# DELETION CONSTRUCTIONS OF SYMMETRIC 4-CONFIGURATIONS. PART I.

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ABSTRACT. By *deletion constructions* we mean several methods of generation of new geometric configurations by the judicious deletion of certain points and lines, and introduction of other lines or points. A number of such procedures have recently been developed in a systematic way. We present here one family of such constructions, and will describe other families in the following parts.

## 1. INTRODUCTION

Geometric 3-configurations—that is, families of points and lines in the Euclidean or real projective plane, such that each point is on three lines and each line passes through three of the points—have been studied for a century and a quarter, with a variety of results known about them. In contrast, analogously defined 4-configurations have a much shorter history, and the first visually intelligible example was published less than twenty years ago [10]. (If the number of points (and lines) $n$ of a 4-configuration is relevant for a discussion, the configuration is referred to as an $(n_4)$ configuration.) In the short period since [10] was published, a large number of methods of construction of such configurations have been found, and the advances have been in many directions (see, e.g., [1, 2, 3, 4, 5, 7]). A recent survey appears in [8], and a detailed account of many of the results in the theory of all geometric configurations has just been published [9].

Many of these configurations in the Euclidean plane exhibit a very high degree of symmetry, in the sense that isometric maps of the plane map the points and lines of the configuration on themselves. This is easily seen to imply that the number of points (and lines) $n$ must be a composite number, and that the symmetry group consists of rotations about a fixed point, the *center* of the configuration, and possibly reflections in mirrors through the same point. Typically, we assume that the fixed point is at the origin. In most of the constructions the number of points in each orbit (of the symmetry group) is the same, and equals the order of the symmetry group, and these points all lie on a circle whose center is the center of the configuration. We will refer to configurations with $h$ symmetry classes of points

as $h$-ring configurations. Typical examples are shown in Figures 2 and 7; both configurations in Figure 2 shown are 3-ring configurations, while the configurations in Figure 7 are 4-ring configurations.

The present paper is the first in a series devoted to a systematic development of methods of construction that yield configurations in which not all orbits need to have the same number of points (or lines). The characteristic aspects of these constructions is that they start with given geometric configurations (usually quite symmetric) and through judicious deletion of points (and/or lines) and addition of suitable elements lead to new types of geometric configurations.

## 2. Modified celestial configurations: the deletion technique

*Celestial* ($n_4$) configurations, named in [2], were originally developed by Branko Grünbaum [6] and further studied by Marko Boben and Tomaž Pisanski [4] and Grünbaum [8]. They are highly symmetric configurations and have the property that every point has precisely two lines from each of two symmetry classes passing through them, and if there are $m$ points in a symmetry class, then the configuration has the symmetries of a regular $m$-gon; that is, symmetry group $d_m$.

Many ($n_4$) configurations may be constructed by deleting parts of certain symmetry classes of celestial configurations and replacing them with other objects. The movable ($n_4$) configurations discussed in [2] were constructed in this way, yielding configurations with only rotational symmetry. Several other classes of modified celestial configurations will be discussed here. To present these constructions we need notation for the celestial configurations involved. We describe these first.

The most recent notation for celestial configurations is discussed thoroughly in [2], based on the treatment in [8]; an outline is as follows.

A connected celestial configuration has symbol

$$m\#(s_1, t_1; s_2, t_2; \ldots; s_h, t_h);$$

to emphasize the number of symmetry classes of points and lines, we will say that such a configuration is an $h$-celestial configuration (with $h$ symmetry classes of points and of lines). It begins with a set of $m$ points collectively called $v_0$ forming a regular $m$-gon which are labelled[1] cyclically in the counterclockwise direction as

$$(v_0)_0, (v_0)_1, \ldots, (v_0)_{m-1};$$

by convention these are centered at the origin with

$$(v_0)_i = \left( \cos\left(\frac{2\pi i}{m}\right), \ \sin\left(\frac{2\pi i}{m}\right) \right).$$

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[1]The notation in [2] has been modified slightly in the present treatment; there points were labelled, e.g., as $v_{0,0}$, $v_{0,1}$, where here they are labelled as $(v_0)_0$, $(v_0)_1$, and similarly for lines.

Draw all lines of span $s_1$—that is, lines connecting the vertex $(v_0)_i$ with $(v_0)_{i+s_1}$—and label these consecutively as $(L_0)_1, (L_0)_1, \ldots, (L_0)_{m-1}$, or collectively as $L_0$. On the $t_1$-st intersections of these lines, which are each given the label $[[s_1, t_1]]$ (see Figure 1), counting from the center of the line segment $(v_0)_i(v_0)_{i+s_1}$, place a new set of vertices $(v_1)_0, \ldots, (v_1)_{m-1}$. Draw in lines of span $s_2$ (collectively labelled $L_1$) using these vertices, and place the third set of vertices, with label $[[s_2, t_2]]$, at the $t_2$-nd intersection of these new lines. Continue in this fashion until all the $s_i$ and $t_i$ have been used up; if the symbol corresponds to a valid configuration, the points labelled $v_h = [[s_h, t_h]]$ that are the $t_h$-th intersection points on the span $s_h$ lines (which have label $L_{h-1}$) will coincide—as sets—with the original ring of points labelled $(v_0)_0, \ldots (v_0)_{m-1}$, although the point $(v_h)_i$ may not be the same point as $(v_0)_i$. Notice that by construction, for each $i$, the set of points with label $v_i$ forms a regular convex $m$-gon centered at the origin.
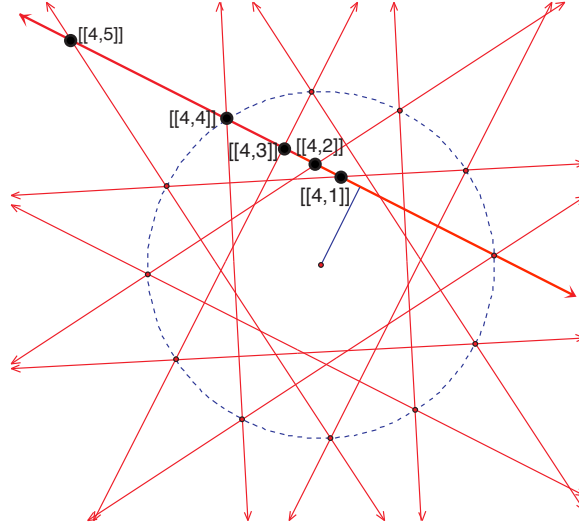


FIGURE 1. The notation $[[s, t]]$ for points on lines of span $s$. Here, $m = 12$, $s = 4$ and $t = 1, 2, 3, 4, 5$.

Several configuration symbols may correspond to the same geometric configuration, and the points labelled $v_0$ need not be the outermost ring of points; Figure 2 shows such a situation for $m = 8$. For a configuration symbol to be valid, two consecutive terms must be distinct, and there are various other constraints on the $s_i$ and $t_i$ as well; see [8, p. 202] for details. Of particular utility are the *trivial* celestial configurations, where $\{s_1, \ldots, s_h\} = \{t_1, \ldots, t_h\}$ as sets. For example, Figure 2 shows two trivial 3-celestial configurations, where in both cases $\{s_1, s_2, s_3\} = \{t_1, t_2, t_3\} = \{1, 2, 3\}$ as sets.

We say that a line is a *diameter* of a celestial configuration if it passes through the center of symmetry of the configuration (conventionally taken to be the origin) and a point from the set $v_0$. If $m$ is even, all diameters connect

(a) $8\#(2, 1; 3, 2; 1, 3)$, with points labelled
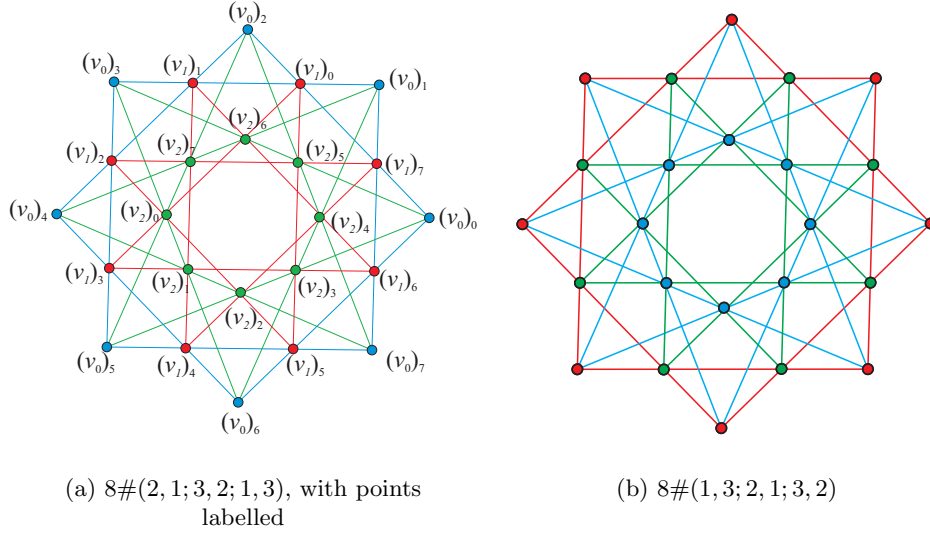
(b) $8\#(1, 3; 2, 1; 3, 2)$

FIGURE 2. Illustrating notation used for celestial configurations. This shows two symbols corresponding to the same celestial configuration. In each configuration, lines $L_0$ and points $v_0$ are blue, $L_1$ and $v_1$ are red, and $L_2$ and $v_2$ are green.

pairs of points $(v_0)_i$ and $(v_0)_{i+\frac{m}{2}}$. We say a line is a *mid-diameter* if it is the rotation by an angle of $\frac{\pi}{m}$ of some diameter. (If $m$ is odd, mid-diameters are themselves diameters.) All diameters and mid-diameters are mirrors of the configuration. If diameters (can) pass through a class of points, that class is said to be *diametral* or of *type D*, and likewise if mid-diameters (can) pass through the points they are said to be *mid-diametral* or of *type MD*. If there are two classes of points and they both are diametral or both are mid-diametral, the classes of points are the same *type*. A configuration symbol $m\#(s_1, t_1; s_2, t_2; \ldots; s_h, t_h); D$ or $m\#(s_1, t_1; s_2, t_2; \ldots; s_h, t_h); MD$ denotes a celestial configuration with diameters or mid-diameters added, respectively. Examples of diameters and mid-diameters are shown in Figure 3; in the underlying configurations, the points labelled $v_0$ (blue) and $v_2$ (green) are type $D$, while points labelled $v_1$ (red) are type $MD$.

Suppose that $m\#(s_1, t_1; s_2, t_2; \ldots; s_h, t_h)$ is a celestial configuration where $m$ is even. To construct modified celestial configurations, we will need the following results from [2], somewhat restated from that context.

We wish to analyze what happens to a configuration when half of a symmetry class of lines or points (that is, every other point or line) is removed. Note that deleting half the points or lines in a symmetry class makes no sense if $m$ is odd! For the remainder of the section, we will assume $m$ is even, and when we say we are *removing half of the elements* of a symmetry

(a) Diameters: $8\#(2, 1; 3, 2; 1, 3); D.$

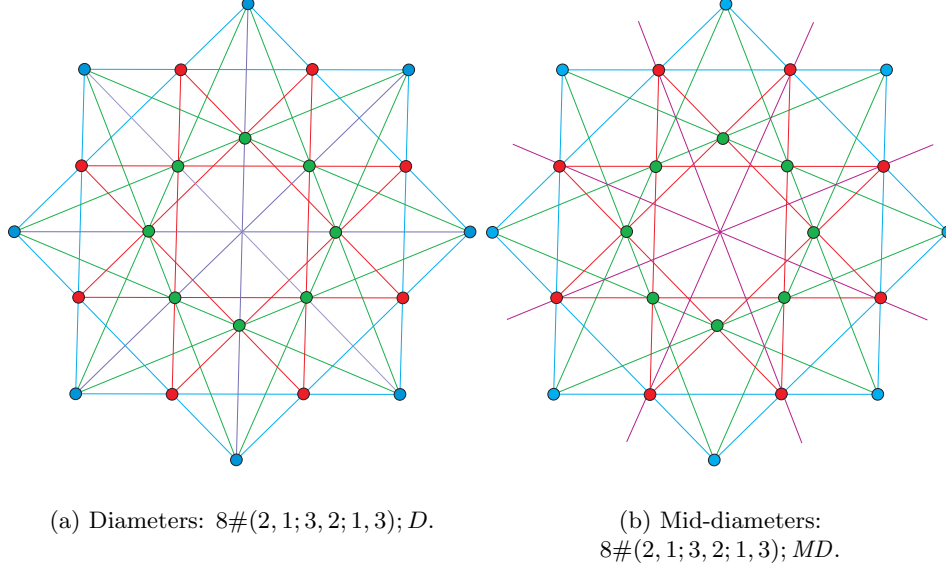(b) Mid-diameters: $8\#(2, 1; 3, 2; 1, 3); MD.$

FIGURE 3.    Diameters and mid-diameters of the configuration $8\#(2, 1; 3, 2; 1, 3)$, shown in Figure 2(a). Neither of these incidence structures is a configuration.

class, we mean that we are removing the elements of the class which are all of even index or all of odd index. For example, if we remove half the lines $L_1$ of even index, we are removing the lines $(L_1)_0, (L_1)_2, (L_1)_4, \ldots, (L_1)_{m-2}$, but if we remove half the lines labelled $L_1$ of odd index, we remove the lines $(L_1)_1, (L_1)_3, (L_1)_5, \ldots, (L_1)_{m-1}$.

A single symmetry class of lines $L_{i-1}$ contains points $v_{i-1}$ and $v_i$; with respect to the points $v_{i-1}$ they are of span $s_i$, while with respect to the points $v_i$ they are of span $t_i$. If $s_i$ is odd, then removing half of the lines labelled $L_{i-1}$ results in each point of $v_{i-1}$ with still exactly one line labelled $L_{i-1}$ passing through it. On the other hand, if $s_i$ is even, then removing half of the lines labelled $L_{i-1}$ means that every other point labelled $v_{i-1}$ has both lines labelled $L_{i-1}$ removed, while the remaining points have no lines removed.

Still considering the lines $L_{i-1}$ but looking at them from the point of view of the points $v_i$ so that they are of span $t_i$, if $t_i$ is odd then removing half the lines $L_{i-1}$ results in exactly one line from the symmetry class passing through each point $v_i$, while if $t_i$ is even, every other point labelled $v_i$ has no lines labelled $L_{i-1}$ passing through it and remaining points still have two lines passing through them.

Similarly, looking backwards, if $t_i$ is odd, removing half of the points (every other point) labelled $v_i = [[s_i, t_i]]$ means that each line $L_{i-1}$ contains a single point labelled $v_i$, while if $t_i$ is even, every other line $L_{i-1}$ contains two points labelled $v_i$ and the remaining lines contain no points. If $s_{i+1}$ is

odd, then removing half the points labelled $v_i$ results in every line labelled $L_i$ having a single point labelled $v_i$ lying on it, whereas if $s_{i+1}$ is even, removing half the points $v_i$ results in every other line $L_i$ containing two points $v_i$ and the rest containing none.

We place an asterisk in front of $t_i$ to indicate that half the points $[[s_i, t_i]]$ have been removed and say that the configuration is *point-deleted*; an asterisk in front of $s_i$ indicates that half the lines of the corresponding symmetry class $L_{i-1}$ have been removed and we say the configuration is *line-deleted*. (Note that in [2] such modified configurations were referred to as *point-modified* and *line-modified*, respectively.) Figure 4 shows examples of such modifications. Note that a point-deleted or line-deleted configuration is not typically itself a configuration, but rather merely an incidence structure.
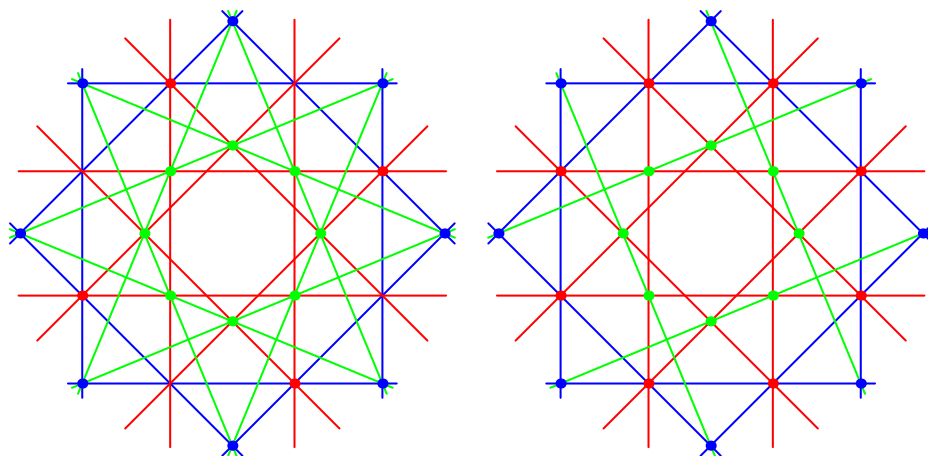


(a) A point-deleted configuration:
$8\#(2, *1; 3, 2; 1, 3)$

(b) A line-deleted configuration:
$8\#(2, 1; 3, 2; *1, 3)$

FIGURE 4. Point- and line-deleted configurations. In both cases, points $v_0$ and lines $L_0$ are blue.

To modify celestial configurations so that we can add diameters or mid-diameters, we need the following observation:

**Lemma 2.1.** *For a given $i$, if $s_i \equiv t_i \bmod 2$, the points labelled $v_i$ (i.e., $[[s_i, t_i]]$) are the same type as the points labelled $v_{i-1}$ (or $[[s_{i-1}, t_{i-1}]]$), with indices taken modulo $h$; if $s_i \not\equiv t_i \bmod 2$, then the points $v_i$ and $v_{i-1}$ are of opposite type.*

That is, for even $m$, the preceding lemma says that parity changes between $s_i$ and $t_i$ correspond to switching between points lying on diameters and on mid-diameters, or vice versa. For example, in Figure 2(a), the points in

classes $v_0$ and $v_1$ are of opposite type, as are those of $v_1$ and $v_2$, but the points of $v_2$ and $v_0$ are of the same type. By convention, points labelled $v_0 = [[s_h, t_h]]$ are diametral.

## 3. Deleting lines and adding diameters:
$$(hm_4) \to (hm_4)$$

If $s_i$ and $t_i$ are both odd, then two facts are true: (1) removing half the lines labelled $L_{i-1}$ results in each point labelled $v_{i-1}$ and $v_i$ containing exactly one line labelled $L_{i-1}$ (and two other lines of a different symmetry class, for a total of three lines remaining per point) and (2) the points labelled $v_i = [[s_i, t_i]]$ and $v_{i-1} = [[s_{i-1}, t_{i-1}]]$ are of the same type. Therefore, we can add in either diameters or mid-diameters, depending on the type of $[[s_i, t_i]]$, and these lines will pass through the points labelled $v_i = [s_i, t_i]]$ and $v_{i-1} = [[s_{i-1}, v_{i-1}]]$. If these are the only two points of that type, then the result will be a $(n_4)$ configuration.

In fact, this construction technique yields two enantiomorphic configurations, depending on whether the half of the lines that were removed were of even or odd index. Figure 5 shows two examples. Both examples have symbol $8\#(2, 1; 3, 2; *1, 3); D$.
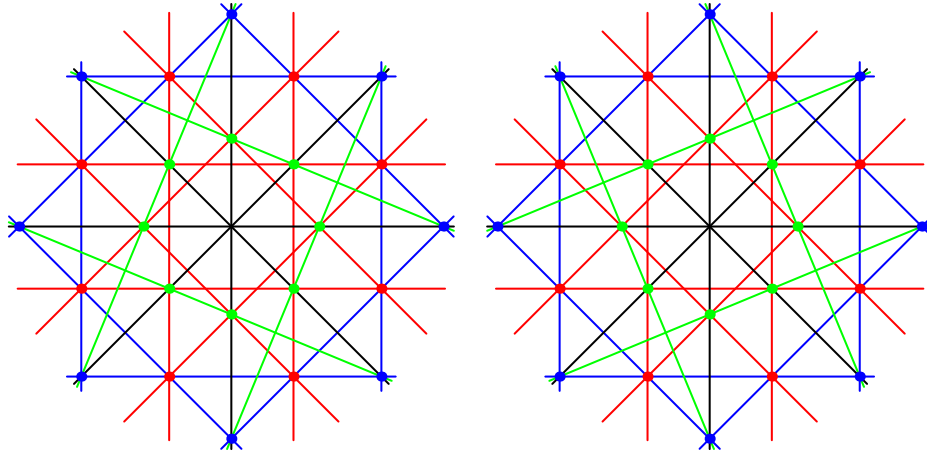


(a) Lines $(L_2)_1$, $(L_2)_3$, $(L_2)_5$, $(L_2)_7$ of odd index removed.

(b) Lines $(L_2)_0$, $(L_2)_2$, $(L_2)_4$, $(L_2)_6$ of even index removed, as in Figure 4(b).

FIGURE 5. Odd deletion: two enantiomorphic configurations, both with symbol $8\#(2, 1; 3, 2; *1, 3); D$. In both cases, points $v_0$ and lines $L_0$ are blue.

Figure 5(a) has lines labelled $L_2$ of odd index removed (that is, lines $(L_2)_1$, $(L_2)_3$, $(L_2)_5$, $(L_2)_7$ were removed), while Figure 5(b) has lines labelled $L_2$ of even index removed (that is, $(L_2)_0$, $(L_2)_2$, $(L_2)_4$, $(L_2)_6$ were removed).

We can trace the parity changes through this example. By convention, the points $v_0$ are type $D$. Since the first symbol $[[s_1, t_1]] = [[2, 1]] = v_1$ has entries of opposite parity, points $v_1$ are type $MD$. Since $[[s_2, t_2]] = [[3, 2]] = v_2$ again has entries of opposite parity, the points $v_2$ are type $D$. Since $[[s_3, t_3]] = [[1, 3]]$ has entries of the same parity, the points $v_0$ are (still!) type $D$. The symbol sequence $*1, 3$ indicates that half of the third class of lines is removed; because $v_0 = [[s_3, t_3]] = [[1, 3]]$ are points of type $D$, diameters must be added. Finally, note that diameters pass through the points $v_0$ and $v_2$, so they contain four points each, and each of the points labelled $v_0$ has passing through it two lines labelled $L_0$, one line labelled $L_2$, and one diameter (and points $v_2$ similarly have four lines passing through them), so the result is indeed a 4-configuration.

In general, care must be taken to ensure that if diameters are added, there are only two classes of points of type $D$ (although there may be as many classes as you like of points of type $MD$), and vice versa if mid-diameters are added. The following result determines allowable symbol sequences where there are two classes of points of one type and the rest of the other type.

**Theorem 3.1.** *(Odd deletion) Suppose for $h \geq 3$ the configuration*

$$m\#(s_1, t_1; s_2, t_2; \ldots; s_h, t_h)$$

*has the following properties:*

(1) $s_h$ *and* $t_h$, *the last two terms in the configuration symbol, are both odd;*

(2) *the pairs* $s_{h-1}, t_{h-1}$ *and* $s_1, t_1$ *are of opposite parity;*

(3) *all other pairs* $s_j, t_j$ *for* $2 \leq j \leq h - 2$ *are of the same parity.*

*Then half the lines labelled* $L_{h-1}$ *may be removed and diameters may be added to form an* $(n_4)$ *configuration.*

Note that the requirement that $h \geq 3$ is not arbitrary; there are no 2-celestial configurations whose last two entries are both odd.

*Proof.* The lines labelled $L_{h-1}$ contain points labelled $v_{h-1}$ and $v_h = v_0$. Since $s_h$ and $t_h$ are both odd, removing half of the lines labelled $L_{h-1}$ results in each point labelled $v_0$ and $v_{h-1}$ containing a single line labelled $L_{h-1}$ By construction, points labelled $v_0$ are type $D$, so points $v_{h-1}$ are also type $D$. Since $s_1$ and $t_1$ are of opposite parity, points $v_0$ and $v_1$ are of the opposite type, so points $v_1$ are type $MD$. Since $s_j$ and $t_j$ are of the same parity for $2 \leq j \leq h - 2$, all the points labeled $v_2, \ldots, v_{h-2}$ are of the same type as $v_1$ and thus are also of type $MD$. Finally, since $s_{h-1}$ and $t_{h-1}$ are of opposite parity, the points $v_{h-1}$ switch type and are of type $D$. Therefore, the only points of type $D$ are $v_0$ and $v_{h-1}$, while the rest are of type $MD$; adding diameters yields an $(n_4)$ configuration. □

For clarity, we present parity patterns for configuration symbols for small values of $h$. The notation $E$ means that the entry is even, $O$ means the entry is odd, $X, X$ in a pair means the pair must be of the same parity, and $X, Y$ in a pair means the pair must be of the opposite parity.

**h = 3:** $m\#(X, Y; X, Y; *O, O); D$

**h = 4:** $m\#(X, Y; X, X; X, Y; *O, O); D$

**h ≥ 5:** $m\#(X, Y; \underbrace{X, X; \ldots; X, X}_{h-3}; X, Y; *O, O); D$

In particular, there are several trivial families of configurations formed by odd deletion, including:

- $m\#(a, b; c, a; *b, c); D$ where $b$ and $c$ are odd and $a$ is even (Figure 5);
- $m\#(a, b; c, d; b, a; *d, c); D$ where $c$ and $d$ are odd and $a$ and $b$ are of opposite parity;
- $m\#(a, b; c, d; e, a; b, c; *d, e); D$ where $b$ is even and $a, c, d$ and $e$ are all odd (Figure 6).
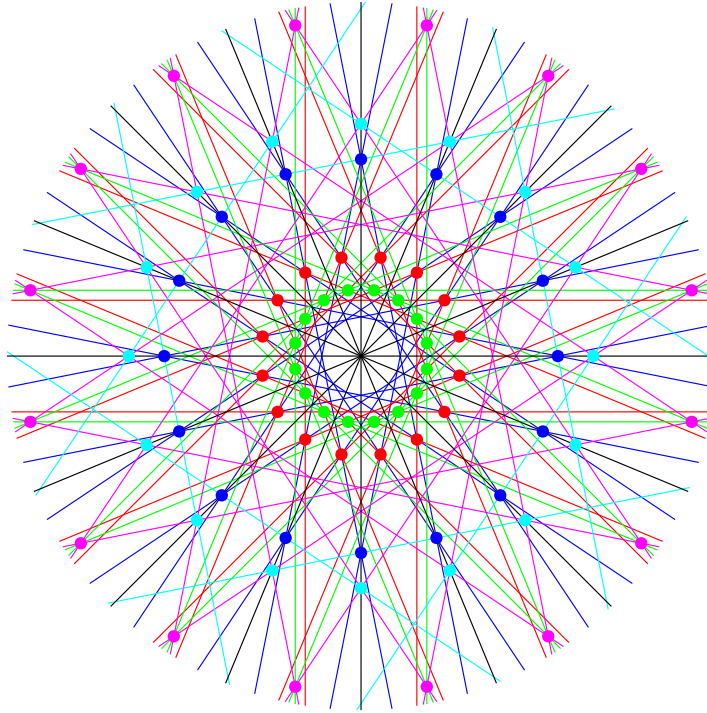


FIGURE 6. The configuration

$$16\#(7, 6; 5, 3; 1, 7; 6, 5; *3, 1); D.$$

The points $v_0$ and lines $L_0$ are shown in blue; half of the lines $L_4$, in cyan, have been deleted.

## 4. Deleting points and lines and adding mid-diameters:
$$(hm_4) \rightarrow \left(\left((h-1)m + \tfrac{m}{2}\right)_4\right)$$

In the previous construction technique, half of a symmetry class of lines was deleted and diameters were added, but it was not necessary to delete any points and (usually) the number of points and lines in the configuration remained constant. In the following construction technique, half of a symmetry class of points and of lines will be deleted and mid-diameters will be added, yielding a configuration with a smaller number of points and lines. An example is shown in Figure 7.
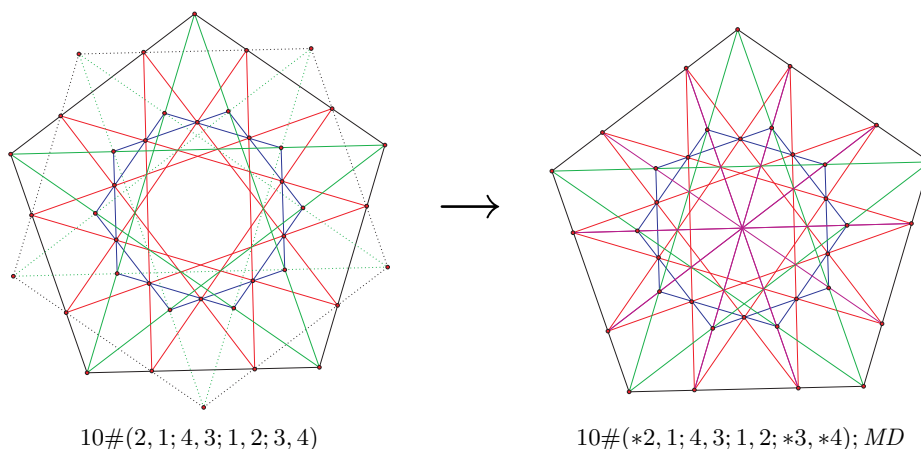


$10\#(2, 1; 4, 3; 1, 2; 3, 4)$          $10\#(*2, 1; 4, 3; 1, 2; *3, *4); MD$

FIGURE 7. Even deletion. The points $v_0$ are the outermost ring of points; lines $L_0$ are black and lines $L_3$ are green.

**Theorem 4.1.** *(Even deletion) Suppose for $h \geq 3$ that*

$$m\#(s_1, t_1; s_2, t_2; \ldots; s_h, t_h)$$

*has the following properties:*

(1) *$t_h$ and $s_1$ are distinct, and both are even (that is, the first and last entries in the symbol are even);*
(2) *$s_h$ and $t_1$ are odd (the second and second-to-last entries);*
(3) *If $h > 3$, the pairs $s_2, t_2$ and $s_{h-1}, t_{h-1}$ are of opposite parity, while if $h = 3$, the pair $s_2, t_2$ is of the same parity;*
(4) *All other pairs $s_j, t_j$, for $3 \leq j \leq h - 2$, are of the same parity.*

*If every other point labelled $v_0$ and all the lines incident with those points are removed—that is, half the symmetry class of lines labelled $L_0$ and half the symmetry class labelled $L_{h-1}$—then mid-diameters, which will pass through points labelled $v_1$ and $v_{h-1}$, may be added to form a $\left(\left((h-1)m + \tfrac{m}{2}\right)_4\right)$ configuration.*

Again, the requirement that $h \geq 3$ is not arbitrary; although there are 2-celestial configurations whose first and last entries are even and second and second-to-last entries are odd, there are too few rings of points for the construction to work.

*Proof.* Recall that lines $L_0$ contain points labelled $v_0$ and $v_1$ and lines $L_{h-1}$ contain points labelled $v_{h-1}$ and $v_0$. Since $t_h$ and $s_1$ are both even, when we remove half the points labelled $v_0$, half of the lines labelled $L_{h-1}$ contain two points labelled $v_0$ and the remainder contain none (by considering $t_h$) and likewise, considering $s_1$, every other line $L_0$ contains two points labelled $v_0$ and the rest none. The theorem instructed us to remove the lines which now contain no points labelled $v_0$ (since before removal, these were the very lines incident with the removed points).

Now consider the lines $L_0$, half of which have been removed. These lines also contain points labelled $v_1$, and in fact the lines labelled $L_0$ are of span $t_1$ with respect to these points. Since $t_1$ is odd, when half of the lines $L_0$ are deleted, each point labelled $v_1$ has one line labelled $L_0$ passing through it. Similarly, since the lines $L_{h-1}$ are of span $s_h$ with respect to the points $v_{h-1}$ and $s_h$ is odd, when half of the lines $L_{h-1}$ are removed, each point $v_{h-1}$ has one line labelled $L_{h-1}$ passing through it.

We need to show that the points $v_1$ and $v_{h-1}$ are the only points of type $MD$ in the configuration, so that when we add mid-diameters, only four points will be incident with those mid-diameters. To do this we will apply Lemma 2.1 multiple times. By definition, points $v_0$ are of type $D$. Since $s_1$ and $t_1$ are opposite parity, points $v_1$ are of type $MD$. Suppose that $h \geq 4$; in this case, by hypothesis, $s_2$ and $t_2$ are also of opposite parity, so points $v_2$ are of opposite type, that is, type $D$. Now, the theorem says all pairs $s_j, t_j$ for $3 \leq j \leq h-2$ are of the same parity, so all points $v_3, \ldots, v_{h-2}$ are of type $D$. Finally, since pairs $s_{h-1}, t_{h-1}$ and $s_h, t_h$ are of opposite parity, points labelled $v_{h-1}$ and $v_h = v_0$ are of type $MD$ and $D$, respectively. In the case where $h = 3$, the situation is even more straightforward: we begin with points $v_0$ of type $D$. Since $s_1, t_1$ are of opposite parity, points $v_1$ are type $MD$. Since $s_2, t_2$ are the *same* parity, points $v_2$ are also type $MD$. Finally, the opposite parity of $s_3, t_3$ switches $v_3 = v_0$ back to type $D$.

In particular, only points labelled $v_1$ and $v_{h-1}$ are of type $MD$; conveniently, these were the very points which, after deletion of half of the lines labelled $L_1$, had only three lines passing through them. Adding in the mid-diameters contributes a fourth line through each point, and each mid-diameter passes through two points labelled $v_1$ and two labelled $v_{h-1}$.

The total number of points has been reduced by $\frac{m}{2}$ as has the total number of lines, since $m$ lines were deleted and $\frac{m}{2}$ mid-diameters were added.      $\square$

Again, for clarity, especially since the patterns differ slightly for small numbers of rings, we present the parity patterns needed for $h = 3, 4$ and $h \geq 5$.

**h = 3:** $m\#(*E, O; X, X; *O, *E); MD$
**h = 4:** $m\#(*E, O; X, Y; X, Y; *O, *E); MD$
**h $\geq$ 5:** $m\#(*E, O; X, Y; \underbrace{X, X; \ldots; X, X}_{h-4}; X, Y; *O, *E); MD$

There are several trivial families of configurations formed by even deletion (that is, families beginning with trivial celestial configurations), including:

- $m\#(*a, b; c, a; *b, *c); MD$ for $a$ and $c$ even and $b$ odd (Figures 8(a) and 8(c));
- $m\#(*a, b; c, d; b, a; *d, *c); MD$ for $a, c$ even and $b, d$ odd (Figure 7);
- $m\#(*a, b; c, d; e, a; b, c; *d, *e); MD$ for $a, c, e$ even and $b, d$ odd (Figure 9).

(Note that while Figure 8(b) is also formed by applying even deletion to a 3-celestial configuration, the original 3-celestial configuration is not trivial.)

Figure 7 shows a four ring configuration formed using even deletion. In the figure shown on the left hand side of Figure 7, half of the points labelled $v_0$ are to be deleted, along with half of the lines labelled $L_0$ and $L_3$ (these are shown dashed); on the right hand side, the points and lines have been deleted, and mid-diameters have been added. Figures 8 and 9 show examples of three and five ring configurations, respectively, formed by even deletion.

This construction method is especially useful for constructing small configurations; three of the known symmetric $(25_4)$ configurations may be constructed in this way, beginning from celestial $(30_4)$ configurations. They are shown in Figure 8.

## 5. Combining deletion techniques

Given an appropriate symbol for a celestial configuration, the two different types of deletion methods may be combined to produce other symmetric 4-configurations.

In a celestial configuration $m\#(s_1, t_1; s_2, t_2; s_3, t_3; s_4, t_4)$ where pairs $s_i, t_i$, for $i$ even, are both odd and pairs with odd $i$ are of opposite parity, half of the line classes $L_1$ and $L_3$ may be removed and both diameters and mid-diameters may be added to form the configuration

$$m\#(s_1, t_1; *s_2, t_2; s_3, t_3; *s_4, t_4); D; MD.$$

That is, a slight generalization of the odd deletion technique has been applied twice to the same configuration. Figure 10 shows the configuration $10\#(1, 2; *3, 5; 2, 1; *5, 3); D; MD$.

Both even and odd deletions are combined in Figure 11, which shows a $(42_4)$ configuration with symbol $14\#(*6, 4; *2, *3; 4, 6; *3, *2); MD$. It has had half of the lines labelled $L_0$ (blue) and $L_3$ (magenta) removed, along with half the points labelled $v_0$ (blue), and also half the points labelled $v_1$ (red) and lines labelled $L_1$ (red) removed, and mid-diameters added.
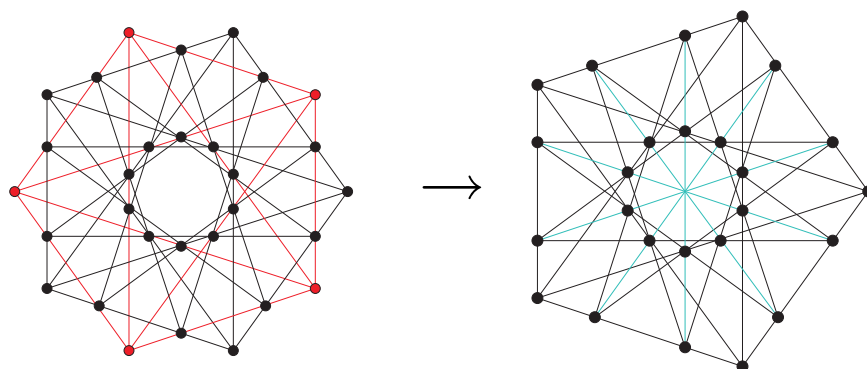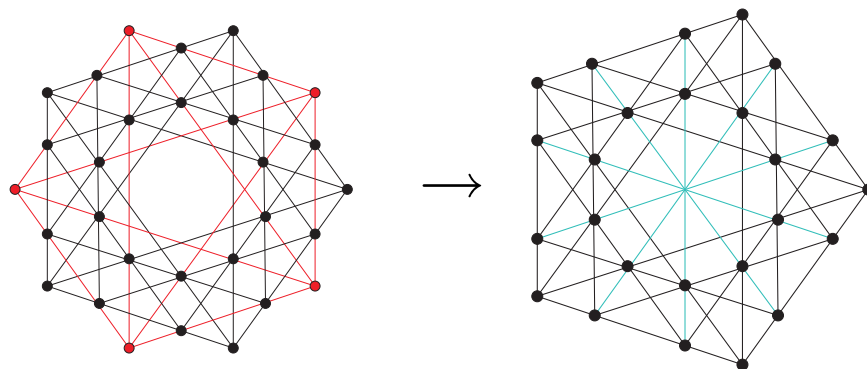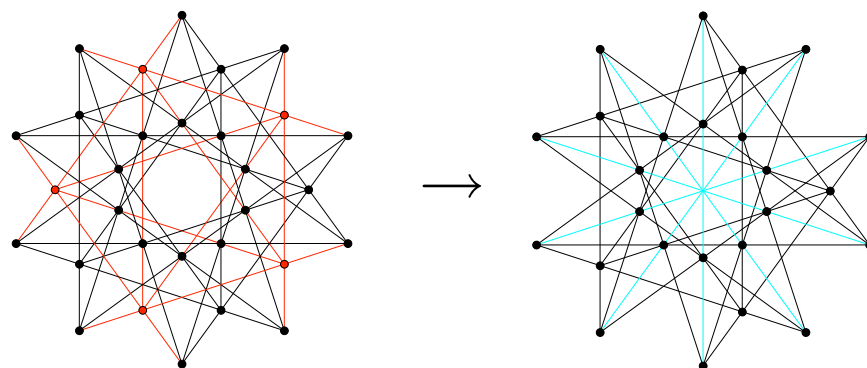
(a) $10\#(*4, 1; 2, 4; *1, *2); MD$.



(b) $10\#(*4, 3; 1, 3; *1, *2); MD$.



(c) $10\#(*4, 3; 2, 4; *3, *2); MD$.

FIGURE 8. Three of the known symmetric $(25_4)$ configurations, all formed by even deletion. In (a) and (b), the points $v_0$ form the outermost ring of points, while in (c), the points $v_0$ are the middle ring of points.
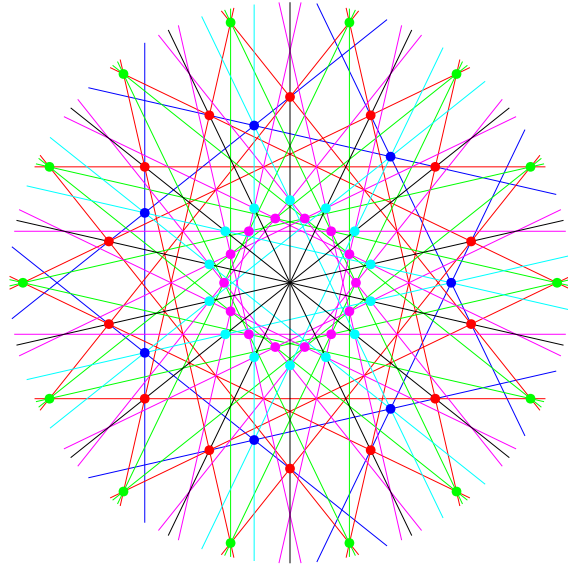
FIGURE 9. More even deletion:

$$14\#(*2, 3; 4, 5; 6, 2; 3, 4; *5, *6).$$

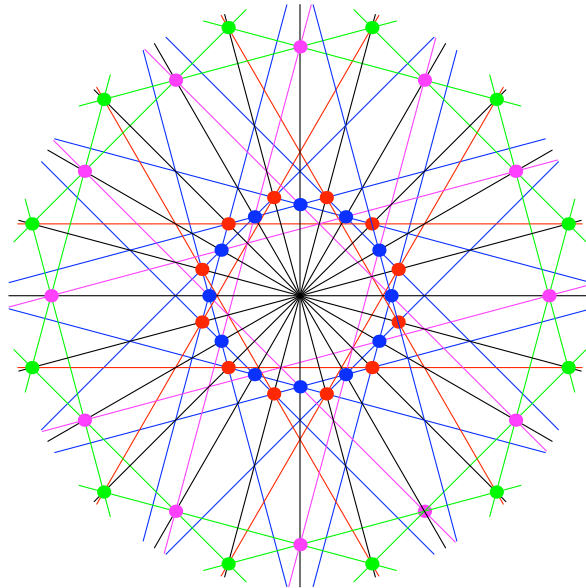The points $v_0$ and the lines $L_0$ are shown in dark blue.



FIGURE 10. The configuration

$$12\#(1, 2; *3, 5; 2, 1; *5, 3); D; MD,$$

which has half of two symmetry classes of lines removed and diameters and mid-diameters added.
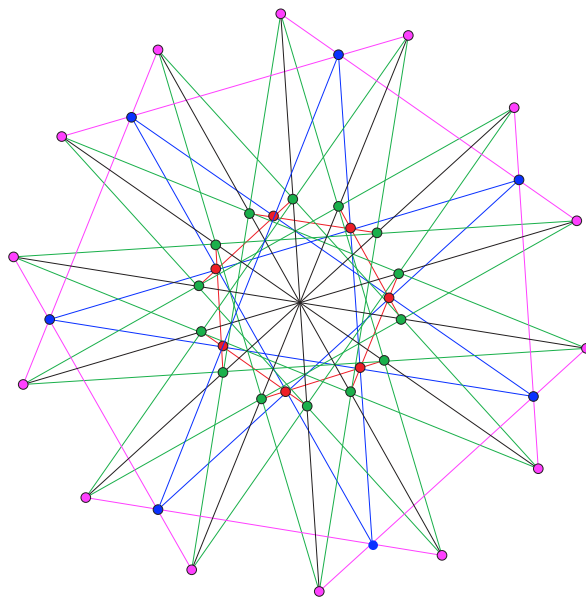
FIGURE 11. Multiple modifications; a $(42_4)$ configuration with symbol $14\#(*6, 4; *2, *3; 4, 6; *3, *2)$; $MD$. Points $v_0$ and lines $L_0$ are blue.

## 6. OPEN QUESTIONS

Undoubtedly, there are many more variants of deletion techniques to be found.

For example, the odd and even deletion constructions do not require beginning with a celestial configuration—celestial configurations simply provide a convenient class of examples to work with. Similar constructions are possible with other classes of 4-configurations. For example, starting from the "floral configuration" (see [3] for details on this kind of configuration) in Figure 12(a), by deleting the lines shown in red and replacing them with the green lines shown in Figure 12(b), we obtain a new $(72_4)$ configuration. Analogous deletion constructions can be carried out with several other types of floral configurations. In the sequel to this paper we shall investigate a number of other deletion constructions.

The deletion constructions presented here are based on eliminating every other point in a ring of points. Are there analogous constructions where every third point is eliminated? Every fourth point?

Are there interesting configurations which can be constructed using deletion techniques or combinations of deletion techniques that do not lead to $(n_4)$ configurations, but rather to configurations with higher numbers of incidences of points and lines (e.g., $(p_4, n_5)$ configurations?).

What configurations may be constructed beginning with other classes of configurations, such as 3-configurations or 6-configurations?
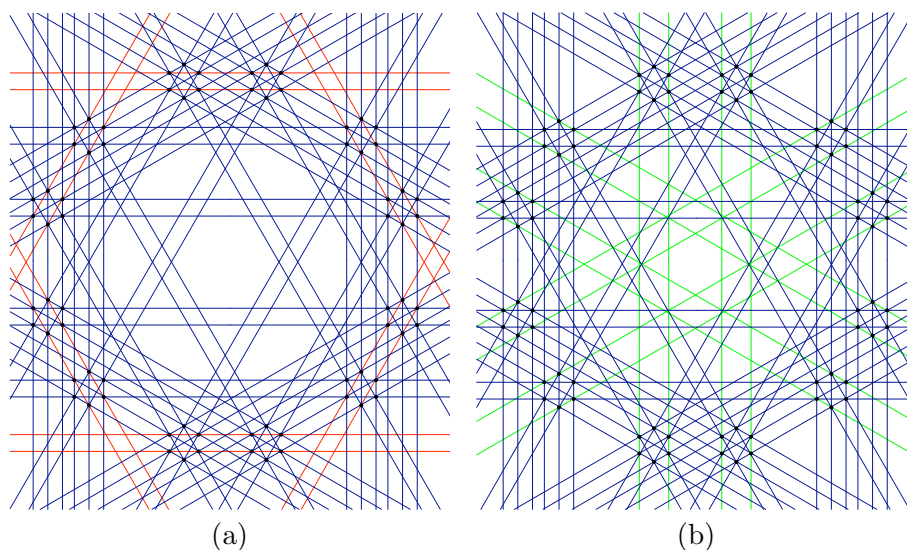
(a)                                        (b)

FIGURE 12. (a) A $(72_4)$ floral configuration. If the red lines
are deleted and replaced with the green lines shown in (b),
we obtain a new $(72_4)$ configuration.

## REFERENCES

1. L. W. Berman, *Even astral configurations*, Electronic Journal of Combinatorics **11** (2004), #R37.
2. _____, *Movable* $(n_4)$ *configurations*, Electronic Journal of Combinatorics **13** (2006), #R104.
3. L. W. Berman, J. Bokowski, B. Grünbaum, and T. Pisanski, *Geometric "floral" configurations*, Canadian Mathematical Bulletin **52** (2009), no. 3, 327–341.
4. M. Boben and T. Pisanski, *Polycyclic configurations*, European Journal of Combinatorics **24** (2003), no. 4, 431–457.
5. J. Bokowski, B. Grünbaum, and L. Schewe, *Topological configurations* $(n_4)$ *exist for all* $n \geq 17$, European Journal of Combinatorics, in press, available online 13 January 2009.
6. B. Grünbaum, *Configurations*, unpublished manuscript / class notes, 1991.
7. _____, *Astral* $(n_4)$ *configurations*, Geombinatorics **9** (2000), 127–134.
8. _____, *Configurations of points and lines*, The Coxeter Legacy: Reflections and Projections, American Mathematical Society, 2006, pp. 179–225.
9. _____, *Configurations of points and lines*, Graduate Studies in Mathematics, vol. 109, American Mathematical Society, 2009.
10. B. Grünbaum and J. F. Rigby, *The real configuration* $(21_4)$, J. London Math. Soc (2) **41** (1990), 336–346.

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