Contributions to Discrete Mathematics

Volume 6, Number 2, Pages 70–90
ISSN 1715-0868

A BIJECTION BETWEEN NONCROSSING AND NONNESTING PARTITIONS OF TYPES A, B AND C

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Abstract. The total number of noncrossing partitions of type Ψ is equal to the nth Catalan number $\frac{1}{n+1}\binom{2n}{n}$ when $\Psi = A_{n-1}$, and to the corresponding binomial coefficient $\binom{2n}{n}$ when $\Psi = B_n$ or C_n. These numbers coincide with the corresponding number of nonnesting partitions. For type A, there are several bijective proofs of this equality; in particular, the intuitive map, which locally converts each crossing to a nesting, is one of them. In this paper we present a bijection between nonnesting and noncrossing partitions of types A, B and C that generalizes the type A bijection that locally converts each crossing to a nesting.

1. Introduction

The poset of noncrossing partitions can be defined in a uniform way for any finite Coxeter group W. More precisely, for $u, w \in W$, let $u \leq w$ if there is a shortest factorization of w as a product of reflections in W having as prefix such a shortest factorization for u. This partial order turns W into a graded poset $Abs(W)$ having the identity 1 as its unique minimal element, where the rank of w is the length of a shortest factorization of w into reflections. Let c be a Coxeter element of W. Since all Coxeter elements in W are conjugate to each other, the interval $[1, c]$ in $Abs(W)$ is independent, up to isomorphism, of the choice of c. We denote this interval by $\mathrm{NC}(W)$ or by $\mathrm{NC}(\Psi)$, where Ψ is the Cartan-Killing type of W, and call it the *poset of noncrossing partitions* of W. It is a self-dual, graded lattice which reduces to the classical lattice of noncrossing partitions of the set $[n] = \{1, 2, \ldots, n\}$ defined by Kreweras in [10] when W is the symmetric group $\mathfrak{S}_n$ (the Coxeter group of type A_{n-1}), and to its type B and C analogues, defined by Reiner [12] when W is the hyperoctahedral group. The elements in $\mathrm{NC}(W)$ are

Received by the editors August 19, 2010, and in revised form May 31, 2011.

2000 *Mathematics Subject Classification.* 05A10; 05A15.

Key words and phrases. Root systems, noncrossing partitions, nonnesting partitions, bijection.

This work was supported by CMUC - Centro de Matemática da Universidade de Coimbra.

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counted by the generalized Catalan numbers,

$$Cat(W) = \prod_{i=1}^{k} \frac{d_i + h}{d_i},$$

where k is the number of simple reflections in W, h is the Coxeter number and $d_1, \ldots, d_k$ are the degrees of the fundamental invariants of W (see [1, 7, 8, 12] for details on the theory of Coxeter groups and noncrossing partitions). When W is the symmetric group, $Cat(\mathfrak{S}_n)$ is equal to the usual nth Catalan number $\frac{1}{n+1}\binom{2n}{n}$, and in types B_n and C_n this number is the binomial coefficient $\binom{2n}{n}$.

Nonnesting partitions were defined by Postnikov (see [12, Remark 2]) in a uniform way for all irreducible root systems associated with Weyl groups. If Φ is such a system, Φ^+ is a choice of positive roots, and Δ is the simple system in Φ^+, define the *root order* on Φ^+ by $\alpha \leq \beta$ if $\alpha, \beta \in \Phi^+$ and $\beta - \alpha$ is in the positive integer span of the simple roots in Δ. Equiped with this partial order, Φ^+ is the *root poset* of the associated Weyl group W. A *nonnesting partition* on Φ is defined as an antichain in root poset $(\Phi^+, \leq)$, that is, a set of pairwise incomparable elements. Denote by $\mathrm{NN}(W)$ or by $\mathrm{NN}(\Psi)$, where Ψ is the Cartan-Killing type of W, the set of all nonnesting partitions of W. Postnikov showed that the nonnesting partitions in $\mathrm{NN}(W)$ are also counted by the generalized Catalan number $Cat(W)$.

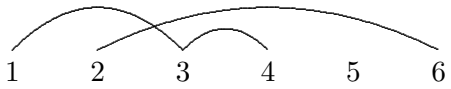
In the case of the root systems of type A, different bijective proofs of the equality between the cardinals $|\mathrm{NN}(A_{n-1})| = |\mathrm{NC}(A_{n-1})|$ are known (see [1, 2, 3, 9, 14]), and more recently several bijections between noncrossing and nonnesting partitions of classical types have been constructed (see [6, 11, 13, 14]) but all of them have different designs and settings. Our contribution in this paper is to present a uniform proof that $|\mathrm{NN}(\Psi)| = |\mathrm{NC}(\Psi)|$, for $\Psi = A_{n-1}, B_n$ or C_n that generalizes the bijection presented by Armstrong in [1]. Our ideas have been used to construct a bijection between noncrossing and nonnesting partitions of type D [5]. This paper is the complete version of the extended abstract [11] and, moreover, contains an extention of the bijection to the type C.

2. Noncrossing and nonnesting set partitions

A *set partition* of $[n]$ is a collection of nonempty disjoint subsets of $[n]$, called *blocks*, whose union is $[n]$. The type of a set partition π of $[n]$ is the integer partition formed by the cardinals of the blocks of π. A set partition of $[n]$ of type $(2, \ldots, 2, 1, \ldots, 1)$ is called a partial matching, and a set partition of $[2n]$ of type $(2, 2, \ldots, 2)$ is said to be a (perfect) matching of $[2n]$.

A set partition can be graphically represented by placing the integers $1, 2, \ldots, n$ along a line and drawing arcs above the line between i and j whenever i and j lie in the same block and no other element between them does so. A *singleton* of a set partition is a block which has only one element, so it corresponds to an isolated vertex in the graphical representation. For instance,

the graphical representation of the set partition $\pi = \{\{1,3,4\},\{2,6\},\{5\}\}$ of type $(3,2,1)$ is displayed below:



Given a set partition π, let

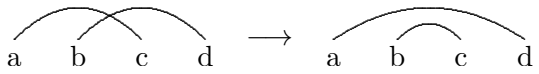
$$\begin{aligned} op(\pi) &= \{\text{least block elements of } \pi\}, \\ cl(\pi) &= \{\text{greatest block elements of } \pi\}, \text{ and} \\ tr(\pi) &= [n] \setminus (op(\pi) \cup cl(\pi)). \end{aligned}$$

The elements of $op(\pi)$, $cl(\pi)$, and $tr(\pi)$ are called openers, closers and transients, respectively. Graphically, the openers correspond to singletons and to vertices from which one arc begin and no arc ends, the closers correspond to singletons and to vertices to which one arc ends, and no arc begins, and the transients are the vertices from which one arc ends an another one begins. In the example above, $op(\pi) = \{1,2,5\}, cl(\pi) = \{4,5,6\}$ and $tr(\pi) = \{3\}$. The triples $\mathfrak{T}(\pi) = (op(\pi), tr(\pi), cl(\pi))$ encode useful information about the partition π. For instance, the number of blocks is $|op(\pi)| = |cl(\pi)|$, the number of singletons is $|op(\pi) \cap cl(\pi)|$, π is a partial matching if and only if $tr(\pi) = \varnothing$, and π is a (perfect) matching if and only if $tr(\pi) = \varnothing$ and $op(\pi) \cap cl(\pi) = \varnothing$.

A noncrossing partition of the set $[n]$ is a set partition of $[n]$ such that there are no $a < b < c < d$, with a, c belonging to some block of the partition and b, d belonging to some other block. The set of noncrossing partitions of $[n]$, denoted by $\mathrm{NC}(n)$, is a lattice for the refinement order. A nonnesting partition of the set $[n]$ is a partition of $[n]$ such that if $a < b < c < d$ and a, d are consecutive elements of a block, then b and c are not both contained in some other block. The set of nonnesting partitions of $[n]$ will be denoted by $\mathrm{NN}(n)$. Graphically, the noncrossing condition means that no two of the arcs cross, while the nonnesting condition means that no two arcs are nested one within the other.

The partition $\pi = \{\{1,3,4\},\{2,6\},\{5\}\}$ represented above is neither noncrossing nor nonnesting since two of the arcs cross, and two of the arcs are nested one within the other. The partitions $\{\{2,3\},\{1,4,5\}\}$ and $\{\{1,3\},\{2,4,5\}\}$, represented below, are examples of noncrossing and nonnesting partitions of [5]:

As pointed out in [1], the intuitive map that locally converts each crossing to a nesting



defines a bijection between noncrossing and nonnesting set partitions that preserves the number of blocks. We will refer to this bijection as the L-map.

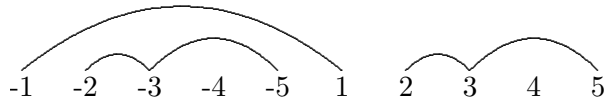
A B_n set partition π is a set partition of $[\pm n] := \{\pm 1, \pm 2, \ldots, \pm n\}$ which has at most one block (called the zero block) fixed by negation and is such that for any block B of π, the set $-B$, obtained by negating the elements of B, is also a block of π. The type of a B_n set partition π is the integer partition whose parts are the cardinalities of the blocks of π, including one part for each pair of nonzero blocks $B, -B$. Given a B_n set partition π, let the set of openers $op(\pi)$ be formed by the least element of all blocks of π having only positive integers; let the set of closers $cl(\pi)$ be formed by the greatest element of all blocks of π having only positive integers and by the absolute values of the least and greatest elements of all blocks having positive and negative integers; and finally let the set of transients $tr(\pi)$ be formed by all elements of $[n]$ which are not in $op(\pi) \cup cl(\pi)$.

Identifying the sets $[\pm n]$ and $[2n]$ through the map $i \mapsto i$ for $i \in [n]$ and $i \mapsto n-i$ for $i \in \{-1, -2, \ldots, -n\}$, we may represent B_n set partitions graphically using the conventions made for its type A analogs, placing the integers $-1, -2, \ldots, -n, 1, 2, \ldots, n$ along a line instead of the usual $1, 2, \ldots, 2n$.

Example 2.1. *The B_5 set partition*

$$\pi = \{\{-1, 1\}, \{2, 3, 5\}, \{-2, -3, -5\}, \{4\}, \{-4\}\},$$

represented below, has type $(3, 2, 1)$, set of openers $op(\pi) = \{2, 4\}$, closers $cl(\pi) = \{1, 4, 5\}$ and transients $tr(\pi) = \{3\}$. The openers, closers and transients can be visualized as in type A by looking only at the arcs on the positive half of the representation of π.



3. Noncrossing and nonnesting partitions of types A, B and C

We will now review the usual combinatorial realizations of the Coxeter groups of types A, B and C, referring to [8] for any undefined terminology. The Coxeter group W of type A_{n-1} is realized combinatorially as the symmetric group $\mathfrak{S}_n$. The permutations in $\mathfrak{S}_n$ will be written in cycle notation. The simple generators of $\mathfrak{S}_n$ are the transpositions of adjacent integers $(i\ i+1)$, for $i = 1, \ldots, n-1$, and the reflections are the transpositions $(i\ j)$ for $1 \leq i < j \leq n$. To any permutation $\pi \in \mathfrak{S}_n$ we associate the partition of the set $[n]$ given by its cycle structure. This defines an isomorphism between the posets $\mathrm{NC}(\mathfrak{S}_n)$ of noncrossing partitions of $\mathfrak{S}_n$, defined in the introduction, and $\mathrm{NC}(n)$, with respect to the Coxeter element $c = (12 \cdots n)$ [4, Theorem 1].

Denoting by $e_1, \ldots, e_n$ the standard basis of $\mathbb{R}^n$, the root system of type A_{n-1} consists of the set of vectors

$$\Phi = \{e_i - e_j : i \neq j, 1 \leq i, j \leq n\},$$

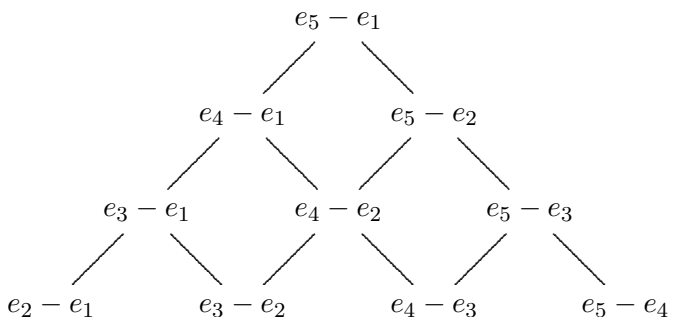
each root $e_i - e_j$ defining a reflection that acts on $\mathbb{R}^n$ as the transposition $(i\ j)$. We shall identify the root $e_i - e_j$ with the corresponding transposition $(i\ j)$. Take

$$\Phi^+ = \{e_i - e_j \in \Phi : i > j\}$$

for the set of positive roots and, defining $r_i := e_{i+1} - e_i$, $i = 1, \ldots, n-1$, we obtain the simple system $\Delta = \{r_1, \ldots, r_{n-1}\}$ for $\mathfrak{S}_n$. Note that

$$e_i - e_j = \sum_{k=j}^{i-1} r_k \qquad \text{if } i > j.$$

The correspondence between the antichains in the root poset $(\Phi^+, \leq)$ and the set of nonnesting partitions of $[n]$ is given by the bijection which sends the positive root $e_i - e_j$ to the set partition of $[n]$ having i and j in the same block. For instance, consider the root poset $(\Phi^+, \leq)$ of type A_4:



The antichain $e_3 - e_1$ corresponds to the transposition (13) in the symmetric group $\mathfrak{S}_5$, and thus to the nonnesting set partition $\{\{1,3\},\{2\},\{4\},\{5\}\}$, while the antichain $\{e_3 - e_1, e_4 - e_2, e_5 - e_4\}$ corresponds to the product of transpositions $(13)(24)(45) = (13)(245)$ in $\mathfrak{S}_5$, and thus to the nonnesting set partition $\{\{1,3\},\{2,4,5\}\}$.

Given a positive root $\alpha = e_i - e_j \in \Phi^+$, define the *support* of α as the set $\mathrm{supp}(\alpha) = \{j, j+1 \ldots, i-1\}$. The elements in $\mathrm{supp}(\alpha)$ correspond to the indices of the simple roots that appear with nonzero coefficient in the expansion of α as a linear combination of simple roots. The integers j and $i-1$ will be called, respectively, the *initial* and *terminal indices* of α. We have the following lemma.

Lemma 3.1. *Let α_1, α_2 be two roots in Φ^+ with initial and terminal indices i_1, j_1 and i_2, j_2, respectively, such that $i_1 \leq i_2$. Then, α_1, α_2 form an antichain if and only if $i_1 < i_2$ and $j_1 < j_2$.*

Consider now the Coxeter group W of type B_n, with its usual combinatorial realization as the hyperoctahedral group of signed permutations of

$$[\pm n] := \{\pm 1, \pm 2, \ldots, \pm n\}.$$

These are permutations of $[\pm n]$ which commute with the involution $i \mapsto -i$. We will write the elements of W in cycle notation, using commas between

elements. The simple generators of W are the transposition $(-1,\ 1)$ and the pairs $(-i-1,\ -i)(i,\ i+1)$, for $i = 1, \ldots, n-1$. The reflections in W are the transpositions $(-i,\ i)$, for $i = 1, \ldots, n$, and the pairs of transpositions $(i,\ j)(-j,\ -i)$, for $i \neq j$.

Identifying the sets $[\pm n]$ and $[2n]$ through the map $i \mapsto i$ for $i \in [n]$ and $i \mapsto n-i$ for $i \in \{-1, -2, \ldots, -n\}$, allows us to identify the hyperoctahedral group W with the subgroup U of $\mathfrak{S}_{2n}$ which commutes with the permutation $(1, n+1)(2, n+2)\cdots(n, 2n)$. For example, the signed permutations $(1, 3)$ and $(2, -3)(-2, 3)$ in the hyperoctahedral group of type B_3 correspond to the permutations (1 3) and (2 6)(5 3) in the symmetric group $\mathfrak{S}_6$. The set $\mathrm{NC}(U)$ is a sublattice of $\mathrm{NC}(\mathfrak{S}_{2n})$, isomorphic to $\mathrm{NC}(W)$ (see [1]). It follows that the map sending a signed permutation of $[\pm n]$ to the B_n set partition of $[\pm n]$ given by its cycle structure defines an isomorphism between $\mathrm{NC}(W)$ and the B_n set partitions in which no two arcs cross.

The type B_n root system consists on the set of $2n^2$ vectors

$$\Phi = \{\pm e_i : 1 \leq i \leq n\} \cup \{\pm e_i \pm e_j : i \neq j, 1 \leq i, j \leq n\},$$

and we take

$$\Phi^+ = \{e_i : 1 \leq i \leq n\} \cup \{e_i \pm e_j : 1 \leq j < i \leq n\}$$

as a choice of positive roots. Changing the notation slightly from the one used for $\mathfrak{S}_n$, let $r_1 := e_1$ and $r_i := e_i - e_{i-1}$, for $i = 2, \ldots, n$. The set

$$\Delta := \{r_1, r_2, \ldots, r_n\}$$

is a simple system for W, and easy computations show that

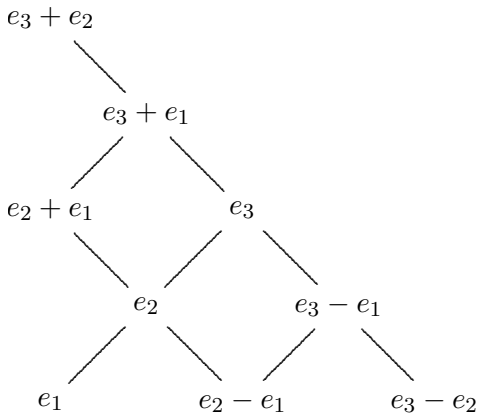
$$\begin{aligned} e_i &= \sum_{k=1}^{i} r_k, \\ e_i - e_j &= \sum_{k=j+1}^{i} r_k \qquad\qquad \text{if } i > j, \\ e_i + e_j &= 2\sum_{k=1}^{j} r_k + \sum_{k=j+1}^{i} r_k \quad \text{if } i > j. \end{aligned}$$

Each root $e_i, e_i - e_j$ and $e_i + e_j$ defines a reflection that acts on $\mathbb{R}^n$ as the permutation $(i,\ -i)$, $(i,\ j)(-i,\ -j)$ and $(i,\ -j)(-i,\ j)$, respectively, and we will identify the roots with the corresponding permutations.

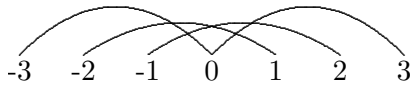
A graphical representation of a nonnesting partition $\pi \in \mathrm{NN}(B_n)$ can be drawn by placing the integers $-n, \ldots, -2, -1, 0, 1, 2, \ldots, n$, in this order, along a line and arcs between them as follows: for $i, j \in [n]$, we include an arc between i and j, and between $-i$ and $-j$, if π contains the root $e_i - e_j$; an arc between i and $-j$, and between $-i$ and j if π contains the root $e_i + e_j$; and arcs between i and 0 and $-i$ and 0 if π contains the root e_i. The presence of 0 in the ground set for nonnesting partitions is necessary to correctly represent (when present) the arc between a positive

number i an its negative (see [2]). The chains of successive arcs in the diagram become the blocks of a B_n set partition, after dropping 0, which is the partition we associate to π. This map defines a bijection between nonnesting partitions of W and B_n set partitions whose diagrams, in the above sense, contain no two arcs nested one within the other. We call this diagram the nonnesting graphical representation of π, to distinguish it from the graphical representation of the B_n set partition associated to π.

Example 3.2. *Consider the root poset of type B_3 is displayed below:*



In this root poset, the antichain $\{e_2+e_1, e_3\}$ corresponds to the B_3 set partition $\pi = \{\{3,-3\},\{1,-2\},\{-1,2\}\}$, and thus to signed permutation $(3,-3)(1,-2)(-1,2)$. The nonnesting graphical representation of π is given by



Given the positive root e_i+e_j, with $i>j$, define its support by

$$\operatorname{supp}(e_i+e_j) = \{1^d, 2^d, \ldots, j^d, j+1, \ldots, i\},$$

and let $\overline{\operatorname{supp}}(e_i+e_j) = \{1,2,\ldots,i\}$. As in type A, the integers $1,\ldots,i$ correspond to the set of indices of the simple roots that appear with nonzero coefficient in the expansion of α as a linear combination of simple roots, and the symbol k^d indicates that the coefficient of r_k in such decomposition is 2. The positive root e_i+e_j is said to have *double coefficients*, and the integer j is called the *terminal double index* of e_i+e_j. We will need also to consider the set $D_{e_i+e_j} = \{2,\ldots,j\}$. For the other two kinds of positive roots, e_i and e_i-e_j, we define the correspondent support by $\overline{\operatorname{supp}}(e_i) = \operatorname{supp}(e_i) = \{1,\ldots,i\}$, $\overline{\operatorname{supp}}(e_i-e_j) = \operatorname{supp}(e_i-e_j) = \{j+1,\ldots,i\}$, and $D_{e_i} = D_{e_i-e_j} = \varnothing$. The initial and terminal indices of $\alpha \in \Phi^+$ are, respectively, the least and greatest elements in $\overline{\operatorname{supp}}(\alpha)$. We have the following lemma.

Lemma 3.3. *Let α and β be two roots in Φ^+ with initial indices i, i' and terminal indices j, j', respectively, such that $i \le i'$. If neither α nor β have*

double coefficients, then $\{\alpha, \beta\}$ *is an antichain if and only if* $i < i'$ *and* $j < j'$. *If* α *has double coefficients, then* $\{\alpha, \beta\}$ *is an antichain if and only if* $j < j'$ *and* $D_\alpha \supset D_\beta$.

The root system of type C_n is obtained from the root system of type B_n by replacing the roots e_i by $2e_i$,

$$\Phi = \{\pm 2e_i : 1 \leq i \leq n\} \cup \{\pm e_i \pm e_j : i \neq j, 1 \leq i, j \leq n\}.$$

For $n \geq 3$, the two root systems are not congruent, but their corresponding Weyl groups, generated by reflections orthogonal to the roots, are clearly the same. Thus, the relations between roots of type C_n and signed permutations of $[\pm n]$ is the same as in type B_n. We let

$$\Phi^+ = \{2e_i : 1 \leq i \leq n\} \cup \{e_i \pm e_j : 1 \leq j < i \leq n\}$$

as a choice of positive roots, and put $r_1 := 2e_1$ and $r_i := e_i - e_{i-1}$, for $i = 2, \ldots, n$. The set

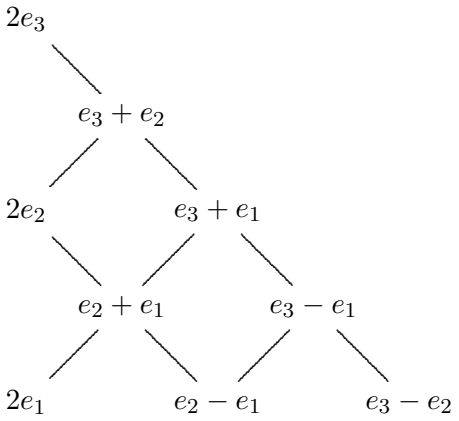
$$\Delta := \{r_1, r_2, \ldots, r_n\}$$

is a simple system for W, and easy computations show that

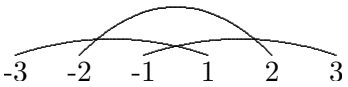
$$\begin{aligned}
2e_i &= r_1 + \sum_{k=2}^{i} 2r_k, \\
e_i - e_j &= \sum_{k=j+1}^{i} r_k \qquad \text{if } i > j, \\
e_i + e_j &= r_1 + \sum_{k=2}^{j} 2r_k + \sum_{k=j+1}^{i} r_k \quad \text{if } i > j, \\
e_i + e_1 &= \sum_{k=1}^{i} r_k.
\end{aligned}$$

The notions of C_n set partition and noncrossing partitions of type C_n coincide with its type B_n analogs, as well as the notions of opener, closer and transient. Also, the nonnesting diagram associated to a nonnesting partition $\pi \in \mathrm{NN}(C_n)$ is determined as in type B_n, except that i and $-i$ are connected by an arc if π contains $2e_i$ and that 0 does not appear in the diagram. Again, this map defines a bijection between nonnesting partitions of type C_n and C_n set partitions whose diagrams, in the above sense, contain no two arcs nested one within the other.

Example 3.4. *Consider the root poset of type C_3:*



In this root poset, the antichain $\{2e_2, e_3+e_1\}$ corresponds to the C_3 set partition $\pi = \{\{2,-2\},\{1,-3\},\{-1,3\}\}$, and thus to the signed permutation $(2,-2)(1,-3)(-1,3)$. Its nonnesting diagram is represented by



Note that this set partition π is not a nonnesting partition of type B_3, since, considered in type B_3, there would be an arc linking 0 to 2 which would be nested by the arc linking -1 to 3.

Import to type C the notions of double coefficient and support of a positive root. We have the following lemma.

Lemma 3.5. *Let α and β be two roots in Φ^+ with first indices i, i' and last indices j, j', respectively, such that $i \leq i'$. If neither α nor β have double coefficients, then $\{\alpha, \beta\}$ is an antichain if and only if $i < i'$ and $j < j'$. If α has double coefficients, then $\{\alpha, \beta\}$ is an antichain if and only if $j < j'$ and $D_\alpha \supset D_\beta$.*

Although the root systems of type B_n and C_n are not congruent, there is a simple connection between the antichains in $\mathrm{NN}(B_n)$ and those in $\mathrm{NN}(C_n)$.

Proposition 3.6. *There is a bijection τ between antichains in the root posets of types B_n and C_n which preserves the triples (op, cl, tr).*

Proof. First, notice that if $\pi \in \mathrm{NN}(B_n)$ does not have the positive root e_k, for $k \in [n]$, then the set $\tau(\pi) := \pi$ is also an antichain in $\mathrm{NN}(C_n)$ which does not have the positive root $2e_k$, and both π and $\tau(\pi)$ correspond to the same B_n (C_n) set partition. In particular, $\mathfrak{T}(\pi) = \mathfrak{T}(\tau(\pi))$.

Similarly, if π possesses the root e_k, but it does not have any root with double coefficients, then the set $\tau(\pi)$, obtained by replacing e_k by $2e_k$, is again an antichain in $\mathrm{NN}(C_n)$ and both π and $\tau(\pi)$ correspond to the same B_n (C_n) set partition. Again, we find that $\mathfrak{T}(\pi) = \mathfrak{T}(\tau(\pi))$.

Assume now that $\pi \in \mathrm{NN}(B_n)$ has a positive root with double coefficients and also has the root e_k, for some $k \in [n]$. Notice that in this case, by lemma 3.5, the B_n (C_n) set partition associated with π does not correspond to an antichain in $\mathrm{NN}(C_n)$. Let $\pi' = \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$ be the antichain formed by all the roots in π having double coefficients and by the root e_k. If

$$\alpha_1 = e_a + e_b,\ \alpha_2 = e_c + e_d,\ \alpha_3 = e_e + e_f,\ \ldots,\ \alpha_{\ell-1} = e_g + e_h,\ \alpha_\ell = e_k$$

with $g < \cdots < e < c < a < b < d < f < \cdots < h < k$, then define the following C_n roots

$$\alpha'_1 = 2e_b,\ \alpha'_2 = e_a + e_d,\ \alpha'_3 = e_c + e_f,\ \ldots,\ \alpha'_\ell = e_g + e_k.$$

Let $\tau(\pi)$ be the set formed by replacing each root α_i, $i = 1, \ldots, \ell$, in π by the C_n roots α'_i, $i = 1, \ldots, \ell$. By lemma 3.5, $\tau(\pi)$ is an antichain in $\mathrm{NN}(C_n)$ and it is straightforward to see that $\mathfrak{T}(\pi) = \mathfrak{T}(\tau(\pi))$. Also, notice that by lemma 3.3, the B_n (C_n) set partition associated with $\tau(\pi)$ does not correspond to an antichain in $\mathrm{NN}(B_n)$.

This map τ is clearly injective, and since both $\mathrm{NN}(B_n)$ and $\mathrm{NN}(C_n)$ have the same cardinality, it establishes a bijection between antichains in the root posets of types B_n and C_n which preserves the triples (op, cl, tr). □

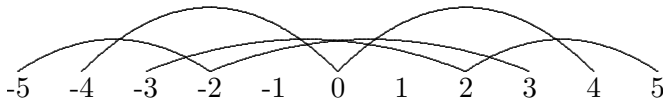
Example 3.7. *Consider the antichain*

$$\pi = \{e_2 + e_3, e_4, e_5 - e_2\}$$

is $\mathrm{NN}(B_5)$, *associated to the* B_5 *(*C_5*) set partition*

$$\{\{4, -4\}, \{-3, 2, 5\}, \{-5, -2, 3\}, \{1\}, \{-1\}\}.$$

Its nonnesting diagram is represented by



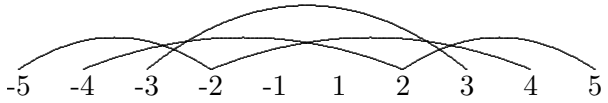
Note that this set partition does not correspond to a nonnesting diagram of type C_5*, since in this case the arc linking* -4 *to* 4 *would nest the arc between* -3 *and* 2*. Applying the construction given in the proposition above, we get the antichain*

$$\tau(\pi) = \{2e_3, e_4 + e_2, e_5 - e_2\}$$

in $\mathrm{NN}(C_5)$*, corresponding to the* B_5 *(*C_5*) set partition*

$$\{\{3, -3\}, \{-4, 2, 5\}, \{-5, -2, 4\}, \{1\}, \{-1\}\},$$

whose nonnesting diagram is represented by



As before, the B_5 *(*C_5*) set partition associated with* $\tau(\pi)$ *does not corresponds to a nonnesting diagram of type* B_5*, since in this case the arc linking* -3 *to* 0 *would be nested by the arc linking* -4 *to* 2*.*

In what follows we will identify antichains in the root posets of types B_n and C_n by the bijection τ, *i.e.*, if π is an antichain in $\mathrm{NN}(C_n)$, we will consider its image $\tau(\pi)$ in $\mathrm{NN}(B_n)$.

4. Main result

Let Φ denote a root system of type A, B or C, and let Φ^+ and Δ be defined as above. In view of lemmas 3.1 and 3.3, we consider antichains $\{\alpha_1, \ldots, \alpha_m\}$ in Φ^+ as ordered m-tuples numbered so that if i_ℓ is the terminal index of α_ℓ, then $i_1 < \cdots < i_m$.

Definition 4.1. *Given two positive roots α and β, with β having no double coefficient, and such that the intersection of their supports is nonempty, define their $\oplus$-sum, denoted $\alpha \oplus \beta$, and their $\ominus$-difference, denoted $\alpha \ominus \beta$, as the positive roots with supports*

$$\mathrm{supp}(\alpha \oplus \beta) := \mathrm{supp}(\alpha) \cup (\mathrm{supp}(\beta) \setminus \overline{\mathrm{supp}}(\alpha))$$

and

$$\mathrm{supp}(\alpha \ominus \beta) := \overline{\mathrm{supp}}(\alpha) \cap \mathrm{supp}(\beta),$$

respectively. If moreover α has double coefficients, then define also the $\ominus^d$-difference, denoted $\alpha \ominus^d \beta$, as the positive root with support

$$\mathrm{supp}(\alpha \ominus^d \beta) := D_\alpha \cap \mathrm{supp}(\beta).$$

Example 4.2. *Given the type B_3 positive roots $\alpha = e_3 + e_2$ and $\beta = e_4 - e_1$ we have*

$$\alpha \oplus \beta = e_4 + e_2, \ \alpha \ominus \beta = e_3 - e_1, \ \textit{and} \ \ \alpha \ominus^d \beta = e_2 - e_1.$$

An antichain $(\alpha_1, \ldots, \alpha_m)$ is said to be *connected* if the intersection of the supports of any two adjacent roots α_i, α_{i+1} is non empty. The connected components

$$(\alpha_1, \ldots, \alpha_i), \ (\alpha_{i+1}, \ldots, \alpha_j), \ \ldots, \ (\alpha_k, \ldots, \alpha_m)$$

of an antichain $\alpha = (\alpha_1, \ldots, \alpha_m)$ are the connected sub-antichains of α for which the supports of the union of the roots in any two distinct components are disjoint. For instance, the antichain $(e_2, e_3 - e_1, e_4 - e_3)$ has the connected components $(e_2, e_3 - e_1)$ and $e_4 - e_3$. We will use lower and upper arcs to match two roots in a connected antichain in a geometric manner. Two roots linked by a lower [respectively upper] arc are said to be l-linked [respectively, u-linked]. In what follows we will identify each root with the correspondent permutation.

For clarity of the exposition, we start by presenting the map for type A.

Definition 4.3. *Define the map f from the set $\mathrm{NN}(A_{n-1})$ into $\mathrm{NC}(A_{n-1})$ recursively as follows. When α_1 is a positive root we set $f(\alpha_1) := \alpha_1$.*

If $\alpha = (\alpha_1, \ldots, \alpha_m)$ is a connected antichain with $m \geq 2$, then define

$$f(\alpha) := \left(\bigoplus_{k=1}^{m} \alpha_k \right) f(\overline{\alpha}_2, \ldots, \overline{\alpha}_m),$$

where $\overline{\alpha}_k = \alpha_{k-1} \ominus \alpha_k$ *for* $k = 2, \ldots, m$.

For the general case, if $(\alpha_1, \ldots, \alpha_i)$, $(\alpha_{i+1}, \ldots, \alpha_j)$, ..., $(\alpha_k, \ldots, \alpha_m)$ *are the connected components of* $(\alpha_1, \ldots, \alpha_m)$, *let*

$$f(\alpha_1, \ldots, \alpha_m) := f(\alpha_1, \ldots, \alpha_i) f(\alpha_{i+1}, \ldots, \alpha_j) \cdots f(\alpha_k, \ldots, \alpha_m).$$

By its definition, it is clear that $(\overline{\alpha}_2, \ldots, \overline{\alpha}_m)$ is an antichain, which need not be connected. In the next definition we will generalize the map f to types B_n and C_n. Before, however, lets consider the following example.

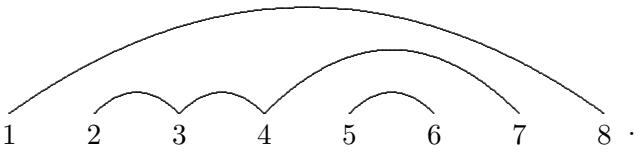
Example 4.4. *Consider the antichain*

$$\alpha = (e_3 - e_1, e_4 - e_2, e_6 - e_3, e_7 - e_4, e_8 - e_5)$$

in the root poset of type A_7, *corresponding to the permutation* $(136)(247)(58)$ *in the symmetric group* $\mathfrak{S}_8$. *Applying the map* f *to* α, *we get the noncrossing partition*

$$\begin{aligned} f(\alpha) &= (e_8 - e_1) f(e_3 - e_2, e_4 - e_3, e_6 - e_4, e_7 - e_5) \\ &= (e_8 - e_1)(e_3 - e_2)(e_4 - e_3) f(e_6 - e_4, e_7 - e_5) \\ &= (e_8 - e_1)(e_3 - e_2)(e_4 - e_3)(e_7 - e_4)(e_6 - e_5) \\ &\equiv (18)(2347)(56), \end{aligned}$$

whose graphical representation is given by



Consider now the general case, where Φ denotes a root system of type A, B or C.

Definition 4.5. *Define the map* f *from the set* $\mathrm{NN}(\Phi)$ *into* $\mathrm{NC}(\Phi)$ *recursively as follows. When* α_1 *is a positive root we set* $f(\alpha_1) := \alpha_1$. *If* $\alpha = (\alpha_1, \ldots, \alpha_m)$ *is a connected antichain with* $m \geq 2$, *we have two cases:*

(a) *If there are no double coefficients in the antichain, then as in the previous definition we set*

$$f(\alpha) := \left(\bigoplus_{k=1}^{m} \alpha_k \right) f(\overline{\alpha}_2, \ldots, \overline{\alpha}_m),$$

where $\overline{\alpha}_k = \alpha_{k-1} \ominus \alpha_k$ *for* $k = 2, \ldots, m$.

(b) *Assume now that* α_1, ..., α_ℓ *have double coefficients, for some* $\ell \geq 1$, *and* $\alpha_{\ell+1}$, ..., α_m *have none. Let* $\Gamma_d := (\alpha_1, \ldots, \alpha_\ell)$ *and* $\Gamma := (\alpha_{\ell+1}, \ldots, \alpha_m)$. *We start by introducing l-links as follows.*

Let m' *be the largest index of elements in* Γ *such that the following holds:* $\alpha_{m'}$ *has initial index* $i \neq 1$, *so that there is a rightmost element in* Γ_d, *say* α_k, *with* $i \in D_{\alpha_k}$. *If there is such an integer* m', *l-link* α_k *with* $\alpha_{m'}$. *Then, ignore* α_k *and* $\alpha_{m'}$ *and proceed with the*

remaining roots as before. This procedure terminates after a finite number of steps (and not all elements of α need to be l-linked).

Next proceed by introducing u-links in α. The starting point of u-links, which we consider drawn from right to left, will be elements in Γ that have no initial index 1 and are not l-linked. We will refer to these elements as admissible roots. So, let m' be the smallest integer such that the following holds: $\alpha_{m'}$ is an admissible root with initial index $i \neq 1$ so that there is a leftmost element, say α_k, not yet u-linked to an element on its right and such that $i \in \overline{\mathrm{supp}}(\alpha_k)$. If there is such an integer m', u-link α_k with $\alpha_{m'}$. Remove $\alpha_{m'}$ from the set of admissible roots and proceed as before. Again this process terminates after a finite number of steps.

Finally, let $T = \{t_1 < \cdots < t_p\}$ be the collection of all terminal double indices of the roots in Γ_d not l-linked, and all the terminal indices of the roots in α not u-linked to an element on its right. Then, define

$$f(\alpha) := \pi_1 \cdots \pi_\ell \, \pi_0 \, \theta_1 \cdots \theta_q \, f(\theta_{q+1}, \ldots, \theta_s),$$

where for $j = 1, \ldots, \ell$, $\pi_j = e_{j'} + e_{j''}$, with j' and j'' respectively the leftmost and rightmost integers in T not considered yet; π_0 is either the root e_{i_j}, if the initial index of $\alpha_{\ell+1}$ is 1, with i_j the only integer in T not used yet for defining the roots π_j, or the identity otherwise; each θ_j, $j = 1, \ldots, q$ is the $\ominus^d$-difference of l-linked roots, starting from the rightmost one in Γ_d, and each θ_j, $j = q+1, \ldots, s$ is the $\ominus$-difference of u-linked roots, starting from the leftmost one in Γ.

(c) *For the general case, if $(\alpha_1, \ldots, \alpha_i)$, $(\alpha_{i+1}, \ldots, \alpha_j)$, …, $(\alpha_k, \ldots, \alpha_m)$ are the connected components of $(\alpha_1, \ldots, \alpha_m)$, let*

$$f(\alpha_1, \ldots, \alpha_m) := f(\alpha_1, \ldots, \alpha_i) f(\alpha_{i+1}, \ldots, \alpha_j) \cdots f(\alpha_k, \ldots, \alpha_m).$$

Remark: (*i*) Notice that the type A case is given by conditions (a) and (c) of the above definition. Also, note that if all roots in α have double coefficients then condition (b) is vacuous and the map f reduces to the identity map. We point out that the number of roots in $f(\alpha)$ is equal to the number of roots in the antichain α.

(*ii*) The sequence $(\overline{\alpha}_2, \ldots, \overline{\alpha}_m)$ obtained in step (a) is a (not necessarily connected) antichain. It is easy to check that after all l-links and all u-links are settled, the set T has an odd number of elements if and only if the initial index of $\alpha_{\ell+1}$ is 1. Thus, the root π_0 given in condition (b) is well defined.

We will show that f establishes a bijection between the sets $\mathrm{NN}(\Psi)$ and $\mathrm{NC}(\Psi)$, for $\Psi = A_{n-1}, B_n$ or C_n. Before, however, we present some examples.

Example 4.7. *Consider now the antichain*

$$\alpha = (e_5 + e_4, e_6 + e_2, e_7, e_8 - e_2, e_9 - e_3)$$

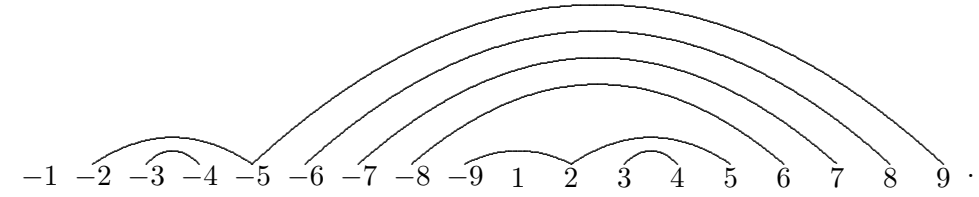
in the root poset B_9. Following definition 4.5, we get the l-links and the u-links

$$\alpha = (e_5 + e_4, e_6 + e_2, e_7, e_8 - e_2, e_9 - e_3) \quad .$$

Therefore, $T = \{2, 6, 7, 8, 9\}$ and the application of f to α yields

$$\begin{aligned} f(\alpha) &= (e_9 + e_2)(e_8 + e_6)(e_7)(e_4 - e_3)f(e_5 - e_2) \\ &\equiv (2, -9)(-2, 9)(6, -8)(-6, 8)(7, -7)(3, 4)(-3, -4)(2, 5)(-2, -5) \\ &= (2, 5, -9)(-2, -5, 9)(6, -8)(-6, 8)(7, -7)(3, 4)(-3, -4). \end{aligned}$$

The image $f(\alpha)$ is a noncrossing partition in $[\pm 9]$, as we may check in its representation



Example 4.8. *For a final example, consider the antichain*

$$\alpha = (e_6 + e_5, e_7 + e_4, e_8 + e_3, e_9 - e_1, e_{10} - e_4, e_{11} - e_6)$$

in the root poset of type B_{11}. The l-links and u-links are shown below, so that $T = \{4, 6, 8, 9, 10, 11\}$:

$$\alpha = (e_6 + e_5, e_7 + e_4, e_8 + e_3, e_9 - e_1, e_{10} - e_4, e_{11} - e_6) \quad .$$

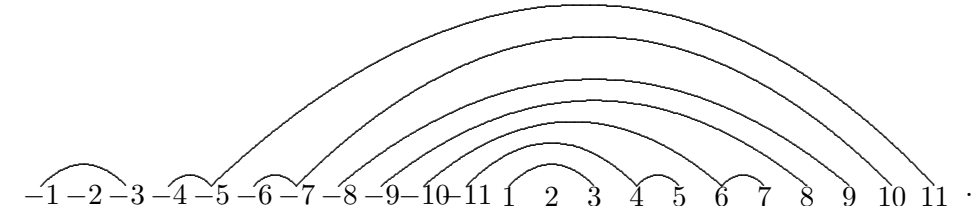
Therefore, the application of f to α gives

$$f(\alpha) = (e_{11} + e_4)(e_{10} + e_6)(e_9 + e_8)(e_3 - e_1)(e_5 - e_4)f(e_7 - e_6)$$

and thus, $f(\alpha)$ is the noncrossing partition

$$\begin{aligned} &(4, -11)(-4, 11)(6, -10)(-6, 10)(8, -9)(-8, 9) \\ &\quad \cdot (1, 3)(-1, -3)(4, 5)(-4, -5)(6, 7)(-6, -7) \\ &\quad = (4, 5, -11)(-4, -5, 11)(6, 7, -10) \\ &\quad \quad \cdot (-6, -7, 10)(8, -9)(-8, 9)(1, 3)(-1, -3) \end{aligned}$$

represented by



Lemma 4.9. *If $\alpha \in \mathrm{NN}(B_n)$ then $f(\alpha) \in \mathrm{NC}(B_n)$, and $\mathfrak{T}(\alpha) = \mathfrak{T}(f(\alpha))$.*

Proof. Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be an antichain in the root poset of type B_n, and let $(\alpha_1, \ldots, \alpha_w)$ be its first connected component. Start by assuming that $1 \notin \overline{\mathrm{supp}}(\alpha_i)$, for all $i = 1, \ldots, m$. We will use induction on $m \geq 1$ to show that in this case $f(\alpha)$ is a noncrossing partition on the set $\{i-1, \ldots, q, -(i-1), \ldots, -q\}$, where i is the initial index of α_1 and q is the terminal index of α_m, and such that each positive integer is sent to another positive integer. The result is clear when $m = 1$. So, let $m \geq 2$ and assume that the result holds for antichains of length less than, or equal to $m-1$. Then, we may write

$$f(\alpha) = \left(\bigoplus_{k=1}^{w} \alpha_k \right) f(\overline{\alpha}_2, \ldots, \overline{\alpha}_w) f(\alpha_{w+1}, \ldots, \alpha_m),$$

where each $\overline{\alpha}_k = \alpha_{k-1} \ominus \alpha_k$, for $k = 2, \ldots, w$. By the inductive step, $f(\overline{\alpha}_2, \ldots, \overline{\alpha}_w) \equiv \pi_1$ and $f(\alpha_{w+1}, \ldots, \alpha_m) \equiv \pi_2$ are noncrossing partitions on the sets

$$\{a-1, \ldots, b, -(a-1), \ldots, -b\} \text{ and } \{p-1, \ldots, q, -(p-1), \ldots, -q\},$$

respectively, where a and p are the initial indices of $\overline{\alpha}_2$ and α_{w+1}, respectively, and b and q are the terminal indices of $\overline{\alpha}_w$ and α_m, respectively. Moreover, all positive integers are sent to positive ones by π_1 and π_2. Denoting by j the terminal index of α_w, we get

$$\bigoplus_{k=1}^{w} \alpha_k = e_j - e_{i-1} \equiv (i-1, j)(-(i-1), -j)$$

with $i - 1 < a - 1 < b < j \leq p - 1 < q$. Therefore

$$f(\alpha) \equiv (i-1, j)(-(i-1), -j)\pi_1\pi_2$$

is a noncrossing partition on the set $\{i-1, \ldots, q, -(i-1), \ldots, -q\}$ sending each positive integer to another positive integer.

Note that for the rest of the proof, we may assume without loss of generality that α is connected, since none of the connected components of an antichain, except possibly for the first one, have double coefficients, and therefore their images are noncrossing partitions sending each positive integer to another positive integer.

Suppose now that $1 \in \mathrm{supp}(\alpha_1)$. We will show that $f(\alpha)$ is a noncrossing partition on the set $\{i-1, \ldots, q, -(i-1), \ldots, -q\}$, where i is the initial index of α_2 and q is the terminal index of α_m, and such that one and only one positive integer is sent to a negative one. The result is certainly true for $m = 1$, and when $m > 1$ we have

$$f(\alpha) = \left(\bigoplus_{k=1}^{m} \alpha_k \right) f(\overline{\alpha}_2, \ldots, \overline{\alpha}_m),$$

where $\oplus_{k=1}^{m} \alpha_k \equiv (q, -q)$, and $\overline{\alpha}_k = \alpha_{k-1} \ominus \alpha_k$ for $k = 2, \ldots, m$. By the previous case, $f(\overline{\alpha}_2, \ldots, \overline{\alpha}_m) \equiv \pi$ is a noncrossing partition on the set $\{i-1, \ldots, j, -(i-1), \ldots, -j\}$, with i the initial index of α_2 and $j < q$

the terminal index of α_{m-1}. Therefore, $f(\alpha) \equiv (q, -q)\pi$ is a noncrossing partition satisfying the desired conditions.

Next, assume that α satisfies condition (b) of definition 4.5, and consider its image

$$f(\alpha) = \pi_1 \cdots \pi_\ell \, \pi_0 \, \theta_1 \cdots \theta_q \, f(\theta_{q+1}, \ldots, \theta_s).$$

By the construction of the set T, it follows that each D_{α_j}, $j = 1, \ldots, \ell$, is contained in D_{π_i}, for some $i = 1, \ldots, \ell$, and that $\pi_1 \cdots \pi_\ell \pi_0$ is a noncrossing partition, sending each nonfixed positive integer to a negative one. Note also that the support of each θ_j, $j = 1, \ldots, q$, is contained in some D_{α_i}, $i = 1, \ldots, \ell$, and therefore, in some D_{π_i}, $i = 1, \ldots, \ell$. Moreover, the supports of any two roots θ_i and θ_j, $1 \leq i, j \leq q$, are either disjoint, or one of them is contained into the other one. Therefore $\theta_1 \cdots \theta_q$ is a noncrossing partition sending each nonfixed positive integer into another positive integer. By the previous cases, $f(\theta_{q+1}, \ldots, \theta_s)$ is also a noncrossing partition sending each nonfixed positive integer into another positive integer. Again by the construction of the set T, we find that the support of each θ_j, $j = q+1, \ldots, s$, is either contained in some D_{π_i}, or it does not intersect D_{π_ℓ}. For each $j = 1, \ldots, q$ and $i = q+1, \ldots, s$, either we have $\mathrm{supp}(\theta_i) \cap \mathrm{supp}(\theta_j) = \varnothing$, or $\mathrm{supp}(\theta_i) \supseteq \mathrm{supp}(\theta_j)$, this last case happening when θ_i arises from the $\ominus$-difference of two u-linked roots $\alpha_u \in \Gamma_d$ and $\alpha_v \in \Gamma$, and there is some $\alpha_{v+k} \in \Gamma$, $k \geq 1$, l-linked to α_u, whose $\ominus^d$-difference gives θ_j. Therefore, it follows that $f(\alpha)$ is noncrossing.

To see that $\mathfrak{T}(\alpha) = \mathfrak{T}(f(\alpha))$, notice that if $a \in op(\alpha)$, then α must have a root $e_b - e_a$, for some $b > a + 1$, and cannot have neither a root with terminal double index equal to a nor a root with terminal index equal to a. By its construction, the same is true for the sequence $f(\alpha)$, and therefore $a \in op(f(\alpha))$. Assume now that $a \in tr(\alpha)$. Then, a must appear either as the terminal index or terminal double index of a root, and $a + 1$ as the initial index of another root. Again the same will happen in the sequence $f(\alpha)$, and thus $a \in tr(f(\alpha))$. It follows that $\mathfrak{T}(\alpha) = \mathfrak{T}(f(\alpha))$. □

With some minor adaptations, the proof of lemma 4.9, in the case where $1 \notin \overline{\mathrm{supp}}(\alpha_i)$ for $i = 1, \ldots, m$, gives the type A analog of the previous result, and its type C analog is obtained through the bijection τ.

Corollary 4.10. *If $\alpha \in \mathrm{NN}(\Psi)$ then $f(\alpha) \in \mathrm{NC}(\Psi)$, for $\Psi = A_{n-1}$ or C_n, and $\mathfrak{T}(\alpha) = \mathfrak{T}(f(\alpha))$.*

We will now construct the inverse function of f, thus showing that f establishes a bijection between the sets $\mathrm{NN}(\Psi)$ and $\mathrm{NC}(\Psi)$, for $\Psi = A_{n-1}, B_n$ or C_n. For that purpose, recall the following property.

Lemma 4.11. *Two distinct transpositions (a, b) and (i, j) in $\mathfrak{S}_n$ commute if and only if the sets $\{i, j\}$ and $\{a, b\}$ are disjoint.*

If $\pi_1 \cdots \pi_p$ is the cycle structure of a signed permutation π, then for each cycle $\pi_i = (ij \cdots k)$ there is another cycle $\pi_j = (-i - j \cdots - k)$. Denote by π_i'

the cycle in $\{\pi_i, \pi_j\}$ having the smallest positive integer (when $\pi_i = \pi_j$ then π_i' is just π_i), and call positive cycle structure to the subword of $\pi_1 \cdots \pi_p$ formed by the cycles π_i'. Extend this definition to permutations in $\mathfrak{S}_n$ by identifying positive cycle structure with cycle structure.

Theorem 4.12. *The map $f : \mathrm{NN}(\Psi) \to \mathrm{NC}(\Psi)$, for $\Psi = A_{n-1}$, B_n or C_n, is a bijection between sets which preserves the triples $(op(\pi), cl(\pi), tr(\pi))$.*

Proof. We will construct the inverse map $g : \mathrm{NC}(\Psi) \to \mathrm{NN}(\Psi)$ of f, for $\Psi = B_n$. The other cases are analogous. Given $\pi \in \mathrm{NC}(B_n)$, let $\pi_1 \cdots \pi_s$ be its positive cycle structure. Replace each cycle $\pi_i = (i_1 i_2 \cdots i_k)$ by $(i_1 i_2)(i_2 i_3) \cdots (i_{k-1} i_k)$, if $i_\ell > 0$ for $\ell = 1, \ldots, k$, or by

$$\pi_i = (i_1 i_{j+1})(i_1 i_2)(i_2 i_3) \cdots (i_{j-1} i_j)(i_{j+1} i_{j+2}) \cdots (i_{k-1} i_k),$$

if $i_\ell > 0$ for $\ell = 1, \ldots, j$, and $i_\ell < 0$ for $\ell = j+1, \ldots, k$. Next, bearing in mind lemma 4.11 and recalling that π is noncrossing, move all transpositions (i, j), with $i > 0$ and $j < 0$ (if any), to the leftmost positions and order them by its least positive element, and order all remaining transpositions (i, j), with $i, j > 0$, by its least positive integer. Replace each transposition (ij) by its correspondent root in the root system of type Ψ, and let

$$(\alpha_1, \ldots, \alpha_k)(\alpha_{k+1}, \ldots, \alpha_\ell) \cdots (\alpha_m, \ldots, \alpha_n) \tag{4.1}$$

be the correspondent sequence of roots, divided by its connected components. Note that given two distinct roots in (4.1), the sets formed by the initial and terminal indices, if there are no double coefficients, or by the terminal and terminal double indices, otherwise, are clearly disjoint.

We start by considering that the sequence (4.1) has only one connected component $(\alpha_1, \ldots, \alpha_k)$. Let $\Gamma_d = (\alpha_1, \ldots, \alpha_r)$ be the subsequence formed by the roots having double coefficients, and denote by $\Gamma = (\alpha_{r+1}, \ldots, \alpha_k)$ the remaining subsequence. Define $\Gamma' = \Gamma_d' = \varnothing$. If Γ_d is not empty and $r \neq k$, apply the following algorithm:

> Let $\overline{\Gamma}$ be the subsequence of Γ obtained by striking out the root α_{r+1} if its initial index is 1. While $\overline{\Gamma} \neq \varnothing$, repeat the following steps:
>
> (i) Let α_i be the leftmost root in $\overline{\Gamma}$ and check if $\mathrm{supp}(\alpha_i) \subseteq D_{\alpha_j}$, for some $\alpha_j \in \Gamma_d \setminus \Gamma_d'$.
>
> (ii) If so, let α_{i_j} be the rightmost root in $\Gamma_d \setminus \Gamma_d'$ with this property. Update Γ' by including in it the rightmost root $\overline{\alpha}$ of $\overline{\Gamma}$ whose support is contained in $\mathrm{supp}(\alpha_i)$. Update $\overline{\Gamma}$ by striking out the root $\overline{\alpha}$ and update Γ_d' by including in this set the root α_{i_j}.
>
> (iii) Otherwise, update $\overline{\Gamma}$ by striking out the root α_i.

Next, let $T = \{t_1 > \cdots > t_r\}$ be the set formed by all terminal double indices of the roots in $\Gamma_d \setminus \Gamma_d'$ and by the terminal indices of the roots in Γ'; let $F_{st} = \{f_{r+1} < \cdots < f_k\}$ be the set formed by the initial indices of the roots in Γ, and let $L_{st} = \{\ell_1 < \cdots < \ell_k\}$ be the set formed by the terminal indices of the roots in $(\Gamma \setminus \Gamma') \cup \Gamma_d$ and by the terminal double indices of

the roots in Γ'_d. By this construction, we have $f_i < \ell_i$ for $i = 1, \ldots, r$, and $f_i < \ell_i$, for $i = r+1, \ldots, k$. Then, define

$$g(\pi) = (\overline{\alpha}_1, \ldots, \overline{\alpha}_k),$$

where for $i = 1, \ldots, r$, $\overline{\alpha}_i = e_{\ell_i} + e_{t_i}$, and for $i = r+1, \ldots, k$, $\overline{\alpha}_i = e_{\ell_i} - e_{f_i - 1}$.

For the general case define

$$g(\pi) = g(\alpha_1, \ldots, \alpha_k) g(\alpha_{k+1}, \ldots, \alpha_\ell) \cdots g(\alpha_m, \ldots, \alpha_n).$$

It is clear from this construction that $g(\pi)$ is an antichain in the root poset of type Ψ. Moreover, a closer look at the construction of the map f shows that g is the inverse of f. Thus, f (and g) establishes a bijection between nonnesting and noncrossing partitions of types A, B and C. □

In the following examples we illustrate the application of the map g.

Example 4.13. *Consider the cycle structure of the noncrossing partition* $\pi = (18)(2347)(56)$ *in the symmetric group* $\mathfrak{S}_8$ *used in example 4.4. Following the proof of Theorem 4.12, write*

$$\begin{aligned} \pi &\equiv (18)(2347)(56) \\ &= (18)(23)(34)(47)(56) \\ &\equiv (e_8 - e_1)(e_3 - e_2)(e_4 - e_3)(e_7 - e_4)(e_6 - e_5). \end{aligned}$$

Note that π *has only one connected component, and there are no double coefficients. Next define the sets*

$$F_{st} = \{1, 2, 3, 4, 5\} \text{ and } L_{st} = \{2, 3, 5, 6, 7\}.$$

Thus, we find that the image of π *by the map* g *is the antichain*

$$g(\pi) = (e_3 - e_1, e_4 - e_2, e_6 - e_3, e_7 - e_4, e_8 - e_5).$$

Example 4.14. *Consider now the noncrossing partition*

$$\pi = (2, 5, -9)(-2, -5, 9)(6, -8)(-6, 8)(7, -7)(3, 4)(-3, -4)$$

obtained in Example 4.7. Its positive cycle structure is

$$(2, -9)(2, 5)(6, -8)(7, -7)(3, 4) = (2, -9)(6, -8)(7, -7)(2, 5)(3, 4),$$

and thus we get

$$\pi \equiv (e_9 + e_2, e_8 + e_6, e_7, e_5 - e_2, e_4 - e_3).$$

Next, construct the sets

$$T = \{4, 2\}, \ F_{st} = \{1, 3, 4\}, \ \text{and} \ \ L_{st} = \{5, 6, 7, 8, 9\}.$$

Therefore, the image of π *by the map* g *is the antichain*

$$(e_5 + e_4, e_6 + e_2, e_7, e_8 - e_2, e_9 - e_3).$$

Finally, in the next result we prove that the map f generalizes the bijection that locally converts each crossing to a nesting.

Theorem 4.15. *When restricted to the type* A_{n-1} *case, the map* f *coincides with the L-map.*

Proof. Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be an antichain in the root poset of type A_{n-1}. The result will be handled by induction over $m \geq 1$. Without loss of generality, we may assume that α is connected, since otherwise there is an integer $1 < k < n-1$ such that each integer less (resp. greater) than k is sent by α to an integer that still is less (resp. greater) that k. Therefore, the same happens with the image of α by either the map f or the L-map.

The result is vacuous when $m = 1$, and when $m = 2$, the only connected nonnesting partition which does not stay invariant under the maps f and L is $\alpha = (e_{i'+1} - e_i)(e_{j'+1} - e_j)$, for some integers $1 \leq i < j < i' < j' \leq n-1$. In this case, the equality between f and the L-map is obvious. So, let $m > 2$ and assume the result for antichains of length $\leq m-1$. Let i and j be, respectively, the first and last indices of α_1 and α_m. Then,

$$f(\alpha) = (e_{j+1} - e_i) f(\overline{\alpha}_2, \ldots, \overline{\alpha}_m),$$

where each $\overline{\alpha}_k = \alpha_{k-1} \ominus \alpha_k$ for $k \geq 2$, and the antichain $(\overline{\alpha}_2, \ldots, \overline{\alpha}_m)$ is clearly nonnesting, and not necessarily connected. By the inductive step, $f(\overline{\alpha}_2, \ldots, \overline{\alpha}_m) = L(\overline{\alpha}_2, \ldots, \overline{\alpha}_m)$. Moreover, note that converting, from left to right, each local crossing between the first root and the leftmost root in α whose arcs cross, into a nesting gives, precisely,

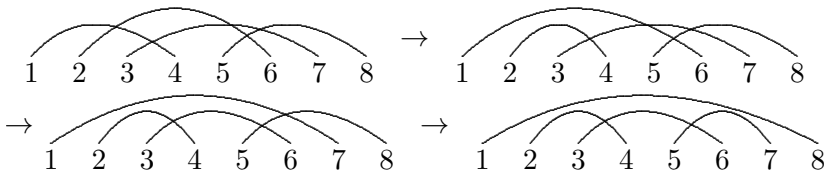
$$(e_{j+1} - e_i) L(\overline{\alpha}_2, \ldots, \overline{\alpha}_m),$$

and this operation may be considered the first step of the L-map. Thus, we find that $f(\alpha) = L(\alpha)$. □

Example 4.16. *Consider the antichain* $\alpha = (e_4 - e_1, e_6 - e_2, e_7 - e_3, e_8 - e_5)$ *in the root poset of type* A_7*. Applying the map* f *we get*

$$\begin{aligned} f(\alpha) &= (e_8 - e_1) f(e_4 - e_2, e_6 - e_3, e_7 - e_5) \\ &= (e_8 - e_1)(e_7 - e_2) f(e_4 - e_3, e_6 - e_5) \\ &= (e_8 - e_1)(e_7 - e_2)(e_4 - e_3)(e_6 - e_5) \equiv (18)(27)(34)(56). \end{aligned}$$

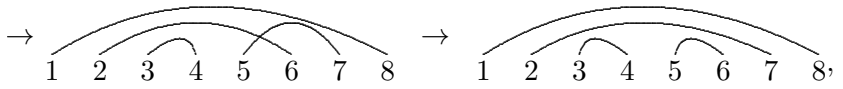
On the other hand, applying the L-map to each crossing between the first root and the leftmost root in α *whose arcs cross, we get successively*



Thus, in the first step of the L-map, we get

$$L(\alpha) = (e_8 - e_1) L(e_4 - e_2, e_6 - e_3, e_7 - e_5).$$

Continuing the application of the L-map, now replacing, by a nesting, each crossing between the second root and the leftmost root in α *whose arcs cross, we get*



and therefore, we have

$$L(\alpha) = (e_8 - e_1)(e_7 - e_2)(e_4 - e_3)(e_6 - e_5) = f(\alpha).$$

5. Concluding remarks

The three bijections [6, 11, 14] between noncrossing and nonnesting partitions are all distinct, and preserve different statistics. While our bijection preserves the triples (op, cl, tr) formed by the openers, closers and transients of the partitions, for the types A, B and C, and therefore also the number of blocks, the one by Alex Fink and Benjamin Giraldo [6] preserves the type of the partitions but not the triples (op, cl, tr). For the types A and B, Stump's bijection does not preserve neither the type nor the triples (op, cl, tr). Our construction coincides with the bijection defined by M. Rubey and C. Stump [13], but both constructions have very different designs and settings.

Acknowledgements

The author would like to thank Alexander Kovacec and Christian Krattenthaler for useful comments and suggestions, and the hospitality of The Combinatorics Group at the Fakultät für Mathematik of the Universität Wien. The author also wishes to thank the referee for helpful comments and suggestions.

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