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A VARIANT OF THE BIPARTITE RELATION THEOREM AND ITS APPLICATION TO CLIQUE GRAPHS

KAZUHIRO KAWAMURA

Abstract. We consider a homological variant of the Bipartite Relation Theorem [1] in the context of the flag complex of the square of a bipartite graph [9]. We apply the results to study the homology and homotopy groups of the flag complexes of clique graphs.

1. Introduction and Preliminaries

All graphs are assumed to be simple and finite. For a graph G, $K(G)$ denotes the clique graph of G: the vertices are the maximal complete subgraphs (called *cliques*) with the edges being the pairs of intersecting cliques. The flag complex of a graph G is denoted by $\Delta(G)$. Larrión, Pizaña and Villarroel-Flores [9] proved that the fundamental group of $\Delta(G)$ is isomorphic to the one of $\Delta(K(G))$ for each graph G. This is a consequence of their general theorem [9, Theorem 3.1] on a simplicial complex associated with a bipartite graph. For a connected bipartite graph $B = (X, Y)$ with the partite sets X and Y, they introduced a graph, called the *square* B^2 of B with its induced subgraphs $B^2[X]$ and $B^2[Y]$, and prove the isomorphisms of the fundamental groups:

$$\pi_1\big(\Delta(B^2[X])\big) \cong \pi_1\big(\Delta(B^2)\big) \cong \pi_1\big(\Delta(B^2[Y])\big).$$

The complexes $\Delta(B^2[X])$ and $\Delta(B^2[Y])$ contain Dowker-Mather complexes DM_X and DM_Y which are known to be homotopy equivalent (the Bipartite Relation Theorem [1], [5] and [10]).

In what follows, $\Delta(B^2[X])$ and $\Delta(B^2[Y])$ are denoted by B_X and B_Y for simplicity. Under this notation, the above situation is summarized in the following diagram.

$$\begin{tikzcd}
B_X \arrow[r, "i_X"] & \Delta(B^2) & B_Y \arrow[l, "i_Y"'] \\
\mathrm{DM}_X \arrow[u, "j_X"] & & \mathrm{DM}_Y \arrow[u, "j_Y"]
\end{tikzcd}$$

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where the inclusions i_X and i_Y induce isomorphisms on fundamental groups and $\mathrm{DM}_X \simeq \mathrm{DM}_Y$.

The present paper takes a close look at the above diagram. We introduce a subcomplex $\mathrm{DM}_{X,Y}$ of $\Delta(B^2)$ which collapses onto DM_X and DM_Y respectively which "closes" the above diagram (see Figure 1), where the inclusions k_X and k_Y are homotopy equivalences.

$$\begin{tikzcd}
B_X \arrow[r, "i_X"] & \Delta(B^2) & B_Y \arrow[l, "i_Y"'] \\
\mathrm{DM}_X \arrow[u, "j_X"] \arrow[r, "k_X"] & \mathrm{DM}_{X,Y} \arrow[u, "j"] & \mathrm{DM}_Y \arrow[u, "j_Y"'] \arrow[l, "k_Y"']
\end{tikzcd}$$

Figure 1.

We prove that i_X and i_Y are homological n-equivalences (see Section 3 for the definition) if and only if so are j_X and j_Y, if and only if so is j (Theorem 3.3).

As in [9], we apply the above result to the flag complexes $\Delta(G)$ and $\Delta(K(G))$ to obtain information on their higher homology and homotopy groups. When a graph G has the "$(n+1)$-bounded clique-Helly property" (see Section 4), the homology and homotopy groups of $\Delta(G)$ and $\Delta(K(G))$ are isomorphic up to dimension $(n-1)$.

In the rest of this section, we make notational convention and state some preliminary results. For a graph G, the vertex set and the edge set are denoted by $V(G)$ and $E(G)$ respectively. For a vertex v of G, $N_G(v)$ denotes the set of all neighbors of v in G. For a subset A of $V(G)$, let

$$CN_G(A) = \bigcap_{a \in A} N_G(a),$$

the set of the *common neighbors* of A. For $S \subset V(G)$, the subgraph induced by S is denoted by $G[S]$.

For a simplicial complex K, $K^{(i)}$ denotes the i-skeleton of K, the set of all simplices of dimension less than or equal to i. For simplicity, a geometric realization of a simplicial complex K is also denoted by the same symbol K, which will cause no confusion in the present paper.

For a graph G, a simplicial complex $\Delta(G)$, called the *flag complex* of G, is defined as follows: the set of vertices of $\Delta(G)$ is the vertex set $V(G)$. A subset σ of $V(G)$ spans a simplex of $\Delta(G)$ if and only if σ induces a complete subgraph of G.

Let $B = B(X, Y)$ be a connected bipartite graph with the partite sets X and Y. The *square* $B^2 = B^2(X, Y)$ of B is a graph with the vertex set $X \amalg Y$ having the edge set

$$\begin{aligned} E(B^2) = E(B) &\cup \{x_1x_2 \mid x_1, x_2 \in X, N_B(x_1) \cap N_B(x_2) \neq \varnothing\} \\ &\cup \{y_1y_2 \mid y_1, y_2 \in Y, N_B(y_1) \cap N_B(y_2) \neq \varnothing\}. \end{aligned}$$

Convention: A simplex of the flag complex $\Delta(B^2)$ with the vertices $U \cup V$ with $U \subset X$ and $V \subset Y$ is denoted by $U \oplus V$.

Notice that $\varnothing \oplus V \in \Delta(B^2)$ if and only if for each $y_1, y_2 \in V$,

$$N_B(y_1) \cap N_B(y_2) \neq \varnothing.$$

The same remark applies to a simplex of the form $U \oplus \varnothing$. Under this notation, the subcomplexes B_X and B_Y are written as $B_X = \{U \oplus \varnothing \mid U \oplus \varnothing \in \Delta(B^2)\}$ and $B_Y = \{\varnothing \oplus V \mid \varnothing \oplus V \in \Delta(B^2)\}$.

Now we define a subcomplex $\mathrm{DM}_{X,Y}$ of $\Delta(B^2)$ as follows:

$$\begin{aligned}\mathrm{DM}_{X,Y} = {} & \{U \oplus V \in \Delta(B^2) \mid U \neq \varnothing \neq V\} \\ & \cup \{U \oplus \varnothing \in \Delta(B^2) \mid CN_B(U) \neq \varnothing\} \\ & \cup \{\varnothing \oplus V \in \Delta(B^2) \mid CN_B(V) \neq \varnothing\}.\end{aligned}$$

The inclusion of $\mathrm{DM}_{X,Y}$ into $\Delta(B^2)$ is denoted by $j : \mathrm{DM}_{X,Y} \to \Delta(B^2)$.

The Dowker-Mather complexes DM_X and DM_Y are subcomplexes of $\mathrm{DM}_{X,Y}$ defined by

$$\begin{aligned}\mathrm{DM}_X &= \{U \oplus \varnothing \in \mathrm{DM}_{X,Y} \mid CN_B(U) \neq \varnothing\} \text{ and} \\ \mathrm{DM}_Y &= \{\varnothing \oplus V \in \mathrm{DM}_{X,Y} \mid CN_B(V) \neq \varnothing\}.\end{aligned}$$

Let $k_X : \mathrm{DM}_X \to \mathrm{DM}_{X,Y}$ and $k_Y : \mathrm{DM}_Y \to \mathrm{DM}_{X,Y}$ be the inclusions. These form the commutative diagram in Figure 1.

All homology groups under consideration are singular (or simplicial) homology groups with integer coefficients.

2. Some auxiliary results

Let us first make an observation which follows immediately from the definitions.

Observation 2.1. *Let $B = B(X, Y)$ be a connected bipartite graph with the partite sets X and Y. We have the following equalities.*

1. $\Delta(B^2) = B_X \cup \mathrm{DM}_{X,Y} \cup B_Y$.
2. $B_X \cap \mathrm{DM}_{X,Y} = \mathrm{DM}_X$ *and* $B_Y \cap \mathrm{DM}_{X,Y} = \mathrm{DM}_Y$.
3. $B_X^{(1)} = \mathrm{DM}_X^{(1)}$ *and* $B_Y^{(1)} = \mathrm{DM}_Y^{(1)}$.

The proof of [9, Theorem 3.1] shows the following.

Theorem 2.2. *Let B be a connected bipartite graph with the partite sets X and Y. There exist retractions*

$$r_X : \Delta(B^2)^{(2)} \to B_X \text{ and } r_Y : \Delta(B^2)^{(2)} \to B_Y$$

which induce isomorphisms on fundamental groups.

Thus we see that $(i_X)_\sharp : \pi_1(B_X) \to \pi_1(\Delta(B^2))$ and $(i_Y)_\sharp : \pi_1(B_Y) \to \pi_1(\Delta(B^2))$ are isomorphisms, the isomorphisms mentioned at the beginning of Section 1.

As is mentioned in the previous section, DM_X and DM_Y are homotopy equivalent. For a proof, see for example, [1, Theorem 10.9]. In what follows, we give another argument by constructing collapses of $\mathrm{DM}_{X,Y}$ onto DM_X and DM_Y respectively. Our proof relies on Discrete Morse Theory [6] and imitates an argument due to Csorba [2, Theorem 8], in which the roles of $\mathrm{DM}_{X,Y}$, DM_X, and DM_Y are played by the box complex and the neighborhood complex of a graph.

Definition 2.3. *Let K be a simplicial complex and let $P = \mathcal{F}(K)$ be the face poset of K.*

1. *The symbol $\succ$ means the covering relation on P. That is, for simplices σ and τ of K, $\tau \succ \sigma$ means that σ is a face of τ such that* $\dim \tau = \dim \sigma + 1$.
2. *A partial matching on P is a pair (Σ, μ) where $\mu : \Sigma \to P \setminus \Sigma$ is an injection such that $\mu(\sigma) \succ \sigma$ for each $\sigma \in \Sigma$. The simplices of $P \setminus (\Sigma \cup \mu(\Sigma))$ are said to be critical.*
3. *A partial matching (Σ, μ) is said to be acyclic if there exists no sequence of distinct elements $\sigma_1, \ldots, \sigma_t$ of Σ such that*

$$\mu(\sigma_1) \succ \sigma_2, \quad \mu(\sigma_2) \succ \sigma_3, \quad \ldots, \quad \mu(\sigma_t) \succ \sigma_1.$$

For the proof of the following theorem, see, for example, [7, Proposition 6.4] or [8, Theorem 11.3].

Theorem 2.4. *Let K be a simplicial complex and let (Σ, μ) be an acyclic matching on the face poset $P = \mathcal{F}(K)$. If the set L of all critical simplices of the matching forms a subcomplex of K, then K collapses onto L.*

Now we are ready to prove the following result.

Theorem 2.5. *Let B be a connected bipartite graph with the partite sets X and Y. The simplicial complex $\mathrm{DM}_{X,Y}$ collapses onto DM_X and DM_Y respectively. Thus the inclusions*

$$k_X : \mathrm{DM}_X \to \mathrm{DM}_{X,Y} \text{ and } k_X : \mathrm{DM}_X \to \mathrm{DM}_{X,Y}$$

are homotopy equivalences.

Proof. We follow the proof of [2, Theorem 8] to show that $\mathrm{DM}_{X,Y}$ collapses onto DM_X. At the outset, we fix a linear order $<$ on X. For a simplex $\sigma = U \oplus V$ of $\mathrm{DM}_{X,Y}$ with $V \neq \varnothing$, let $x_\sigma = \min CN_B(V)$, where the minimum is taken with respect to the above order $<$. Also let

$$\Sigma = \{\sigma = U \oplus V \mid V \neq \varnothing,\ x_\sigma \notin U\} \subset \mathrm{DM}_{X,Y}$$

and, for $\sigma = U \oplus V \in \Sigma$, define a simplex $\mu(\sigma)$ of $\mathrm{DM}_{X,Y}$ by

$$\mu(\sigma) = \big(U \cup \{x_\sigma\}\big) \oplus V = \sigma \cup \big(\{x_\sigma\} \oplus \varnothing\big).$$

This gives an injection $\mu : \Sigma \to \mathrm{DM}_{X,Y}$ such that

$$\mu(\Sigma) = \{\sigma = U \oplus V \mid V \neq \varnothing, x_\sigma \in U\}$$

and

$$\mathrm{DM}_{X,Y} \setminus (\Sigma \cup \mu(\Sigma)) = \{U \oplus \varnothing \mid CN_B(U) \neq \varnothing\} = \mathrm{DM}_X.$$

In order to prove that (Σ, μ) is an acyclic partial matching, it suffices to verify the following claim.

Claim. *Let $\sigma = U \oplus V$ and $\tau = U' \oplus V'$ be distinct elements of Σ with $\mu(\sigma) \succ \tau$. Then we have $V \supsetneqq V'$.*

> *Proof.* By the assumption, we have $x_\sigma \notin U$, $U \cup \{x_\sigma\} \supset U'$ and $V \supset V'$. Suppose that $V = V'$. Then $x_\sigma = x_\tau \notin U'$ and this implies $U \supset U'$. Hence $\mu(\sigma) \supset \sigma \supset \tau$ and the covering relation $\mu(\sigma) \succ \tau$ implies $\sigma = \tau$. This contradicts the hypothesis $\sigma \neq \tau$ and completes the proof. □

This completes the proof of theorem. □

Passing to the homology groups, we obtain, from the diagram Figure 1, the diagram shown in Figure 2, where $(k_X)_*$ and $(k_Y)_*$ are isomorphisms for each $q \geq 0$ and $(i_X)_*$ and $(i_Y)_*$ are isomorphisms for $q \geq 1$ by Theorem 2.2 and the Hurewicz Theorem [13].

$$\begin{tikzcd}
\mathrm{H}_q(B_X) \arrow[r, "(i_X)_*"] & \mathrm{H}_q(\Delta(B^2)) & \mathrm{H}_q(B_Y) \arrow[l, "(i_Y)_*"'] \\
\mathrm{H}_q(\mathrm{DM}_X) \arrow[u, "(j_X)_*"] \arrow[r, "(k_X)_*"] & \mathrm{H}_q(\mathrm{DM}_{X,Y}) \arrow[u, "j_*"] & \mathrm{H}_q(\mathrm{DM}_Y) \arrow[u, "(j_Y)_*"'] \arrow[l, "(k_Y)_*"']
\end{tikzcd}$$

FIGURE 2.

In the next section, we consider the situation in which $(i_X)_*$ and $(i_Y)_*$ are isomorphisms up to a certain dimension n (≥ 2) and study the connection to the homomorphisms j_*, $(j_X)_*$ and $(j_Y)_*$.

3. ON n-EQUIVALENCES OF INCLUSIONS

Definition 3.1. *Let $f : S \to T$ be a continuous map of connected topological spaces.*

1. *The map f is called an n-equivalence if the induced homomorphism $f_\sharp : \pi_q(S) \to \pi_q(T)$ is an isomorphism for each $q < n$ and an epimorphism for $q = n$.*
2. *The map f is called a homological n-equivalence if the induced homomorphism $f_* : \mathrm{H}_q(S) \to \mathrm{H}_q(T)$ is an isomorphism for each $q < n$ and an epimorphism for $q = n$.*

The Whitehead Theorem [13] implies that every n-equivalence is a homological n-equivalence. When $f : S \hookrightarrow T$ is an inclusion map, f is an n-equivalence (resp. a homological n-equivalence) if and only if the relative homotopy group $\pi_q(T, S)$ (resp. the relative homology group $\mathrm{H}_q(T, S)$) is trivial for each $q \leq n$.

Proposition 3.2. *We have the following isomorphisms for each* $q \geq 1$*:*

$$\mathrm{H}_q(\Delta(B^2), B_X) \cong \mathrm{H}_q(B_Y, \mathrm{DM}_Y) \text{ and } \mathrm{H}_q(\Delta(B^2), B_Y) \cong \mathrm{H}_q(B_X, \mathrm{DM}_X).$$

Proof. By Observation 2.1 and the Excision Isomorphism Theorem [13], we have

$$\begin{aligned}\mathrm{H}_q(\Delta(B^2), B_X) &= \mathrm{H}_q(B_X \cup \mathrm{DM}_{X,Y} \cup B_Y, B_X)\\ &\cong \mathrm{H}_q(\mathrm{DM}_{X,Y} \cup B_Y, (\mathrm{DM}_{X,Y} \cup B_Y) \cap B_X)\\ &= \mathrm{H}_q(\mathrm{DM}_{X,Y} \cup B_Y, \mathrm{DM}_X). \qquad (3.1)\end{aligned}$$

By Theorem 2.5, there exist retractions $\rho_X : \mathrm{DM}_{X,Y} \to \mathrm{DM}_X$ and $\rho_Y : \mathrm{DM}_{X,Y} \to \mathrm{DM}_Y$ which are both homotopy equivalences. Let $r_Y : \mathrm{DM}_{X,Y} \cup B_Y \to B_Y$ be the map defined by $r_Y|\mathrm{DM}_{X,Y} = \rho_Y$ and $r_Y|B_Y = \mathrm{id}_{B_Y}$. Notice that r_Y and its restriction $r_Y|\mathrm{DM}_X = \rho_Y \circ k_X : \mathrm{DM}_X \to \mathrm{DM}_Y$ are both homotopy equivalences (recall Theorem 2.5).

Under the above notation, we consider the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}_q(\mathrm{DM}_X) \arrow[r] \arrow[dd, "(r_Y|\mathrm{DM}_X)_*"'] & \mathrm{H}_q(\mathrm{DM}_{X,Y} \cup B_Y) \arrow[r] \arrow[dd, "(r_Y)_*"'] & \cdots \\
 & & & (*) \\
\cdots \arrow[r] & \mathrm{H}_q(\mathrm{DM}_Y) \arrow[r] & \mathrm{H}_q(B_Y) \arrow[r] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}_q(\mathrm{DM}_{X,Y} \cup B_Y, \mathrm{DM}_X) \arrow[r, "\partial"] \arrow[dd, "(r_Y)_*"'] & \cdots \\
(*) & & (**) \\
\cdots \arrow[r] & \mathrm{H}_q(B_Y, \mathrm{DM}_Y) \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & \mathrm{H}_{q-1}(\mathrm{DM}_X) \arrow[r] \arrow[dd, "(r_Y|\mathrm{DM}_X)_*"'] & \mathrm{H}_{q-1}(\mathrm{DM}_{X,Y} \cup B_Y) \arrow[r] \arrow[dd, "(r_Y)_*"'] & \cdots \\
(**) & & & \\
\cdots \arrow[r, "\partial"] & \mathrm{H}_{q-1}(\mathrm{DM}_Y) \arrow[r] & \mathrm{H}_{q-1}(B_Y) \arrow[r] & \cdots
\end{tikzcd}$$

where the horizontal sequences are the homology long exact sequences of the pairs $(\mathrm{DM}_{X,Y} \cup B_Y, \mathrm{DM}_X)$ and (B_Y, DM_Y) whose connecting homomorphisms are denoted by ∂.

By the remark preceding to the above diagram, we see that all four homomorphisms $(r_Y|\mathrm{DM}_X)_*$ and $(r_Y)_*$, except for

$$(r_Y)_* : \mathrm{H}_q(\mathrm{DM}_{X,Y} \cup B_Y, \mathrm{DM}_X) \to \mathrm{H}_q(B_Y, \mathrm{DM}_Y),$$

are isomorphisms. By the Five Lemma [13] we see that the above homomorphism is an isomorphism as well. Combining this with (3.1), we obtain the first isomorphism. The second isomorphism is proved in exactly the same way. This completes the proof. □

Theorem 3.3. *Let $n \geq 2$ be an integer. The following three conditions are equivalent.*

1. *The inclusions $i_X : B_X \to \Delta(B^2)$ and $i_Y : B_Y \to \Delta(B^2)$ are both homological n-equivalences.*
2. *The inclusions $j_X : \mathrm{DM}_X \to B_X$ and $j_Y : \mathrm{DM}_Y \to B_Y$ are both homological n-equivalences.*
3. *The inclusion $j : \mathrm{DM}_{X,Y} \to \Delta(B^2)$ is a homological n-equivalence.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows immediately from Proposition 3.2 and the remark after Definition 3.1. From the equality $(i_X)_* \circ (j_X)_* = j_* \circ (k_X)_*$ (see the Figure 2) and the fact that $(k_X)_*$ is an isomorphism, the implication (1) ($\Leftrightarrow$ (2)) $\Rightarrow$ (3) follows easily.

It remains to prove (3) $\Rightarrow$ (1) ($\Leftrightarrow$ (2)). Assume that $j_* : \mathrm{H}_q(\mathrm{DM}_{X,Y}) \to \mathrm{H}_q(\Delta(B^2))$ is an isomorphism for each $q < n$ and an epimorphism for $q = n$. We show that $(i_X)_* : \mathrm{H}_q(\mathrm{DM}_X) \to \mathrm{H}_q(\Delta(B^2))$ is an isomorphism for each $q < n$ and an epimorphism for $q = n$.

Notice $(i_X)_* \circ (j_X)_* = j_* \circ (k_X)_*$ and $(i_Y)_* \circ (j_Y)_* = j_* \circ (k_Y)_*$. Since $(k_X)_*$ and $(k_Y)_*$ are isomorphisms, we see

1. $(i_X)_*$ and $(i_Y)_*$ are epimorphisms for each $q \leq n$, and
2. $(j_X)_*$ and $(j_Y)_*$ are monomorphisms for each $q < n$.

So it suffices to verify that $(i_X)_* : \mathrm{H}_q(\mathrm{DM}_X) \to \mathrm{H}_q(\Delta(B^2))$ and $(i_Y)_* : \mathrm{H}_q(\mathrm{DM}_Y) \to \mathrm{H}_q(\Delta(B^2))$ are isomorphisms for each $q < n$.

The proof is by induction on q. Theorem 2.2 guarantees the validity of the first step $q = 1$. Let $2 \leq q < n$ and assume that, for each $r < q$, $(i_X)_*$ and $(i_Y)_*$ are isomorphisms. Then we have $\mathrm{H}_r(\Delta(B^2), B_X) = \mathrm{H}_r(\Delta(B^2), B_Y) = 0$ for each $r \leq q$ (we use (i) for $r = q$). By Proposition 3.2, we obtain

$$\mathrm{H}_r(B_X, \mathrm{DM}_X) = \mathrm{H}_r(B_Y, \mathrm{DM}_Y) = 0$$

for each $r \leq q$. In particular, $(j_X)_*$ and $(j_Y)_*$ are epimorphisms in dimension q. Combining this with (ii) above, we see that $(j_X)_*$ and $(j_Y)_*$ are isomorphisms in dimension q. The equalities $(i_X)_* \circ (j_X)_* = j_* \circ (k_X)_*$ and $(i_Y)_* \circ (j_Y)_* = j_* \circ (k_Y)_*$ imply that $(i_X)_*$ and $(i_Y)_*$ are isomorphisms in dimension q. This completes the inductive step and completes the proof of the theorem. □

As in Observation 2.1(3), B_X and DM_X (resp. B_Y and DM_Y) have the same 1-skeletons. Hence the induced homomorphism $(j_X)_\sharp : \pi_1(\mathrm{DM}_X) \to \pi_1(B_X)$ and $(j_Y)_\sharp : \pi_1(\mathrm{DM}_Y) \to \pi_1(B_Y)$ are epimorphisms. So the simple connectivity of DM_X (resp. DM_Y)) implies that of B_X (resp. B_Y). This observation leads to the following corollary.

Corollary 3.4. *Let $B = B(X, Y)$ be a connected bipartite graph with the partite sets X and Y. Assume that $\mathrm{DM}_X(\simeq \mathrm{DM}_Y)$ is simply connected.*

1. *If j is an n-equivalence, then so are i_X, i_Y, j_X and j_Y.*
2. *If j_X and j_Y are n-equivalences, then so are j, i_X and i_Y.*

Proof. (1) By the assumption, j is a homological n-equivalence. Hence by Theorem 3.3, we see that i_X, i_Y, j_X and j_Y are all homological n-equivalences. Since the complexes DM_X and DM_Y are simply connected, so are B_X and B_Y. Thus we obtain the desired conclusion via the Whitehead Theorem [13].

(2) If j_X and j_Y are n-equivalences, then they are homological n-equivalences. By Theorem 3.3, j, i_X and i_Y are all homological n-equivalences. The simple connectivity of DM_X and DM_Y implies that of B_X, B_Y and $\mathrm{DM}_{X,Y}(\simeq \mathrm{DM}_X \simeq \mathrm{DM}_Y)$, hence the Whitehead Theorem again finishes the proof. □

4. An application to clique graphs

As in [9], we apply Theorem 3.3 and Corollary 3.4 to the clique graph $K(G)$ of a graph G. First let us recall the definition.

Definition 4.1. *Let G be a graph.*

1. *A clique of G is a maximal complete subgraph of G.*
2. *Let $K(G)$ be the clique graph defined as follows: the vertex set is the set of all cliques of G; two cliques C_1 and C_2 are adjacent in $K(G)$ if and only if C_1 and C_2 have a vertex in common.*
3. *Let $BK(G)$ be the vertex-clique bipartite graph of G defined as follows: the partite sets are $V(G)$ and $V(K(G))$ and the edge set is defined by*

$$E(BK(G)) = \{vQ \mid v \in V(G), Q \in K(G), \text{and } v \in Q\}.$$

As was mentioned in Section 1, Larrión, Pizaña and Villarroel-Flores proved in [9] that $\pi_1(\Delta(G)) \cong \pi_1(\Delta(K(G)))$. Also they pointed out that, when a graph G has the clique-Helly property, $\Delta(G)$ is homotopy equivalent to $\Delta(K(G))$. When G does not have the clique-Helly property, but has a "Helly-like" property with respect to cliques, we may apply Theorem 3.3 and Corollary 3.4 to obtain more information on higher homology and homotopy groups.

Definition 4.2. *Let $n \geq 3$ be an integer. A graph G is said to have the n-bounded clique-Helly Property if the collection $\mathcal{C}$ of cliques of G satisfies the following condition: if any two distinct elements of $\mathcal{C}$ have non-empty intersection, then each subcollection $\mathcal{C}'$ of $\mathcal{C}$ with $|\mathcal{C}'| \leq n$ has the total intersection*

$$\bigcap_{C' \in \mathcal{C}'} C' \neq \varnothing.$$

When $n = \infty$, then the above coincides with the standard clique-Helly property. The notion above was first introduced by R. S. Roberts and J. H. Spencer [12] (see also [4]).

Theorem 4.3. *Let $n \geq 2$ be an integer and let G be a graph with the $(n+1)$-bounded clique-Helly property. Then*

(1) *We have an isomorphism $\mathrm{H}_q(\Delta(G)) \cong \mathrm{H}_q(\Delta(K(G)))$ for each $q \leq n-1$*

(2) *If $\Delta(G)$ is simply connected, then $\pi_q(\Delta(G)) \cong \pi_q(\Delta(K(G)))$ for each $q \leq n-1$*

Proof. We apply Theorem 3.3 to the vertex-clique bipartite graph $BK(G)$ with the partite set $X = V(G)$ and $Y = V(K(G))$. As is pointed out in [9, p. 293], we obtain the equality

$$\mathrm{DM}_X = B_X. \tag{4.1}$$

We show that

$$B_Y^{(n)} \subset \mathrm{DM}_Y. \tag{4.2}$$

Take an n-simplex $\sigma = \{Q_0, Q_1, \ldots, Q_n\}$ of B_Y. Then $Q_i \cap Q_j \neq \varnothing$ for distinct i and j. By the $(n+1)$-bounded clique-Helly property of G, we see that the total intersection $\cap_{i=0}^{n} Q_i$ contains a vertex v. This means that $v \in CN_{BK(G)}(\sigma) \neq \varnothing$ and hence $\sigma \in \mathrm{DM}_Y$. This shows the above inclusion.

By (4.1) and (4.2), we see that the inclusion $j_X : \mathrm{DM}_X \to B_X$ and $j_Y : \mathrm{DM}_Y \to B_Y$ are n-equivalences and hence are homological n-equivalences. Also notice that

$$B_X = \Delta(G) \text{ and } B_Y = \Delta(K(G)).$$

Theorem 3.3 tells us that i_X and i_Y are homological n-equivalences and in particular $\mathrm{H}_q(\Delta(G)) \cong \mathrm{H}_q(\Delta(BK(G)^2)) \cong \mathrm{H}_q(\Delta(K(G)))$ for each $q \leq n-1$. This proves (1). When $\Delta(G) = B_X = \mathrm{DM}_X$ is simply connected, then Corollary 3.4 is applied to prove (2). This completes the proof of the theorem. □

A *complete edge cover* $\mathcal{F} = \{G_i \mid i \in I\}$ of a graph G is a family of complete subgraphs of G such that any vertex and any edge of G lie in some G_i. For such a family, we may consider its intersection graph $\Omega(\mathcal{F})$. As in [9, Section 5], we may define the bipartite graph $B(\mathcal{F})$ with the partite sets $V(G)$ and $V(\Omega(G))$ whose edge set $E(B(\mathcal{F}))$ is defined by $\{vG_i \mid v \in V(G), G_i \in \mathcal{F}, \text{and } v \in G_i\}$. As was in [9], Theorem 3.3 and Corollary 3.4 are applied to obtain information on homology and homotopy groups of the complexes $\Delta(G)$ and $\Delta(\Omega(G))$, when $\mathcal{F}$ has the bounded Helly property.

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Institute of Mathematics, University of Tsukuba,
Tsukuba, Ibaraki 305-8571, Japan
E-mail address: `kawamura@math.tsukuba.ac.jp`