation) corresponding to the affine points of an* $(h+2)$*-dimensional subspace* M*. Then one can partition all incident (point,line) pairs with the points chosen from* U *and lines not through a fixed infinite point* Y*, into* q^{h+1} *strong representative systems.*

Proof. Take the q^3 minimal blocking sets considered in Lemma 3.2. Each of them gives rise to q^{h-2} strong representative systems as follows. Using the notations of Remark 2.6 (after fixing a blocking set) choose an infinite point from each of the I_i-s and consider all tangents (and points of tangencies) through each of them. This is a strong representative system of size q^{h+1}. Let the chosen points run through all points of the corresponding I_i-s simultaneously to find q^{h-2} strong representative systems. Finally, repeat this for all q^3 blocking sets.

By Lemmas 3.2 and 3.3 every point of U will occur in q^2 blocking sets and all tangents will be different, hence a point is in $q^2 \cdot q^{h-2} = q^h$ flags, this is the number of lines through the point (except for the one joining the point to Y). The number of strong representative systems used is $q^3 \cdot q^{h-2} = q^{h+1}$. $\square$

We are ready to prove the main result of the present paper.

Theorem 3.5. *The flags of* $\mathrm{PG}(2, q^h)$*,* $h \geq 2$*, can be partitioned into* $q^{2h-1} + 2q^h$ *strong representative systems.*

Proof. Denote by H the hyperplane at infinity of $\mathrm{PG}(2h, q)$ and partition the affine part with $(h+2)$-dimensional subspaces through a fixed $(h+1)$-dimensional subspace within H. This (through the André, Bruck-Bose representation) gives rise to a partitioning of the affine part of $\mathrm{AG}(2, q^h)$ into q^{h-2} sets corresponding to affine parts of $(h+2)$-spaces like in Corollary 3.4. Taking strong representative systems guaranteed by the corollary, we find a partition of almost all the flags of the affine plane (into $q^{h+1}q^{h-2} = q^{2h-1}$ strong representative systems) except for one parallel class of lines. Hence to finish the proof we have to add strong representative systems partitioning the uncovered flags as in the second part of Lemma 1.1 giving an additional $2q^h$ strong representative systems. $\square$

Corollary 3.6. *The strong chromatic index of the bipartite graph corresponding to* $\mathrm{PG}(2, q^h)$*,* $h \geq 2$*, is at most* $q^{2h-1} + 2q^h$*.*

4. Concluding remarks

There are larger minimal blocking sets in $\mathrm{PG}(2, q^h)$ than the one given in Construction 2.1 (see [5, 6, 8]), but the method presented in this paper does not easily admit a generalization to these.

It remains unknown whether an improvement on the trivial estimate is possible when the order of the finite plane is prime. If one wants to use the original idea of Illés, Szőnyi and Wettl a suitable large minimal blocking set is needed. However, some of the largest known minimal blocking sets in the prime case at present come from the parabola construction [7]. But even the parabola construction itself only guarantees the existence of minimal blocking sets of size $cq \log q$ for q prime. Considering that there are approximately q^3 flags this could give roughly a $c \log q$ improvement over the trivial estimate at best, which may be far off from the described improvement for the non-prime case. It also seems difficult to choose the parabolas in such a way that this mentioned best solution could be achieved.

Acknowledgements

The author wishes to thank T. Szőnyi and A. Gács for their continued support of the author's advancement in finite geometries.

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Department of Computer Science, Eötvös Loránd University
H-1117, Budapest, Pázmány Péter sétány 1/C, Hungary
E-mail address: mengyanc@bkv.hu