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ON GEOMETRIC CONSTRUCTIONS OF (k, g)-GRAPHS

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Dedicated to the centenary of the birth of Ferenc Kárteszi (1907–1989).

Abstract. We give new constructions for k-regular graphs of girth $6, 8$ and 12 with a small number of vertices. The key idea is to start with a generalized n-gon and delete some lines and points to decrease the valency of the incidence graph.

1. Introduction

Noting that the smallest regular graph of valency 3 and girth 5 is the Petersen graph, Ferenc Kárteszi posed the question to determine the least number $c(k, g)$ of vertices a regular graph of valency k and girth g can have. The *girth* of a graph is the length of the shortest cycle in it. In 1960 Ferenc Kárteszi [15] proved the following theorem.

Theorem 1.1. *A regular graph with valency k and girth 6 has at least $2((k-1)^2 + (k-1) + 1)$ vertices where equality holds if and only if it is the incidence graph of a projective plane of order $k-1$.*

In 1963 Erdös and Sachs [11] showed that for every integer $k \geq 2$ and $g \geq 3$ there exists a regular graph (without loops and multiple edges) of valency k and girth g. Such graphs are called *(k, g)-graphs*. A (k, g)-graph with $c(k, g)$ points is called a *(k, g)-cage*. The problem of determining the exact value of $c(k, g)$ is still open for most of the cases, for a survey, we refer to Wong [20] or to the website of Royle [18]. By counting the number of vertices at distance $1, 2, \ldots$ from a vertex or an edge, the following lower bound for $c(k, g)$ is easily proved (see [6], page 180).

Proposition 1.2 (Moore bound)**.**

$$c(k,g) \geq \begin{cases} 1 + k + k(k-1) + \cdots + k(k-1)^{\frac{g-1}{2}-1} & \text{for } g \text{ odd;} \\ 2\left(1 + (k-1) + (k-1)^2 + \cdots + (k-1)^{\frac{g}{2}-1}\right) & \text{for } g \text{ even.} \end{cases}$$

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We shall refer to this as the Moore bound, though originally this name came from an upper bound for the number of vertices a regular graph can have with bounded valency and diameter. We will call a (k, g)-graph a *Moore graph* if its number of vertices satisfy equality in the Moore bound (some authors only use this term for graphs with g odd). There is extensive literature on Moore graphs, and it turns out that for $k \geq 3$, Moore graphs may exist only if $g = 3, 4, 5, 6, 8, 12$. This result is due to Damerell [10], Bannai and Ito [5] and Feit and Higman [12]. Furthermore, the Hoffmann-Singleton theorem [14] says that a Moore graph of girth $g = 5$ may only have valency $k = 2, 3, 7, 57$. Note that for $g = 3$ and 4 the problem is trivial: for $g = 3$ the unique Moore graph is the complete graph on $k + 1$ vertices, while for $g = 4$ the unique Moore graph is the complete bipartite graph on $2k$ vertices.

There are several constructions implying upper bounds on $c(k, g)$. One useful idea is to look for small regular subgraphs of known Moore graphs. For example the unique $(7, 5)$-cage, the Hoffman-Singleton graph (which is a Moore graph) contains the $(6, 5)$-cage, a $(5, 5)$-cage and the $(3, 5)$-cage (i.e., the Petersen graph) as induced subgraphs (see [20]). In this paper we apply this idea to obtain (k, g)-graphs for $g = 6, 8, 12$. In these cases there are infinite series of Moore graphs.

When $g = 2n \geq 6$, one can characterize Moore graphs as the incidence graph of certain generalized n-gons. The *incidence graph* of a set system in general is a bipartite graph, where the two vertex classes correspond to points and sets respectively, and edges correspond to incident point-set pairs.

Definition 1.3. *Let $\mathcal{P}$ be a finite set and $\mathcal{L}$ a set of subsets of $\mathcal{P}$ called points and lines, respectively. The pair $(\mathcal{P}, \mathcal{L})$ is called a generalized n-gon of order (s, t), if it satisfies the following axioms:*

- *there are $s + 1$ lines through every point;*
- *every line contains $t + 1$ points;*
- *the diameter and the girth of the incidence graph are n and $2n$, respectively.*

It is straightforward to check that a regular graph G with girth $2n$ and valency k is a Moore graph if and only if it is the incidence graph of a generalized n-gon of order $(k - 1, k - 1)$. Feit and Higman [12] proved that generalized n-gons with $s, t \geq 2$ can exist only if $n \in \{3, 4, 6, 8\}$ and that for $n = 8$, $s = t$ cannot occur. Hence Moore graphs with $k \geq 3$ and g even can exist only if $g \in \{6, 8, 12\}$. There are examples whenever $k - 1$ is a prime power and it is wide open if there exist examples for other values of k.

When $g = 6, 8$ or 12, but $k-1$ is not a prime power (i.e., there is no known generalized $g/2$-gon of order $(k - 1, k - 1)$), then one can do the following. Start from a Moore graph with valency $q + 1$, where q is the smallest prime power bigger than or equal to k, and delete vertices from the graph to make it k-regular. The first one to use this idea (for $g = 6$) seems to be Brown [8]. In [1] Abreu, Funk, Labbate and Napolitano use a method which is in fact equivalent to Brown's method applied for the projective plane $\mathrm{PG}(2, q)$. In a recent paper by Araujo, Gonzalez, Montellano and Oriol [2], the same idea was used for the $g = 8$ and 12 cases, too.

In this paper we apply the same method, but use more from the geometrical structure of generalized n-gons to improve the previous constructions, hence the upper bounds for $c(k, g)$, $g \in \{6, 8, 12\}$. (In fact, our main improvements are for $g = 6$ and $g = 8$.)

In Section 2 we explain the construction method, and give two constructions that work for every generalized n-gon.

In Section 3 we consider the $g = 6$ case, i.e., projective planes. In this case the best construction we will find is when k is close to the square of a prime power. Furthermore we will also prove that in this case one cannot hope for a better construction by deleting vertices from a Moore graph.

Section 4 is devoted to the $g = 8$ case, i.e., generalized quadrangles. We only achieve an improvement when k is a prime power (in this case a Moore graph of valency $k + 1$ exists) and we cannot prove that this construction is best possible.

In Section 5 we list the cases when our constructions yield a new upper bound on $c(k, g)$.

2. The construction method

In all the constructions of this paper we will look for regular subgraphs (of valency k) of the incidence graph of a generalized n-gon (or order q) by deleting a set of points and a set of lines. The girth of a graph of this kind is at least $2n$, since the original girth is exactly $2n$. In all interesting cases (i.e., for k not not much smaller than q) it is the direct consequence of the Moore bound that the girth of the resulting graph cannot be larger than $2n$.

Definition 2.1. *The pair $(\mathcal{P}_0, \mathcal{L}_0)$ in the generalized n-gon $(\mathcal{P}, \mathcal{L})$ is called a t-*good structure*, if there are t lines of $\mathcal{L}_0$ through any point not in $\mathcal{P}_0$, and there are t points of $\mathcal{P}_0$ on any line not in $\mathcal{L}_0$.*

General construction method. Suppose $(\mathcal{P}_0, \mathcal{L}_0)$ is a t-good structure in the generalized n-gon $(\mathcal{P}, \mathcal{L})$ of order q. Deleting points and lines of $\mathcal{P}_0$ and $\mathcal{L}_0$, respectively, the incidence graph of the resulting structure is $(q+1-t)$-regular with girth at least $2n$. To obtain small subgraphs, we need to find large t-good structures.

Complements of $\mathcal{P}_0$ and $\mathcal{L}_0$ will be denoted by $\mathcal{P}_1$ and $\mathcal{L}_1$, respectively. We will also call the points of $\mathcal{P}_0$ *deleted points* and the lines of $\mathcal{L}_0$ *deleted lines*. Note that since we start with and get a regular bipartite graph, $|\mathcal{P}_0| = |\mathcal{L}_0|$ and $|\mathcal{P}_1| = |\mathcal{L}_1|$ holds.

We end this section with two constructions that work in any generalized n-gon $(\mathcal{P}, \mathcal{L})$. For a point p or line l of the generalized n-gon, p and l will also denote the corresponding vertices of the incidence graph. We define the distance $d(x, y)$ for a pair $x, y \in \mathcal{P} \cup \mathcal{L}$ to be the distance in the incidence graph, i.e., the length of the shortest path connecting x and y. The following construction can be found in [2] in a slightly different form (proof of Theorem 1). We will show that the given constructions are t-good if n is even (i.e., $n = 4, 6$), the $n = 3$ case will be proved in Section 3.

Construction 2.2. *Take a generalized n-gon $(\mathcal{P}, \mathcal{L})$ of order (q, q). Let $p_1, \ldots, p_t \in \mathcal{P}$ all incident with a line l_1 and let $l_2, \ldots, l_t$ be lines through p_1. Delete every line and point at distance at most $n-2$ from p_i or l_i, $i \in \{1, \ldots, t\}$. This gives a t-good structure of size $tq^{n-2} + q^{n-3} + \ldots + q + 1$.*

Proof. It suffices to show that for any point $p \in \mathcal{P}_1$ there are exactly t lines through p in $\mathcal{L}_0$. The analogous statement for the lines not deleted can be seen dually. Since p is not deleted and n is even, $d(p_i, p) = n$, $d(l_i, p) = n-1$ for every $i \in \{1, \ldots, t\}$. Thus the lines incident with p are at distance $n-1$ from the p_is and at distance n or $n-2$ from the l_is. There is a unique path connecting l_i and p of length $n-1$ and hence there is a unique line e_i through p that is at distance $n-2$ from a fixed l_i. These e_is are pairwise distinct, since $e_i = e_j$ would imply that the union of the paths l_ie_i, l_je_i, l_il_j contain a cycle of length at most $(n-2) + (n-2) + 2 < 2n$, which is impossible. (Here we used that the intersection of l_i and l_j, namely p_1 is at distance $n-1$ from p, so it cannot be on the paths l_ie_i or l_je_i.) Therefore there are exactly t deleted lines through p, the e_is. The calculation of the size is not difficult. □

The upper bound for $c(k, 6)$ and $c(k, 8)$ coming from the above construction was already proved (with another method) by Lazebnik, Ustimenko and Woldar [16], for the case when the smallest prime power greater than or equal to k is odd.

Construction 2.3. *Take a generalized n-gon $(\mathcal{P}, \mathcal{L})$ of order (q, q). Let $p \in \mathcal{P}$ and $l \in \mathcal{L}$, where p is not on l. Deleting every line and point that are at distance at most $n-2$ from p or l, we get a 1-good structure of size $q^{n-2} + 2q^{n-3} + q^{n-4} + \ldots + q + 1$.*

Proof. One can see that $(\mathcal{P}_0, \mathcal{L}_0)$ is 1-good using the same ideas as in Construction 2.2. We only calculate the size for the $n = 6$ case, the proof of the $n = 4$ case is similar. Denote by l_1 the vertex incident to p in the unique path between p and l (in the incidence graph). Let A_i denote the vertices of the graph, which are of distance i from p and $i+1$ from l_1 $(i = 1, ..., 5)$, and similarly, denote by B_i the vertices of the graph, which are at distance i from l_1 and $i+1$ from p $(i = 1, ..., 5)$ (see Figure 1). Then l is either in B_2 or in B_4.

Let $A_0 = \{p\}$ and $B_0 = \{l_1\}$. Each vertex of A_i or B_i $(0 \le i \le 4)$ is incident to q vertices of A_{i+1} or B_{i+1}, respectively; and each vertex of A_i or B_i $(1 \le i \le 5)$ is incident to a unique vertex of A_{i-1} or B_{i-1}, respectively. The only remaining edges (besides the one between p and l_1) are those between A_5 and B_5; here we have a regular bipartite graph of valency q.

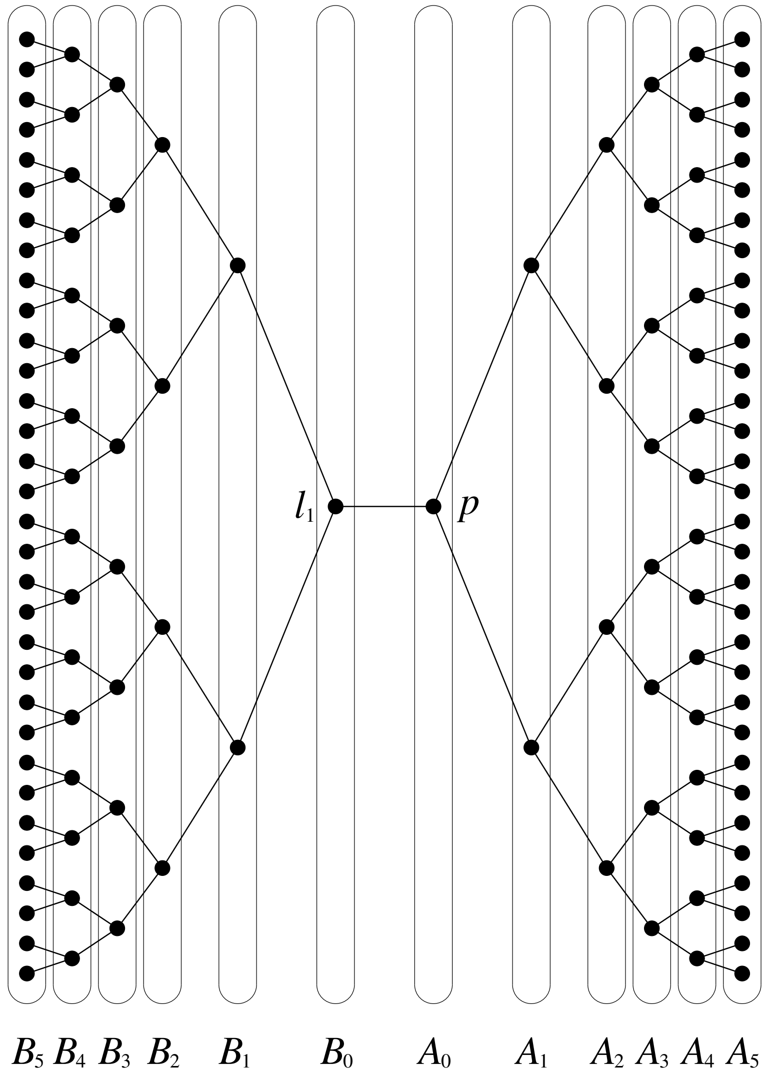

FIGURE 1

Note that the vertex sets corresponding to points of the generalized hexagon are B_5, B_3, B_1, A_0, A_2, A_4. These are all of distance at most 4 from p, except for B_5. Hence all non-deleted points are in B_5. Since all points from B_3, B_1, A_0, A_2, A_4 are deleted, $|\mathcal{P}_0| = q^3 + q + 1 + q^2 + q^4 +$ the number of vertices deleted from B_5. Hence to finish the proof, we have to count the vertices of B_5 at distance 1 or 3 from l. We distinguish two cases according to whether l is in B_2 or B_4.

For $l \in B_2$, all we have is $l - B_3 - B_4 - B_5$ paths, so the number in question is q^3.

For $l \in B_4$, there are three different types of paths: $l - B_5$, $l - B_3 - B_4 - B_5$ and $l - B_5 - A_5 - B_5$. The number of vertices (of B_5) reached from these paths is q, $1 \cdot (q-1) \cdot q$ and $q \cdot q \cdot (q-1)$, respectively. This gives again q^3 vertices. (Note that the girth of the graph assures that we did not count any vertex more than once.) □

Note that the second construction is better than the first one for all three cases, but improvement is achieved only for $t = 1$. As we shall see in the next section, for $n = 3$ one can generalize Construction 2.3 to $t > 1$, and this was already done in [1].

3. The $g = 6$ case: constructions from a projective plane

This section is devoted to the $g = 6$ case, that is, generalized triangles. These are usually called projective planes. It is easy to see that the following definition is equivalent to that of a generalized 3-gon of order (q, q).

Definition 3.1. *Let $\mathcal{P}$ be a finite set and $\mathcal{L}$ a set of subsets of $\mathcal{P}$ called points and lines, respectively. The pair $(\mathcal{P}, \mathcal{L})$ is called a projective plane of order q, if it satisfies the following axioms.*

- *there are $q + 1$ lines through every point and $q + 1$ points on every line;*
- *there is a unique line through any two distinct points and a unique intersection point of any two distinct lines.*

Note that the role of lines and points is symmetric in the definition, hence for every definition and result we also have a dual definition and result by changing the words point and line to each other. It is easy to see (either from the above definition, or from the Moore bound) that the number of points and lines is $q^2 + q + 1$.

First we give two constructions which only use the definition of projective planes. They are not new; see the remarks after the constructions.

Construction 3.2. *Choose lines $l_1, \ldots, l_t$ through a point p_1 and let p_2, …, p_t be $t - 1$ other points on l_1. Let $\mathcal{P}_0$ be the union of points on the l_is ($i = 1, \ldots, t$) and $\mathcal{L}_0$ be the set of lines through any p_i ($i = 1, \ldots, t$). Then $(\mathcal{P}_0, \mathcal{L}_0)$ is t-good of size $tq + 1$.*

Proof. Take a line $e \in \mathcal{L}_1$. Then e intersects every line l_i in one point, therefore e contains t points of $\mathcal{P}_0$. Take a point $p \in \mathcal{P}_1$. We deleted the t lines through p going through some p_i ($1 \le i \le t$). The calculation of the size is easy. □

This construction is the $n = 3$ case of Construction 2.2 (from [2]) and seems to be originally due to Brown [8]. In a recent paper, though with a different terminology, Abreu, Funk, Labbate and Napolitano give the same construction [1, Construction (i), p. 126], see Remark 3.5.

Construction 3.3. *Let l_1 be a line, $p_1 \notin l_1$, $p_2,\dots,p_t \in l$, finally, let $l_2,\dots,l_t$ be the lines joining p_1 to the p_is $(2 \le i \le t)$. Let $\mathcal{P}_0$ consist of all points on the l_is and let $\mathcal{L}_0$ consist of all the lines through the p_is $(1 \le i \le t)$. Then $(\mathcal{P}_0,\mathcal{L}_0)$ is t-good of size $tq+3-t$.*

Proof. Lines in $\mathcal{L}_1$ do not contain any p_i $(1 \le i \le t)$, hence they meet the l_is in t different points, while from points in $\mathcal{P}_1$ (which are not on any l_i $(1 \le i \le t)$) we deleted the t lines which connect the point with some p_i. The calculation of the size is easy.

□

This construction, though with a different terminology, can be found in [1, Construction (ii), p. 126], see Remark 3.5.

Remark 3.4. Note that for $t = 1$, the second construction is slightly better than the first one (recall that we need t-good sets as large as possible). If $t = 2$, then the two constructions above are the same. This proves a conjecture of the just mentioned paper [1, Remark 3.7, p. 127], see Remark 3.5.

Remark 3.5. Now we explain the connection of the above constructions to the ones cited from [1]. For unexplained facts or definitions from finite geometry we refer to [13]. First let us consider and rephrase the constructions in [1]. Let $A = A(q)$ be the addition table of the finite field $\mathrm{GF}(q)$, i.e., the rows and columns are indexed by the elements of the field and $A_{i,j} = i + j$. Similarly, let $M = M(q)$ be the multiplication table of the multiplicative group $\mathrm{GF}(q)^*$ of $\mathrm{GF}(q)$, i.e., $M_{i,j} = ij$. Let H be an arbitrary matrix over $\mathrm{GF}(q)$ and let $z \in \mathrm{GF}(q)$. Define the $0-1$ matrix $P_z(H)$ by $P_z(H)_{i,j} = 1$ if and only if $H_{i,j} = z$. Now the matrices corresponding to the two constructions $G_*(q,1)$ and $G_+(q,1)$ in [1, p. 126], are the following: substitute every element $M_{i,j}$ by $P_{M_{i,j}}(A)$ in M, and respectively, substitute every element $A_{i,j}$ by $P_{A_{i,j}}(M)$ in A. Let these "blow ups" be denoted by $\overline{M}$ and $\overline{A}$, respectively. The conjecture in [1], page 127 says that the incidence graphs of the incidence structures corresponding to these incidence matrices are isomorphic.

Let us consider $\overline{M}$. It is natural to index its rows and columns by pairs (a,b), $a \in \mathrm{GF}(q)^*$, $b \in \mathrm{GF}(q)$. Now by its definition $\overline{M}_{(x,y),(m,b)} = 1$ exactly when $xm = y+b$. Now we can see that the rows and columns of $\overline{M}$ naturally correspond to the points and lines of the affine plane $\mathrm{AG}(2,q)$: the row (x,y) corresponds to the point (x,y) in $\mathrm{AG}(2,q)$, while the column (m,b) corresponds to the line defined by the equation $y = mx-b$ (i.e., the line with slope m and y-intercept $-b$), and a 1 entry in $\overline{M}$ corresponds to an incident point-line pair. Since the first coordinates are from $\mathrm{GF}(q)^*$, we do not have lines having slope 0 or ∞ (i.e., horizontal and vertical lines), furthermore we do not have points on the y axis. Since $\mathrm{PG}(2,q)$ can be viewed as $\mathrm{AG}(2,q)$ and a line at infinity, one can see that the structure related to $\overline{M}$ comes from $\mathrm{PG}(2,q)$ according to Construction 3.2 with l_1 and l_2 being the line at infinity and the y axis, and p_1 and p_2 being the points on the line at infinity corresponding to the parallel classes of vertical and horizontal lines. One may check easily that $\overline{A}$ has the same meaning, so the graphs defined this way (the incidence graphs of the structures described by the incidence matrices $\overline{M}$ and $\overline{A}$) are isomorphic.

Furthermore, Construction 3.2 (or Construction 3.3) give rise to isomorphic structures and graphs for $t = 2$, independently from the choice of p_1, p_2, l_1, l_2, since the automorphism group of $\mathrm{PG}(2, q)$ is well known to be transitive on the quadruples of points in general position, and so there are many automorphisms that bring p_1, p_2 and an arbitrary third point p_3 on l_2 to p'_1, p'_2 and an arbitrary third point p'_3 on l'_2, and this implies that $\mathcal{P}_0$ and $\mathcal{L}_0$ are transformed into $\mathcal{P}'_0$ and $\mathcal{L}'_0$, where $(\mathcal{P}_0, \mathcal{L}_0)$ and $(\mathcal{P}'_0, \mathcal{L}'_0)$ are 2-good structures given by Construction 3.2.

Remark 3.6. In the second construction $(\{p_1, \dots, p_t\}, \{l_1, \dots, l_t\})$ is a so called degenerate subplane. This can be generalized by taking a subplane S of order k and deleting all the lines through the points of S and all the points on the lines which meet S in $k+1$ points. We do not give any details, since this gives rise to smaller t-good sets than the previous ones.

We continue with a construction that is better than the previous ones, but only works when q is a square prime power. First some definitions and basic facts. A subset B of the points of a projective plane is called a *Baer subplane*, if it has size $q + \sqrt{q} + 1$ and meets every line in 1 or $\sqrt{q}+1$ points. Easy calculation shows that through a point out of the set there is a unique $(\sqrt{q} + 1)$-secant, while through points in the set the number of $(\sqrt{q} + 1)$-secants is $\sqrt{q} + 1$. Hence the number of $(\sqrt{q} + 1)$-secants is $q + \sqrt{q} + 1$. After this, one can easily deduce that B, together with its intersections with $(\sqrt{q}+1)$-secants, forms a projective plane of order $\sqrt{q}$. The $(\sqrt{q}+1)$-secants are sometimes called *the lines of* B.

Construction 3.7. *Suppose that in our projective plane there are t disjoint Baer subplanes $B_1, \dots, B_t$ with the property that no two of them has a common $(\sqrt{q} + 1)$-secant. Let $\mathcal{P}_0$ consist of the union of the B_is, and $\mathcal{L}_0$ of all lines intersecting one of the B_is in $\sqrt{q} + 1$ points. Then $(\mathcal{P}_0, \mathcal{L}_0)$ is t-good of size $t(q + \sqrt{q} + 1)$.*

Proof. First of all note that by the above listed properties, all lines meet $\mathcal{P}_0$ in either t or $\sqrt{q + t}$ points. Lines in $\mathcal{L}_1$ meet any of the t deleted subplanes in one point, hence we deleted t points from them. Let $p \in \mathcal{P}_1$ be an arbitrary point not deleted. For every $1 \le i \le t$ there is a unique line through p meeting B_i in $\sqrt{q} + 1$ points, and these lines are different for different i's, so there are t lines deleted from p. The calculation of the size is easy. □

In general, it is not true (or at least not known) that any projective plane of square order has a Baer subplane, but it is true for the ones coordinatized by the finite field $\mathrm{GF}(q)$. These are denoted by $\mathrm{PG}(2, q)$ and can be defined as follows. Let V denote a 3-dimensional vector space over GF(q). Let $\mathcal{P}$ and $\mathcal{L}$ consist of the 1- and 2-dimensional subspaces of V, respectively, and define incidence as inclusion. To make lines become subsets of points, one can identify lines with the set of 1-dimensional subspaces it contains. The pair $(\mathcal{P}, \mathcal{L})$ is a projective plane of order q. When q is square, PG$(2, q)$ does contain Baer subplanes, all of them are isomorphic to PG$(2, \sqrt{q})$. Moreover, any two disjoint Baer subplanes have distinct $(\sqrt{q} + 1)$-secants. This is a particular case of a theorem due to Sved [19]. Even more is true: PG$(2, q)$ can be partitioned into $q - \sqrt{q} + 1$ disjoint Baer subplanes. For more about projective planes, Baer subplanes and for the proofs of the listed properties, we refer to [13].

Theorem 3.8. *For any square prime power q and $t \leq q - \sqrt{q} + 1$, Construction 3.7 works in the plane PG$(2, q)$.*

Proof. By the listed facts about PG$(2, q)$, one can find $q - \sqrt{q} + 1$ disjoint Baer subplanes. Choosing only t of these will be appropriate, since all we need is that the $(\sqrt{q}+1)$-secants are distinct, and this follows from the above mentioned result by Sved. □

In [1, Section 4], there is a construction for $q = 4, 9$ and 16 giving a graph of the same size as the one in Construction 3.7 here. The authors make a conjecture which would imply that their construction works for every square prime power q. Theorem 3.8 shows that a construction giving the same size exists.

After this, it is natural to ask if one could improve this construction by finding larger t-good structures. We will prove that, at least for $t \leq 2\sqrt{q}$, Construction 3.8 is the best possible. We also want to study, if there are more constructions. We will prove the following theorems.

Theorem 3.9. *In an arbitrary projective plane of order q, every t-good structure with $t \leq 2\sqrt{q}$ has size at most $t(q + \sqrt{q} + 1)$.*

Theorem 3.10. *In any projective plane a 1-good pair $(\mathcal{P}_0, \mathcal{L}_0)$ is one of those given by Constructions 3.2, 3.3, and 3.7.*

Theorem 3.11. *In PG$(2, q)$, $q > 256$, every 2-good structure is one of those given by Constructions 3.2, 3.3, and 3.7.*

In the proof of Theorem 3.9 we will use the so called *standard equations.* For any point set S in a projective plane of order q, denote by n_i the number of i-secants to S. Recall that both the number of points and lines is q^2+q+1. By counting the total number of lines, incident pairs (P, l) with $P \in S$, and triples (P, Q, l) with $P \neq Q \in S$, we obtain the following three equations:

$$\sum_{i=0}^{q+1} n_i = q^2 + q + 1,$$

$$\sum_{i=0}^{q+1} i n_i = |S| (q+1),$$

$$\sum_{i=0}^{q+1} i(i-1) n_i = |S| (|S| - 1).$$

Proof of Theorem 3.9. For a t-good structure $(\mathcal{P}_0, \mathcal{L}_0)$, let n_i^0 denote the number of i-secants to $\mathcal{P}_0$ in $\mathcal{L}_0$ and n_i^1 the number of i-secants to $\mathcal{P}_0$ in $\mathcal{L}_1$. Then the total number of i-secants to $\mathcal{P}_0$ is $n_i = n_i^0 + n_i^1$. Since $(\mathcal{P}_0, \mathcal{L}_0)$ is t-good, by definition

$$n_i^1 = \begin{cases} q^2 + q + 1 - |\mathcal{L}_0| & \text{for } i = t, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

Using 3.1, the standard equations and $|\mathcal{P}_0| = |\mathcal{L}_0|$, we obtain

$$\sum_{i=0}^{q+1} n_i^0 = |\mathcal{L}_0|,$$

$$\sum_{i=0}^{q+1} i n_i^0 = |\mathcal{L}_0| (q + 1 + t) - t(q^2 + q + 1),$$

$$\sum_{i=0}^{q+1} i(i-1) n_i^0 = |\mathcal{L}_0|^2 + |\mathcal{L}_0| (t^2 - t - 1) - t(t-1)(q^2 + q + 1).$$

Using the three equations above we get

$$\begin{aligned}
0 &\leq \sum_{i=0}^{q+1} (i - (\sqrt{q} + t))^2 n_i^0 \\
&= \sum_{i=0}^{q+1} i(i-1) n_i^0 - \sum_{i=0}^{q+1} (2(\sqrt{q} + t) - 1) i n_i^0 + \sum_{i=0}^{q+1} (\sqrt{q} + t)^2 n_i^0 \\
&= |\mathcal{L}_0|^2 + |\mathcal{L}_0| \left[t^2 - t - 1 - (2(\sqrt{q} + t) - 1)(q + 1 + t) + (\sqrt{q} + t)^2 \right] \\
&\qquad + (q^2 + q + 1) \left[(2t(\sqrt{q} + t) - t) - t(t-1) \right] \\
&= |\mathcal{L}_0|^2 - 2 \left[(q+1)t + \sqrt{q}(q - \sqrt{q} + 1) \right] |\mathcal{L}_0| + (q^2 + q + 1)(2t\sqrt{q} + t^2) \\
&= (|\mathcal{L}_0| - t(q + \sqrt{q} + 1)) (|\mathcal{L}_0| - (t + 2\sqrt{q})(q - \sqrt{q} + 1));
\end{aligned}$$

hence either $|\mathcal{L}_0| \leq t(q + \sqrt{q} + 1)$ or $|\mathcal{L}_0| \geq (t + 2\sqrt{q})(q - \sqrt{q} + 1)$ (it is easy to check that the first root is smaller than the second one). Assuming

$0 \le t \le 2\sqrt{q}$ and $|\mathcal{L}_0| \ge (t + 2\sqrt{q})(q - \sqrt{q} + 1)$, the number of vertices in the $(q + 1 - t)$-regular graph induced by $\mathcal{L}_1$ and $\mathcal{P}_1$ would be

$$\begin{aligned} |\mathcal{L}_1| + |\mathcal{P}_1| &\le 2\left(q^2 + q + 1 - (t + 2\sqrt{q})\,(q - \sqrt{q} + 1)\right) \\ &< 2\left(q^2 + q + 1 - t(2q - t + 1)\right) \\ &= 2\left((q - t)^2 + (q - t) + 1\right), \end{aligned}$$

contradicting the Moore-bound. Therefore $|\mathcal{L}_0| \le t(q + \sqrt{q} + 1)$ must hold. □

One can characterize equality in the previous bound for $\mathrm{PG}(2, q)$ using the following result due to Blokhuis, Storme and Szőnyi. A subset of the points of a projective plane is called a *t-fold blocking set*, if it meets every line in at least t points. For $t = 1$, it is simply called a *blocking set*.

Theorem 3.12. *(Blokhuis, Storme, Sznyi [7]) In $PG(2, q)$ a t-fold blocking set has at least $t\left(q + \sqrt{q} + 1\right)$ points for $t < \sqrt[4]{q}/2$, and equality holds if and only if the set is the union of t disjoint Baer-subplanes.*

In the proof of Theorem 3.9 equality holds exactly when $n_i^0 \ne 0 \iff i = \sqrt{q} + t$, which means that every line in $\mathcal{L}_0$ intersects $\mathcal{P}_0$ in $\sqrt{q} + t$ points. The lines in $\mathcal{L}_1$ meet $\mathcal{P}_0$ in t points, hence $\mathcal{P}_0$ is a t-fold blocking set.

Corollary 3.13. *If $t < \sqrt[4]{q}/2$ and $(\mathcal{P}_0, \mathcal{L}_0)$ is a t-good structure in $PG(2, q)$ with $|\mathcal{P}_0| = t\left(q + \sqrt{q} + 1\right)$, then $\mathcal{P}_0$ is the union of t disjoint Baer-subplanes and the lines in $\mathcal{L}_0$ are those that intersect one of the Baer-subplanes in $\sqrt{q} + 1$ points, hence we have Construction 3.7.*

For the proofs of Theorems 3.10 and 3.11, we need some more definitions and results about projective planes.

It is easy to check that any blocking set contains at least $q + 1$ points, with equality if and only if it is a line.

Theorem 3.14 (Bruen [9])**.** *In any projective plane of order q a blocking set not containing a line has size at least $q + \sqrt{q} + 1$ with equality if and only if it is a Baer subplane.*

Lemma 3.15. *Let $(\mathcal{P}_0, \mathcal{L}_0)$ be a t-good structure, $t < \sqrt{q}$. Then $\mathcal{P}_0$ is a blocking set.*

Proof. Assume that there exists a line l not meeting $\mathcal{P}_0$. Then l must be in $\mathcal{L}_0$. Since any point p on l is in $\mathcal{P}_1$, there has to be exactly $t - 1$ lines from $\mathcal{L}_0$ different from l through p, therefore $|\mathcal{L}_0| = 1 + (q+1)(t-1) = tq - q + t$. On the other hand, taking a line $e \in \mathcal{L}_1$, we can see at least $(q + 1 - t)t$ deleted lines intersecting e, thus $tq + t - t^2 \le tq - q + t$, which cannot occur for $t < \sqrt{q}$. □

Proof of Theorem 3.10. First note that by Lemma 3.15, $\mathcal{P}_0$ is a blocking set. Since a line not deleted meets $\mathcal{P}_0$ in exactly $t = 1$ points, every line joining two deleted points has to be deleted, and dually, the intersection of two deleted lines is in $\mathcal{P}_0$. We distinguish three cases according to the maximum number ν of points in $\mathcal{P}_0$ such that no three of them is collinear:

Case 1. $\nu = 2$.

Then $\mathcal{P}_0$ is contained in a line, but since it is a blocking set, it has to be the full line. It is easy to see that this is Construction 3.2.

Case 2. $\nu = 3$.

In this case $|\mathcal{P}_0| \leq q + 2$, since it cannot contain two pairs of points on two different lines, since that would imply $\nu \geq 4$. By Theorem 3.14, $\mathcal{P}_0$ has to contain a line, thus $|\mathcal{P}_0| = q + 2$. It is easy to see that this is Construction 3.3.

Case 3. $\nu \geq 4$.

Assume that $\mathcal{P}_0$ contains a full line l. Then by $\nu \geq 4$, there must be at least two points of $\mathcal{P}_0$ not on l, but then the lines joining these two points to the points of l are all deleted, thus $|\mathcal{L}_0| \geq 2q+2$, contradicting the upper bound of Theorem 3.9. Therefore $\mathcal{P}_0$ is a blocking set that does not contain a full line, hence by Theorem 3.14 and Theorem 3.9 it is a Baer-subplane, i.e., we have Construction 3.7.

□

For the proof of Theorem 3.11, we need one more lemma.

Lemma 3.16. *If $t = 2$ and $q \geq 5$, then $|\mathcal{P}_0| = |\mathcal{L}_0| \geq 2q+1$ with equality if and only if we have Construction 3.2.*

Proof. Let $p \in \mathcal{P}_1$. There are $q - 1$ lines from $\mathcal{L}_1$ through p all containing exactly 2 points of $\mathcal{P}_0$, hence $|\mathcal{P}_0| = |\mathcal{L}_0| = 2q - 2 + c$, where c denotes the number of deleted points on the two deleted lines through p. By Lemma 3.15, $\mathcal{P}_0$ is a blocking set, so we can deduce that $c \geq 2$. Hence $|\mathcal{P}_0| \geq 2q$ with equality if and only if the two deleted lines through p meet $\mathcal{P}_0$ in 1 point. One can repeat this counting from any $p \in \mathcal{P}_1$ to deduce that if $|\mathcal{P}_0| = 2q$, then all lines from $\mathcal{L}_0$ meet $\mathcal{P}_0$ in 1 or $q + 1$ points. It is easy to see that this cannot be true for a set of $2q$ points.

Finally, suppose that $|\mathcal{P}_0| = 2q + 1$. The above counting shows that through a point of $\mathcal{P}_1$, the two deleted lines meet $\mathcal{P}_0$ in 1 and 2 points, respectively. Hence lines of $\mathcal{L}_0$ are 1-,2-, or $(q + 1)$-secants to $\mathcal{P}_0$. Let $p \in \mathcal{P}_0$. There are $q + 1$ lines through p, so even if they all belong to $\mathcal{L}_0$, one of them has to have at least 2 more points of $\mathcal{P}_0$, so we can deduce that there is a line $l \in \mathcal{L}_0$ with all of its point in $\mathcal{P}_0$. Take any two points from $\mathcal{P}_0$ not on l. The line through them contains at least 3 points from $\mathcal{P}_0$, hence all of its points are in $\mathcal{P}_0$. Hence the deleted points are exactly the points of two lines. The dual of this argument implies that the deleted lines are the lines going through two points. It is easy to see that we have Construction 3.2. □

Recall that for $t = 2$, Constructions 3.2 and 3.3 are the same. Now we are ready to prove Theorem 3.11.

Proof of Theorem 3.11. By Lemma 3.16, a possible counterexample would have $|\mathcal{P}_0| \geq 2q+2$. But this implies that $\mathcal{P}_0$ is a double blocking set: if a line l had at most 1 point from $\mathcal{P}_0$, then, since through the non-deleted points of l there is exactly one more deleted line, we would have $|\mathcal{L}_0| \leq q+1+q$, a contradiction.

Using the result of Blokhuis, Szőnyi and Storme (Theorem 3.12) for $t = 2$, we deduce that $|\mathcal{P}_0| \geq 2(q+\sqrt{q}+1)$ with equality if and only if $\mathcal{P}_0$ is the union of two Baer subplanes, that is, we have Construction 3.7. Applying Theorem 3.9 completes the proof. □

Note that almost everything goes through for an arbitrary projective plane of order q. The only moment when we had to use that we are in $\mathrm{PG}(2, q)$ is (after deducing that $\mathcal{P}_0$ is a double blocking set) when we used the result of Blokhuis, Storme and Szőnyi.

We end this section by listing some results without proofs, and definitions which are only interesting from the finite geometry point of view.

The lower bound $|\mathcal{L}_0| \geq (q+1-t)t$ is sharp if and only if $t = \sqrt{q}$ and $\mathcal{P}_0$ consists of the points of a maximal $(k, \sqrt{q})$-arc. In this case $\mathcal{P}_0$ is not a blocking set. If $\mathcal{P}_0$ is a blocking set, then one can add t to the lower bound, hence $|\mathcal{P}_0| \geq (q+2-t)t$ for $t < \sqrt{q}$. One can prove that assuming $t < \sqrt{q}$, this is sharp only if $t = 1$. However, for $t = \sqrt{q}+1$, a unital and its tangents form a t-good pair with $|\mathcal{P}_0| = (q+2-t)t$ and in this example $\mathcal{P}_0$ is a blocking set.

Small t-good structures can be constructed using subplanes: delete the lines through the points of a subplane of order s and the points that are on the lines intersecting the subplane in $s+1$ points. This is an (s^2+s+1)-good structure of size $(s^2+s+1)q-(s-1)(s^2+s+1)$.

It is easy to prove that for $t < (q+1)/2$, if $\mathcal{P}_0$ and $\mathcal{L}_0$ consist of all points on t given lines and all lines on t given points, respectively, then $(\mathcal{P}_0, \mathcal{L}_0)$ is t-good if and only if the points and lines in question form a (possibly degenerate) subplane.

There are t-good structures of size larger than $tq+1$ when $t = (q+1)/2$, for example take the external points and the secants of an oval. Note that the graph constructed in this way is quite far from the Moore bound, since t is large. However, considering this in $\mathrm{PG}(2, q)$, where the oval is a conic arising from a polarity, one can identify the secants and the external points using the polarity. The graph obtained is regular of girth five exactly when $q \equiv 3 \pmod 4$. Replacing the external points with internal points and secants with skew lines, we get a similar example which works for $q \equiv 1 \pmod 4$. This construction is due to Jason Williford (see [18]).

4. The $g = 8$ case: constructions from a generalized quadrangle

In this section we first give the necessary definitions and recall some results about generalized quadrangles. It is straightforward to check that the following definition is equivalent to the one given in the introduction (for $g = 8$).

Definition 4.1. *Let $\mathcal{P}$ be a finite set and $\mathcal{L}$ a set of subsets of $\mathcal{P}$ called points and lines, respectively. The pair $(\mathcal{P}, \mathcal{L})$ is called a generalized quadrangle of order (s, t), if it satisfies the following axioms:*

- *there are $s + 1$ lines through every point;*
- *every line has $t + 1$ points;*
- *for any point p and line l not through p, there is a unique line through p intersecting l.*

Note that the role of points and lines in the definition of a generalized quadrangle is symmetric, hence interchanging the role of points and lines, one finds another generalized quadrangle (of order (t, s)). This generalized quadrangle is not necessarily isomorphic to the original one even if $s = t$ holds. Taking any definition or result, one can integerchange the words point and line to find a dual definition or result.

The point-line incidence graph of such a structure is a Moore graph (with $g = 8$ and $k = s + 1$) if and only if $s = t$. So from now on we suppose that $s = t$; in this case one usually says that *the generalized quadrangle has order s* and denote the structure by GQ(s).

For any subset of the points U, $U^{\perp}$ denotes the set of points collinear with all points of U, and $U^{\perp\perp}$ the set of points collinear with all points of $U^{\perp}$. One can similarly define $W^{\perp}$ and $W^{\perp\perp}$ for a set W of lines. Next we summarize some easy consequences of the definition.

Lemma 4.2. *Let $GQ(s)$ be a generalized quadrangle of order s. Then*

(i) *there are $(s + 1)(s^2 + 1)$ points (respectively lines);*
(ii) *for any two non-collinear points u and v, $\left|\{u, v\}^{\perp}\right| = s + 1$;*
(iii) *for any two non-collinear points u and v, $\left|\{u, v\}^{\perp\perp}\right| \leq s + 1$;*
(iv) *for any two skew lines l and m, $\left|\{l, m\}^{\perp}\right| = s + 1$;*
(v) *for any two skew lines l and m, $\left|\{l, m\}^{\perp\perp}\right| \leq s + 1$.*

Proof.

(i) Fix a point p. There are $s + 1$ lines through p, hence the number of collinear points to p is $1 + (s + 1)s$. By the third axiom of GQ-s, all lines not through p have a unique point collinear to p, hence the number of lines is $s + 1 + (s + 1)s^2 = (s + 1)(s^2 + 1)$. The number of points is the same by duality.
(ii) There are $s + 1$ lines through v, all of them have a unique point collinear to u.
(iii) Choose two different points $a, b \in \{u, v\}^{\perp}$. Then $\{u, v\}^{\perp\perp} \subseteq \{a, b\}^{\perp}$, hence (ii) implies (iii).
(iv),(v) These are the dual of (ii) and (iii).

□

A non-collinear point-pair u, v is called *regular* if $\left|\{u,v\}^{\perp\perp}\right| = s+1$ holds. One can similarly define a regular line-pair. Next we list some properties of regular pairs that will be needed for our constructions.

Lemma 4.3. *Suppose the point-pair (u_0, u_1) is regular and let $\{u_0, u_1\}^{\perp} = \{v_0, \ldots, v_s\}$, $\{u_0, u_1\}^{\perp\perp} = \{u_0, \ldots, u_s\}$. Denote by L' the set of lines joining a point u_i to a point v_j.*

(i) *Any u_i is collinear to any v_j, but no different u_i and u_j or v_i and v_j can be collinear.*
(ii) *L' contains $(s+1)^2$ lines;*
(iii) *for any u_i, u_j $(i \neq j)$, $\{u_i, u_j\}^{\perp} = \{v_0, \ldots, v_s\}$, and for any v_i, v_j $(i \neq j)$, $\{v_i, v_j\}^{\perp} = \{u_0, \ldots, u_s\}$;*
(iv) *all lines through an u_i or v_i are in L';*
(v) *through any point not in $\{u_0, \ldots, u_s\} \cup \{v_0, \ldots, v_s\}$, there is a unique line in L'.*

Proof.

(i),(ii) Any u_i is collinear to any v_j by definition of the orthogonal of a set. If an u_i and an u_j were collinear, then the line joining them and any v_k would contradict the third axiom of GQ-s.
(iii) This follows from (i) and Lemma 4.2 (ii).
(iv) Note that there are $s+1$ lines through a point, and we see $s+1$ lines through any u_i or v_i in L'.
(v) First suppose that there are at least two lines in L' through a point $p \notin \{u_0, \ldots u_s\} \cup \{v_0, \ldots v_s\}$. Without loss of generality suppose that p is collinear to u_i and u_j. Then $\{u_i, u_j\}^{\perp}$ contains at least $s+2$ points, contradicting Lemma 4.2 (ii). Hence the number of points on the lines of L' is $2(s+1) + (s+1)^2(s-1) = (s+1)(s^2+1)$, this is the number of points of GQ(s), hence every point is on a line of L'.

□

Construction 4.4. *Suppose the $GQ(s)$ has a regular point-pair (u, v). Fix a point $p \notin \{u,v\}^{\perp} \cup \{u,v\}^{\perp\perp}$. Let $\mathcal{P}_0 = \{u,v\}^{\perp} \cup \{u,v\}^{\perp\perp} \cup p^{\perp}$. Let $\mathcal{L}_0$ consist of lines joining a point of $\{u,v\}^{\perp}$ to a point of $\{u,v\}^{\perp\perp}$ together with lines through p. Then $(\mathcal{P}_0, \mathcal{L}_0)$ is 1-good with $|\mathcal{P}_0| = |\mathcal{L}_0| = s^2+3s+1$.*

Proof. By Lemma 4.3, through a point $q \in \mathcal{P}_1$ there is exactly one line joining a point of $\{u,v\}^{\perp}$ to a point of $\{u,v\}^{\perp\perp}$ (and no lines through p, since points collinear to p were deleted). For a line $l \in \mathcal{L}_1$, there is a unique line through p meeting l by the definition of generalized quadrangles. For the size, note that by Lemma 4.3, there is a unique line through p joining a point of $\{u,v\}^{\perp}$ to a point of $\{u,v\}^{\perp\perp}$. Hence $|\mathcal{L}_0| = (s+1)^2 + s + 1 - 1 = s^2 + 3s + 1$. □

Construction 4.5. *Suppose the $GQ(s)$ has a regular point-pair (u, v) and a regular line pair (l, m). Suppose also that there are no points from $\{u,v\}^{\perp} \cup$*

$\{u,v\}^{\perp\perp}$ on the lines of either $\{l,m\}^{\perp}$ or $\{l,m\}^{\perp\perp}$. Let $\mathcal{P}_0$ consist of the points from $\{u,v\}^{\perp} \cup \{u,v\}^{\perp\perp}$ together with points from lines in $\{l,m\}^{\perp} \cup \{l,m\}^{\perp\perp}$. Dually, let $\mathcal{L}_0$ consist of the lines from $\{l,m\}^{\perp} \cup \{l,m\}^{\perp\perp}$ together with lines through points of $\{u,v\}^{\perp} \cup \{u,v\}^{\perp\perp}$. Then $(\mathcal{P}_0, \mathcal{L}_0)$ is 1-good with $|\mathcal{P}_0| = |\mathcal{L}_0| = s^2 + 4s + 3$.

Proof. Let $p \in \mathcal{P}_1$. By Lemma 4.3, there is a unique line joining a point of $\{u,v\}^{\perp}$ to a point of $\{u,v\}^{\perp\perp}$, and since p is not in $\mathcal{P}_0$, there is no line in $\{l,m\}^{\perp} \cup \{l,m\}^{\perp\perp}$ through p. Hence there are s lines in $\mathcal{L}_1$ through p. The dual of this argument (using the dual of Lemma 4.3) implies that on any line in $\mathcal{L}_1$ there are exactly s points. The calculation of the size is easy. □

Looking through the literature of generalized quadrangles, it turns out that examples with both regular point- and line-pairs only exist for q even. Here we show an example where our constructions work.

Definition 4.6. *The symplectic generalized quadrangle of order q denoted by $W(q)$ is the following: as point-set, we take all points of the 3-dimensional projective geometry $PG(3,q)$. The lines are the totally isotropic lines with respect to a symplectic polarity of $PG(3,q)$.*

W(q) is a generalized quadrangle of order q. For the proof of this last statement and further properties of W(q), we refer to [4].

Theorem 4.7. *In $W(q)$, Construction 4.4 always works. Construction 4.5 works if and only if q is even.*

Proof. By [17], all point-pairs are regular of W(q), and the sets $\{u,v\}^{\perp}$ and $\{u,v\}^{\perp\perp}$ consist of points of a non-symplectic line l and points of $l^{\perp}$, respectively. There is at least one regular line-pair if and only if all line pairs are regular if and only if q is even.

If q is even, then for two skew lines l and m of W(q), the sets $\{l,m\}^{\perp}$ and $\{l,m\}^{\perp\perp}$ are the two opposite reguli on a hyperbolic quadric. Hence after choosing l and m for Construction 4.5, all we have to do is choose u and v to be two points determining a non-isotropic line disjoint from the hyperbolic quadric in question. □

5. Order of (k,g)-cages

In this section we summarize the consequences of our constructions. All improvements depend on how close a prime power is to k.

Theorem 5.1. *Denote by q the smallest prime power greater or equal to $k-1$. If q is a square, then*

$$c(k,6) \leq 2\left(kq - (q-k)(\sqrt{q}+1) - \sqrt{q}\right).$$

Proof. We need to delete t Baer subplanes from PG$(2,q)$ using Construction 3.7 (see also Theorem 3.8) with $t = q+1-k$. Hence the number of points of the incidence graph of the resulting structure is

$$2\left((q^2+q+1) - (q+1-k)(q+\sqrt{q}+1)\right).$$

A little calculation shows that this equals the formula stated. □

If the smallest prime power $q \geq k-1$ is not a square, then one can use (the previously known) Construction 3.3 to find an upper bound on $c(k, 6)$.

Note that it is very rare that the smallest prime power $q \geq k-1$ is a square. If q is not a square, then even if $q+1$ is a square prime power, and Constructions 3.3 and 3.2 starting from a plane of order q are better than Construction 3.7 starting from a plane of order $q+1$.

By Theorem 3.9, one cannot hope for a better bound on $c(k, 6)$ using the same construction method. However, there is one example known when $c(k, 6)$ is smaller than the one coming from Theorem 5.1: there is a construction due to Baker [3] (see also [18]) for a $(7, 6)$ graph (which is a regular graph of valency 7 and girth 6) with 90 vertices. Our method would start with a plane of order 7, and even if there was a Baer subplane of order $\sqrt{7}$, Construction 3.7 would give a graph on $2\left((7^2+7+1)-(7+\sqrt{7}+1)\right) \approx 92.7$ vertices.

Theorem 5.2. *Suppose that k is a prime power. If k is even, then $c(k, 8) \leq 2(k^3-3k-2)$. If k is odd, then $c(k, 8) \leq 2(k^3-2k)$.*

Proof. One should start with $W(k)$ and use Construction 4.4 or 4.5 according to whether k is odd or even (see also Theorem 4.7). Hence the number of points of the incidence graph of the resulting structure is $2(k^3+k^2+k+1) - 2\,|\mathcal{P}_0|$. □

Finally, our slight improvement for the $g = 12$ case is the following.

Theorem 5.3. *Suppose k is a prime power. Then $c(k, 12) \leq 2(k^5-k^3)$.*

Proof. One should start with a generalized hexagon of order k and use Construction 2.3. □

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