Contributions to Discrete Mathematics

Volume 2, Number 2, Pages 133–144
ISSN 1715-0868

THE COP DENSITY OF A GRAPH

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ABSTRACT. We consider the game of Cops and Robber played with infinitely many cops on countable graphs. We give a sufficient condition—the strongly 1-e.c. property—for the cop number to be infinite. The cop density of a finite graph, defined as the ratio of the cop number and the number of vertices, is investigated. In the infinite case, the limits of the cop densities of chains of finite graphs are studied. For a strongly 1-e.c. graph, any real number in $[0, 1]$ may be realized as a cop density of the graph. We prove that if the cop number is infinite, then there is a chain with cop density 1; however, we give an example with cop number 1 and cop density 1. We consider the cop density of finite connected graphs, and prove that for the $G(n, p)$ random graph, almost surely the cop density is around approximately $(\ln n)/n$.

1. INTRODUCTION

All graphs we consider are simple, undirected, and, in sections beyond the first, countably infinite (we will use *countable* to mean *countably infinite*). The game of *Cops and Robber* is a vertex pursuit game played on a (possibly infinite) graph $G = (V, E)$. There are two players, a set of k *cops* (or *searchers*) $\mathcal{C}$, where $k > 0$ is a fixed cardinal, and the *robber* $\mathcal{R}$. The cops begin the game by occupying a set of k vertices, and the cops and robber move in alternate *rounds*. More than one cop is allowed to occupy a vertex, and the players may *pass*; that is, remain on their current vertex. Hence, we consider the *passive* version of the game; the active version will not alter our results, as we will explain at the end of Section 2. A *move* in a given round for either player consists of a pass or moving to one of its neighbours. The game is played with perfect information, that is, the players' positions are known to all at any time. The cops win and the game ends if at least one of the cops can eventually occupy the same vertex as the robber; otherwise, $\mathcal{R}$ wins. A *strategy* for the cops is an initial position $c \in V^{\kappa}$ together with

Received by the editors April 16, 2006, and in revised form June 24, 2006, August 29, 2006, and March 29, 2007.

2000 *Mathematics Subject Classification.* 05C75, 05C99, 91A43, 91A13.

Key words and phrases. Cops and Robber game, cop number, cop density, infinite graph, random graph.

The first and second authors gratefully acknowledge support from NSERC and MITACS grants.

a mapping $s : V^\kappa \times V \longrightarrow V^\kappa$ that leads them to a win no matter what the robber's initial position is (a precise definition is easy but cumbersome and unnecessary for our purposes). Cops and Robber may be thought of as one model for network security.

As placing a cop on each vertex guarantees that the cops win, we may define the *cop number*, written $c(G)$, as $\min\{\kappa : \kappa$ cops have a winning strategy on $G\}$. We call a graph G *κ-cop-win* if $c(G) = \kappa$. The cop number was introduced for finite graphs by Aigner and Fromme [1] who proved that if G is finite and planar, then $c(G) \leq 3$. When G is finite, then $c(G)$ is polynomial time computable (as a function of the number of vertices); see [6, 16, 14]. So-called *cop-win* graphs (that is, graphs G with $c(G) = 1$) were structurally characterized in [19, 21]. Recall that the closed neighbourhood $N[x]$ of a vertex x is x along with the vertices joined to x. The finite cop-win graphs $G = (V, E)$ are exactly those graphs which are *dismantlable*: there exists a linear ordering of $V = \{x_j : 1 \leq j \leq n\}$ so that for each $i \in \{1, \ldots, n-1\}$, there is a $j \in \{i+1, \ldots, n\}$ such that $N[x_i] \cap \{x_i, \ldots, x_n\} \subseteq N[x_j] \cap \{x_i, \ldots, x_n\}$. Clearly, this characterization does not apply to infinite cop-win graphs. For example, an infinite one-way ray has such an ordering, but is not cop-win. See [19] for a characterization of infinite cop-win graphs. See [9] for results on infinite dismantlable graphs. Finite chordal and bridged graphs are cop-win; see [4]. No analogous structural characterization of finite κ-cop-win graphs with (finite) cop number $\kappa > 1$ is known. For a survey of results on the cop number and related search parameters for finite graphs, see [2]. See also [3, 12, 13, 22].

The case for $c(G)$ infinite was considered in [15, 19, 23]. No characterization of graphs with infinite cop number is known. In this paper, we consider the case where G is countable, and so at most countably many cops are needed to catch the robber. Hence, $c(G)$ is either a positive integer or $\aleph_0$.

It is usual to think of the countably infinite *random graph* R as a graph having $\mathbb{N}$ as the vertex set, with each pair of vertices joined with (independent) probability 1/2. Erdős and Rényi showed in [11] that, with probability 1, a countably infinite random graph has a unique isomorphism type written R. The unique isomorphism type is that of the countable graph satisfying the *existentially closed* (or *e.c.*) property: for all finite disjoint sets of vertices A, B, there is a vertex $z \notin (A \cup B)$ such that z is joined to each vertex of A and to no vertex of B. Equivalently, a graph satisfies the e.c. property if and only if it satisfies the *strongly n-e.c. property* for each $n \in \mathbb{N}$, that is, the e.c. property with the added restriction that $|A| \leq n$ (while $|B|$ remains arbitrary and finite). See the survey [8] for further discussion.

It is straightforward to see that $c(R) = \aleph_0$; that is, R is *infinite-cop-win*. For a sketch of the proof, note that if the cops occupy a set A of vertices, with the robber at a vertex b, a vertex z in R joined to b and not A supplies an "escape route" for the robber. This sketch will be made more precise in Theorem 2.1, where we generalize this result to the uncountable class of strongly 1-e.c. graphs.

We consider the *cop density* of a finite graph, defined by

$$D_c(G) = \frac{c(G)}{|V(G)|}.$$

Note that $D_c(G)$ is a rational number in $[0,1]$. The parameter D_c measures how dense the cops are in the graph. We extend the definition of D_c to infinite graphs by considering limits of chains of finite graphs. In this way, the cop density for infinite graphs is a real number in $[0,1]$.

A *chain of graphs* is a sequence $(G_n : n \in \mathbb{N})$, each G_n is an induced subgraph of G_{n+1}, for all $n \in \mathbb{N}$. Given a chain $(G_n : n \in \mathbb{N})$ of induced subgraphs of G, we write $G = \lim_{n\to\infty} G_n$ if $V(G) = \bigcup_{n\in\mathbb{N}} V(G_n)$ and $E(G) = \bigcup_{n\in\mathbb{N}} E(G_n)$. Note that every countable graph G is the limit of a chain of finite graphs, and there are infinitely many distinct chains with limit G. Suppose that $G = \lim_{n\to\infty} G_n$, where $C = (G_n : n \in \mathbb{N})$ is a fixed chain of induced subgraphs of G. We say that C is a *full chain for* G. Define

$$D(G,C) = \lim_{n\to\infty} D_c(G_n),$$

if the limit exists (and then it is a real number in $[0,1]$). This is the *cop density of* G *relative to* C; if C is clear from context, we refer to this as the *cop density of* G. We will only consider graphs and chains where this limit exists. Indeed, if the cop number of G is infinite, then for some chain the cop density equals 1 (see Theorem 2.1 and Theorem 2.5). The *upper cop density of* G, written $\mathrm{UD}(G)$, is defined as

$$\sup\{D(G,C) : C \text{ is a full chain for } G\}.$$

We illustrate these parameters with some examples. If G is a ray (that is, a path on the vertex set $\{v_i : i \in \mathbb{N}\}$ with the edge set $\{v_i v_{i+1} : i \in \mathbb{N}\}$), then we may take C to be $(P_n : n \in \mathbb{N})$, where P_n is the path induced by $\{v_0, v_1, \ldots, v_n\}$. As $c(P_n) = 1$, we have that $D(G,C) = 0$. Let G be the disjoint union of infinitely many 4-cycles $\{C_4^{(i)} : i \in \mathbb{N}\}$, and let G_n be the disjoint union of the first n $C_4^{(i)}$. If $C = (G_n : n \in \mathbb{N})$, then $D(G,C) = 1/2$. If G is an infinite clique, then $\mathrm{UD}(G) = 0$, while $\mathrm{UD}(H) = 1$ if H is an infinite co-clique (that is, a graph with no edges).

Since infinitely many cops are needed to catch the robber in R, one may suspect the cop density always equals 1, or is bounded below by some positive constant. Quite to the contrary, we prove in Theorem 2.3 that for all strongly 1-e.c. graphs G, such as R, and all $r \in [0,1]$, we can find full chains C such that $D(G,C) = r$. In Theorem 2.5 we prove that there is a full chain with upper cop density 1 if and only if G satisfies the 0-e.c. property (that is, for each finite set S of vertices, there is a vertex not in S that is not joined to any vertex of S). Indeed, we will show $\mathrm{UD}(G)$ always equals either 0 or 1. In Theorem 2.6, we prove that if a graph has infinite cop number, then it has upper density 1. However, this does not characterize infinite-cop-win graphs: in Theorem 2.7 for each $r \in [0, 1]$ we supply a graph with cop number 1 but with density r.

We consider $D_c(G)$ for finite connected graphs G. If G has n vertices and is connected, then $D_c(G) \leq n/2$. However, as we will prove in Corollary 3.6, almost all graphs have their cop density equal approximately $(\ln n)/n$.

2. Cop density of infinite graphs

Our first result finds connections between infinite-cop-win graphs and the strongly 0- and 1-e.c. properties. Recall from the introduction that a graph satisfies the *strongly* 0-*e.c. property* if for all finite sets of vertices S, there is a vertex $x \notin S$ that is not joined to any vertex of S. A graph G has the *strongly* 1-*e.c. property* if for all finite sets of vertices S and all vertices x not in S, there is a vertex $y \notin S \cup \{x\}$ so that y is joined to x, but not joined to a vertex of S. Note that the strongly 1-e.c. property implies the strongly 0-e.c. property.

Theorem 2.1.

(1) *If G is strongly* 1-*e.c., then $c(G) = \aleph_0$.*
(2) *If $c(G) = \aleph_0$, then G satisfies the strongly* 0-*e.c. property. In particular, G is a spanning subgraph of R.*

Proof. (1) Given only finitely many cops in G, we describe a winning strategy for the robber $\mathcal{R}$. Let the cops first occupy a set of vertices S_0. By the strongly 0-e.c. property, $\mathcal{R}$ may first occupy a vertex u_0 that is not joined to a vertex of S_0. Hence, $\mathcal{R}$ may evade capture on the cops first move, where the cops move to the set S_1. Suppose after the cops nth move, $\mathcal{C}$ occupies the set S_n, and $\mathcal{R}$ occupies the vertex u_{n-1}. By the strongly 1-e.c. property, there is a vertex u_n joined to u_{n-1} and not joined to a vertex of S_n. The robber moves to u_n and may evade capture on the cops $(n+1)^{\text{st}}$ move. In this way, by induction, the robber may indefinitely evade capture.

(2) The robber has a winning strategy if there are only finitely many cops. Hence, no matter what finite set of vertices S the cops first choose to occupy, the robber can evade capture. It follows that there is a vertex $x \notin S$ that is not joined to any vertex of S. The second statement of (2) follows from the well-known fact that a strongly 0-e.c. graph is a spanning subgraph of R (it may be proved directly by a back-and-forth argument; see [8]). □

We will use the following result in the proof of our next theorem. The lemma follows directly, or from Theorems 3.1 and 3.2 of [6]. An *endvertex* is a vertex of degree 1.

Lemma 2.2. *Let H result from G by adding a single endvertex. Then $c(G) = c(H)$.*

In the next theorem, we prove that if G is strongly 1-e.c., then the cop density of G may be *any* real number in $[0, 1]$. The strongly 1-e.c. property is fairly weak: by Erdős, Rényi [11], almost all countable graphs satisfy it. In fact, for each $n \geq 0$, there are $2^{\aleph_0}$ non-isomorphic countable graphs that are strongly n-e.c. but not strongly $(n+1)$-e.c.; see Theorem 4.1 of [7].

Theorem 2.3. *If G is strongly 1-e.c., then for all $r \in [0,1]$, there is a chain C in G such that $D(G,C) = r$.*

Proof. Let $(p_n : n \in \mathbb{N})$ be a sequence of rationals in $[0,1]$ such that $\lim_{n\to\infty} p_n = r$, with $p_0 = 1$. We construct a chain $C = (G_n : n \in \mathbb{N})$ in G such that $G = \lim_{n\to\infty} G_n$, and with the property that $D_c(G_n) = p_n$. Enumerate $V(G)$ as $\{x_n : n \in \mathbb{N}\}$.

We proceed inductively on n. For $n = 0$, let G_0 be the subgraph induced by x_0. Then $c(G_0)/|V(G_0)| = 1 = p_0$.

Fix $n \geq 1$, suppose the induction hypothesis holds for all $k \leq n$, and let $p_{n+1} = a/b$, where a, b are positive integers. Further suppose for an inductive hypothesis that $\{x_0, \ldots, x_n\} \subseteq V(G_n)$. Without loss of generality, as $r \in [0,1]$ we may assume $a < b$, and $\gcd(a,b) = 1$.

We add vertices to G_n in several ways. Let, G'_{n+1} be the graph induced by $V(G_n) \cup \{x_{n+1}\}$. Suppose that $c(G'_{n+1}) = a'$ and $|V(G'_{n+1})| = b'$. If $a'/b' = a/b$, then let $G_{n+1} = G'_{n+1}$. Otherwise, we add some new vertices to adjust the parameter $D_c(G'_{n+1})$.

Each time an isolated vertex is added to a graph, the cop number increases by 1. This follows since a cop must be on each isolated vertex for the cops to win, otherwise the robber chooses the isolated vertex as his first position and wins. By Lemma 2.2, adding an endvertex to a graph does not change the cop number. We may assume that $a'/b' < a/b$ by adding an appropriate number of endvertices. In this way, b' will become larger, while a' will remain unchanged.

We may add an arbitrary finite number of isolated vertices and endvertices to G'_{n+1} by the strongly 1-e.c. property. We add x isolated vertices and y endvertices to G'_{n+1} to form G_{n+1} so that $D_c(G_{n+1}) = c(G_{n+1})/|V(G_{n+1})| = a/b$. This is possible if we can solve the equation

$$\frac{a}{b} = \frac{a' + x}{b' + x + y}.$$

which is equivalent to

$$(b-a)x - ay = ab' - a'b. \tag{2.1}$$

Note that $ab' - a'b > 0$, since otherwise, $ab' \leq a'b$ which is contrary to hypothesis. Hence, we obtain a linear Diophantine equation $cx + dy = e$, where $c = (b-a) > 0$, $d = -a < 0$, and $e = ab' - a'b > 0$. As $\gcd(b-a, -a) = \gcd(a,b) = 1$, (2.1) has infinitely many solutions. The general integer solution of (2.1) is

$$x = x_0 - at, y = y_0 - (b-a)t, \tag{2.2}$$

where (x_0, y_0) is a particular fixed solution, and t is an integer. (For example, we may take $(x_0, y_0) = (-a', a' - b')$.) As the coefficients of t in (2.2) are both negative, we may choose an appropriate $t < 0$ to ensure an integer solution of (2.1) (x, y) with $x, y \geq 0$. This completes the induction step in constructing G_{n+1}.

As $\{x_0, \ldots, x_n\} \subseteq V(G_n)$ for all $n \in \mathbb{N}$, we have that C=$(G_n : n \in \mathbb{N})$ is a full chain for G. Further,

$$D(G, C) = \lim_{n\to\infty} p_n = r.$$

□

As the infinite random graph R satisfies the strongly 1-e.c. property, we have the following corollary.

Corollary 2.4. *For all $r \in [0, 1]$, there is a chain C in R such that $D(R, C) = r$.*

Our next result completely characterizes the upper cop density of a graph G : UD(G) takes on one of the two values 0 or 1, and equals 1 exactly when G is strongly 0-e.c.

Theorem 2.5. *The following are equivalent.*

(1) UD$(G) = 1$.
(2) UD$(G) > 0$.
(3) *G is strongly 0-e.c.*
(4) *G is a spanning subgraph of R.*

Proof. As ($1 \Rightarrow 2$) is trivial, and ($3 \Leftrightarrow 4$) is well known (see, for example, [8]), we prove that ($2 \Rightarrow 3$) and ($3 \Rightarrow 1$).

For ($2 \Rightarrow 3$), suppose for the contrapositive that G is not strongly 0-e.c. Then there is some finite set S of vertices of G with the property that each vertex not in S is joined to some vertex of S; in other words, S is a finite *dominating set* for G. Let C=$(G_n : n \in \mathbb{N})$ be a fixed full chain of finite graphs in G, and suppose that n_0 is the least integer n where $S \subseteq V(G_n)$. Fix $t \geq n_0$. Then $c(G_t) \leq |S|$, since S is a dominating set for G and hence, G_t. Thus, $D_c(G) \leq |S|/|V(G_t)|$, and the latter term tends to 0 as $t \to \infty$. Hence, UD$(G) = 0$.

For ($3 \Rightarrow 1$) enumerate $V(G)$ as $\{x_i : i \in \mathbb{N}\}$. Fix a countable sequence of real numbers $\epsilon_n \in (0, 1)$, such that $\lim_{n\to\infty} \epsilon_n = 1$ and $\epsilon_0 = 1$. It is sufficient to inductively construct a full chain C=$(G_n : n \in \mathbb{N})$ of finite graphs in G satisfying the following conditions for all $n \in \mathbb{N}$:

(1) $x_n \in V(G_n)$;
(2) $\dfrac{c(G_n)}{|V(G_n)|} \geq \epsilon_n$.

If items 1 and 2 hold, then

$$\text{UD}(G) \geq \lim_{n\to\infty} D_c(G_n) \geq \lim_{n\to\infty} \epsilon_n = 1,$$

and so UD$(G) = 1$.

Let G_0 be the subgraph induced by $\{x_0\}$. Then G_0 satisfies items 1 and 2 above. Suppose G_n has been constructed. Add x_{n+1} to G_n to form the induced subgraph G'_{n+1}. If $c(G'_{n+1})/|V(G'_{n+1})| \geq \epsilon_{n+1}$, then let $G_{n+1} = G'_{n+1}$. Otherwise, suppose that $c(G'_{n+1})/|V(G'_{n+1})| = p/q < \epsilon_{n+1}$. By the strongly 0-e.c. property of G, we may add k isolated vertices to G'_{n+1} to form G_{n+1}, where k is a positive integer that is to be determined. Then

$$\frac{c(G_{n+1})}{|V(G_{n+1})|} = \frac{p+k}{q+k}.$$

We choose k so that $(p+k)/(q+k) \geq \epsilon_{n+1}$, which holds so long as

$$k \geq \frac{\epsilon_{n+1}q - p}{1 - \epsilon_{n+1}}.$$

□

The following corollary gives a necessary condition for G to have infinite cop number, and follows directly by Theorems 2.1 and 2.5.

Corollary 2.6. *If* $c(G) = \aleph_0$*, then* $\mathrm{UD}(G) = 1$*.*

The converse of Theorem 2.6, however, is false in a strong sense.

Theorem 2.7. *For each real number* $r \in [0,1]$*, there is a graph* $G(r)$ *with* $c(G(r)) = 1$*, so that for some full chain* C *in* $G(r)$*,* $D(G(r), C) = r$*.*

Proof. Fix $r \in [0,1]$, and let $(p_n : n \in \mathbb{N})$ be a sequence of rationals such that $\lim_{n\to\infty} p_n = r$, with $p_0 = 1$. We construct a full chain $C = (G_n : n \in \mathbb{N})$ for $G = G(r)$ with the property that $D_c(G_n) = p_n$.

The vertex set of G is $X \cup Y$, where $X = \bigcup_{i\in\mathbb{N}} X_i$ and $Y = \bigcup_{j\in\mathbb{N}} Y_j$ are families of finite disjoint sets of vertices. The set X induces a clique, while Y induces a co-clique. Each vertex of X_n is joined to all vertices of Y_m, where $m < n$. The cardinalities of the finite sets X_i and Y_j are to be determined, while we take X_0 and Y_0 to have one element. For all $n \in \mathbb{N}$, let G_n be the subgraph induced by

$$\{X_i : i \leq n\} \cup \{Y_j : j \leq n\}.$$

Suppose that $D_c(G_n) = p_n$ and $p_{n+1} = a/b$, with $a < b$ and $\gcd(a,b) = 1$. Suppose that $|V(G_n)| = k_n$.

Note that $c(G_{n+1}) = 1 + |Y_{n+1}|$, since one cop is needed to capture the robber if he is in $V(G_n)$ or X_{n+1}, while $|Y_{n+1}|$ cops are needed if the robber is in Y_{n+1}. Let $|X_{n+1}| = x_n$, and $|Y_{n+1}| = y_n$. Then

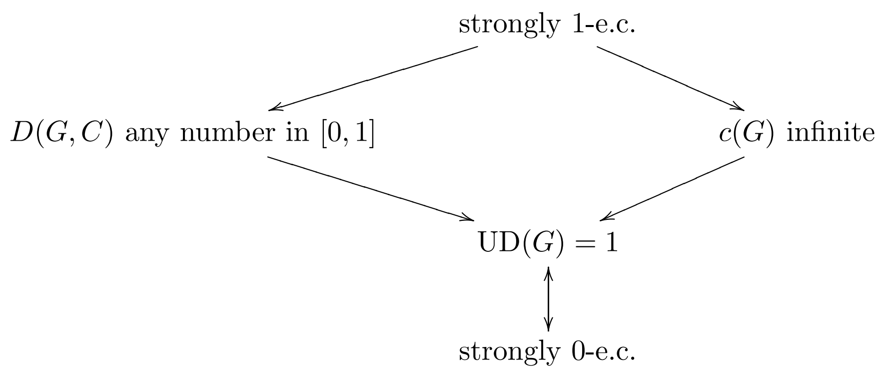
$$\frac{c(G_{n+1})}{|V(G_{n+1})|} = \frac{1+y_n}{k_n + x_n + y_n}.$$

By a method similar to the one used in Theorem 2.3, we may find non-negative integers x_n and y_n such that $c(G_{n+1})/|V(G_{n+1})| = a/b$. Hence,

$$\lim_{n\to\infty} D_c(G_n) = r.$$

One cop may win on G in at most three rounds. To see this, let $\mathcal{C}$ first move to X_1. If $\mathcal{R}$ moves to a vertex of X, then he is captured in the next round. Suppose that $\mathcal{R}$ moves to a vertex of Y_j. Then $\mathcal{C}$ moves to X_{j+1}. As vertices of Y_j are only joined to vertices of X, to avoid capture in the second round, $\mathcal{R}$ must move to some X_k. But then $\mathcal{R}$ is captured in the third round, as X is a clique. $\square$

The diagram below summarizes the logical correspondence between the concepts studied in Theorems 2.1, 2.3, 2.5, and 2.6.



None of the one-way arrows reverse. It can be shown that an infinite one-way ray is infinite-cop win and has cop density any real number in $[0, 1]$, but is not strongly 1-e.c. The infinite co-clique G has upper cop density 1, but $D(G, C) = 1$ for all full chains C for G. The example $G(1)$ of Theorem 2.7 illustrates that $\mathrm{UD}(G)$ may be 1 with $c(G) = 1$.

We may vary the problem by considering the *active* version of the game, where cops and robber must always change vertices at each time step. Define $c'(G)$ for the cop number of this active version. It is straightforward to see that $c'(G) \leq \lfloor |V(G)|/2 \rfloor$. However, nothing new is gained for us in this direction. To see this, consider a full chain

$$C = (G_n : n \in \mathbb{N})$$

in G. As proved in [18],

$$c(G) - 1 \leq c'(G) \leq c(G).$$

Hence,

$$\lim_{n\to\infty} \frac{c'(G_n)}{|V(G_n)|} = \lim_{n\to\infty} \frac{c(G_n)}{|V(G_n)|}.$$

3. The cop density of connected graphs

A natural variation of the cop density parameter is to only consider full chains where G_n is connected for all $n \in \mathbb{N}$. In the case when G is connected, $c(G) \leq \lfloor |V(G)|/2 \rfloor$, so $D_c(G), D(G, C) \leq 1/2$. The *domination number* $\gamma(G)$ of G is the order of a dominating set of smallest cardinality. It is clear that $c(G) \leq \gamma(G)$. Few known general upper bounds exist for the density $c(G)/|V(G)|$ (we repeat that this only applies if G is connected, since disconnected finite graphs may have density 1). For example, an open problem is to find integers $k > 2$, constants c_k, and provide infinitely many examples of connected graphs G of order k where $D_c(G) = c_k/k$. If the minimum degree $\delta(G)$ of the graph is fairly large, then upper bounds exist. By [5, 20], if G is connected, then

$$\gamma(G) \leq \frac{|V(G)|(1+\ln(\delta(G)+1))}{\delta(G)+1},$$

and so this supplies an upper bound for the cop number. The smallest value of $\delta(G)$ is 5 where this gives an upper bound for D_c better than 1/2.

Lemma 3.1. *If G is connected with $\delta(G) = k \geq 5$, then*

$$D_c(G) \leq \frac{(1+\ln(k+1))}{k+1}.$$

The main result of this section is that random graphs have cop density in $\Theta((\ln n)/n)$. For this, we note that Dreyer [10] proves the following result.

Theorem 3.2. *Let $0 < p < 1$ be fixed, and let $q = 1/(1-p)$. For any real $\epsilon > 0$, with probability 1 as $n \to \infty$, there exists a set of vertices of cardinality at most $(1+\epsilon)\log_q n$ in $G \in G(n,p)$ which is a dominating set.*

By Theorem 3.2 we have the following.

Corollary 3.3. *Let $0 < p < 1$ be fixed and $q = 1/(1-p)$. For any real $\epsilon > 0$,with probability 1 as $n \to \infty$, for $G \in G(n,p)$,*

$$c(G) \leq (1+\epsilon)\log_q n.$$

We will prove the following result for the cop number of a finite random graph.

Theorem 3.4. *Let $0 < p < 1$ be fixed and $q = 1/(1-p)$. For every real $\epsilon > 0$, with probability 1 as $n \to \infty$, for $G \in G(n,p)$*

$$(1-\epsilon)\log_q n \leq c(G) \leq (1+\epsilon)\log_q n.$$

The proof will follow by establishing the lower bound for the cop number of $G(n,p)$. We need the following lemma.

Lemma 3.5. *Let $0 < p < 1$ and $r > 0$ be fixed. For any fixed $0 < \epsilon < 1$, if $(\ln 1/(1-p))\, d = 1-\epsilon$, then*

$$\lim_{n\to\infty} n^{\lfloor d\ln n\rfloor+1}\left(1-r(1-p)^{\lfloor d\ln n\rfloor}\right)^{n-\lfloor d\ln n\rfloor-1} = 0. \tag{3.1}$$

Proof. As

$$n^{\lfloor d\ln n\rfloor+1}(1-r(1-p)^{\lfloor d\ln n\rfloor})^{n-\lfloor d\ln n\rfloor-1} \leq n^{d\ln n+1}(1-r(1-p)^{d\ln n})^{n-d\ln n-1},$$

it is enough to prove that

$$\lim_{n\to\infty} n^{d\ln n+1}\left(1-r(1-p)^{d\ln n}\right)^{n-d\ln n-1} = 0. \tag{3.2}$$

If $q = 1 - p$ and $c = \ln(1/q)$, then $c, d > 0$ and $0 < cd < 1$. Now

$$\begin{aligned} n^{d\ln n+1}(1-r(1-p)^{d\ln n})^{n-d\ln n-1} &= n^{d\ln n+1}(1-rq^{d\ln n})^{n-d\ln n-1} \\ &= n^{d\ln n+1}\left(1-\frac{r}{n^{cd}}\right)^{n-d\ln n-1} \\ &= \exp\left(f(n)\right), \end{aligned} \tag{3.3}$$

where

$$f(n) = (d\ln n + 1)\ln(n) + (n - d\ln n - 1)\ln\left(1 - \frac{r}{n^{cd}}\right).$$

For a sufficiently large n, $\ln\left(1 - r/n^{cd}\right) < 0$, we have that

$$\lim_{n\to\infty} f(n) = -\infty. \tag{3.4}$$

By (3.3) and (3.4), (3.2) follows. □

For k a positive integer, a graph is $(1, k)$-*e.c.* if for each k-element subset S of vertices of G and vertex u, there is a vertex $z \notin S$ not joined to any vertex in S and joined to u.

Proof of Theorem 3.4. It is easy to see that if G is $(1, k)$-e.c., then $c(G) \geq k$. Let $0 < p < 1$, and $0 < \epsilon < 1$ be fixed, and let $k = \lfloor (1-\epsilon)\log_q n \rfloor$, where $q = 1/(1-p)$. The probability that G is not $(1, k)$-e.c. is at most

$$f(n) = n^{k+1}(1 - p(1-p)^k)^{n-k-1}.$$

Setting $r = p$ in (3.1), we have that

$$\lim_{n\to\infty} f(n) = 0,$$

by Lemma 3.5. Hence, with probability 1 as $n \to \infty$,

$$c(G) \geq (1-\epsilon)\log_q n.$$

□

If Q_k is the hypercube of dimension k, then Theorem 2.4 of [18] proves that $c(Q_k) \in \Theta(k)$. Hence, the cop density of Q_k is in $\Theta((\ln n)/n)$, where $n = 2^k$. As almost all finite graphs have diameter 2, almost all finite connected graphs have density around $(\ln n)/n$, as the following result demonstrates.

Corollary 3.6. *Fix p and ϵ in $(0, 1)$. Then with probability 1 as $n \to \infty$, for $G \in G(n, p)$,*

$$\frac{(1-\epsilon)\log_q n}{n} \leq D_c(G) \leq \frac{(1+\epsilon)\log_q n}{n}$$

An open problem is to determine the cop number of the random graph $G(n, p)$ when p is a function of n.

4. Acknowledgements

The authors would like to thank Jeannette Janssen and Richard Nowakowksi for helpful discussions. Part of the research for this paper was conducted while the first and third authors were visiting the Fields Institute for Research in Mathematical Sciences in Toronto, and while the first and second authors were visiting the Bellairs Research Institute of McGill University.

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