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A GRAPH THEORETIC PROOF OF THE COMPLEXITY OF COLOURING BY A LOCAL TOURNAMENT WITH AT LEAST TWO DIRECTED CYCLES

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Abstract. In this paper we give a graph theoretic proof of the fact that deciding whether a homomorphism exists to a fixed local tournament with at least two directed cycles is NP-complete. One of the main reasons for the graph theoretic proof is that it showcases all of the techniques that have been built up over the years in the study of the digraph homomorphism problem.

1. Introduction

Let H be a fixed directed graph. A *homomorphism* from a digraph D to H is a mapping $f : V(D) \to V(H)$ such that $xy \in A(D)$ implies that $f(x)f(y) \in A(H)$. The existence of such a homomorphism is denoted by $D \to H$.

The H-colouring problem (HOM_H) is the problem of deciding whether a homomorphism exists from an input digraph D to the target digraph H.

Problem 1.1 HOM_H

Instance:	A digraph D.
Question:	Does there exist a homomorphism $f : D \to H$?

In the case where H is an undirected graph we have the following result by Hell and Nešetřil.

Theorem 1.1 (Hell and Nešetřil [17, 18]). *Let H be a graph with loops allowed.*

- *If H is bipartite or contains a loop, then the H-colouring problem has a polynomial time algorithm.*
- *Otherwise the H-colouring problem is NP-complete.*

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It has been a goal of researchers to try and extend the result by Hell and Nešetřil to the directed case. This seems to be a very hard problem and only partial results are known. One example of this is the theorem by Bang-Jensen, Hell and MacGillivray [6] on the complexity of colouring by semi-complete digraphs. A *semi-complete* digraph has the property that between every pair of vertices there is at least one arc; parallel arcs and loops are not allowed, but a pair of symmetric arcs is allowed.

Theorem 1.2 (Bang-Jensen, Hell and MacGillivray [6]). *Let H be a semi-complete digraph.*

- *If H contains at most one directed cycle, then H-colouring is polynomial time solvable.*
- *Otherwise H-colouring is NP-complete.*

There is the related notion of a locally semi-complete digraph. A digraph H, is said to be *locally semi-complete* if for every vertex v of H, both the in-neighbours of v and the out-neighbours of v induce semi-complete digraphs (separately). A special case of this is that of a local tournament. A *local tournament*, H, is a digraph such that between every pair of vertices there is at most one arc and that for every vertex v of H both $N^+(v)$ and $N^-(v)$ induce tournaments.

Bang-Jensen introduced the notion of locally semicomplete digraphs in [2] where it was shown that many of the known results on tournaments generalize to this family of digraphs. Since their introduction locally semicomplete digraphs have been studied by many authors, see [4] for a large collection of results on these digraphs.

In [7] the present authors generalized Theorem 1.2 to the class of locally semicomplete digraphs. In order to state the generalization, we need the following notation for a unicyclic locally semicomplete digraph.

Let T be a connected unicyclic locally semicomplete digraph and let C be the cycle in T. Then C is induced and forms the unique non-trivial strong component in T. It is not difficult to check (see e.g. [4]) that $T = H[D_1, D_2, \ldots, D_l]$, where $D_j = C$ for some j and $|D_i| = 1$ for all $i \neq j$ and H is an acyclic local tournament (the composition $H[D_1, D_2, \ldots, D_l]$ is defined on page 111). In particular, if $l \geq 2$ we must have that C is either a 2-cycle or a 3-cycle as every vertex of C either dominates or is dominated by some other vertex. If $T = C$ then T-colouring is polynomial so we may assume that C is either a 2-cycle or a 3-cycle.

The unicyclic locally semicomplete digraph T may also be viewed as follows. Let S be the set of neighbours (in- and out-neighbours) of the cycle in T, including vertices in C. Then $V(T) \setminus S$ is the union of two disjoint sets of vertices: those that come before S in the ordering shown above, call these A, and those that come after S in the ordering above, call these B. Define the following three induced sub-digraphs: $T_1 = T[A]$, $T_2 = T[S]$ and $T_3 = T[B]$. Each T_i is a locally semicomplete digraph and T_1 and T_3 are acyclic as well. Note that T_1 or T_3 may be empty. This general structure is illustrated in Figure 1 (with a 3-cycle) where we have written $D_t = \{y_t\}$ for $t \in \{1, 2, \ldots, l\} \setminus \{j\}$. Note that the arc $y_{j-1}y_{j+1}$ may or may not be present depending on whether T_2 is or isn't semicomplete. Furthermore, since T is a locally semicomplete digraph, $y_{i+1}, \ldots, y_{j-1}$ dominate the cycle and $y_{j+1}, \ldots, y_k$ are dominated by the cycle.

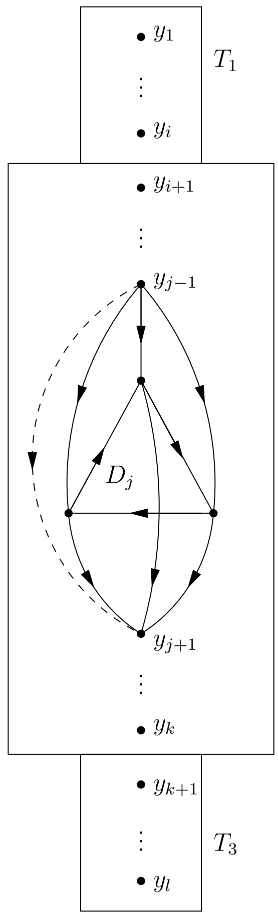

FIGURE 1. The structure of a unicyclic local tournament that is not a directed cycle.

One part of the theorem in [7] follows from a result of Barto, Kozik, and Niven [8] cited below. In order to state their result we need the following concepts.

If H' is a subgraph of H, then a *retraction* of H to H' is a homomorphism $\rho : H \to H'$ such that $\rho(x) = x$ for every $x \in V(H')$. In this case we say that H *retracts to* H' or that H' is a *retract of* H. A digraph H is said to be a *core* if H does not retract to a proper subgraph. It turns out that a digraph is a core if and only if it is not homomorphic to a proper subgraph and that every digraph H has a unique retract that is also a core [18]. This retract is called *the core of* H. If H' is the core of H, then H and H' have equivalent homomorphism problems: $G \to H$ if and only if $G \to H'$.

A digraph H is said to be *smooth* if there are no sources or sinks present in H. The complexity of HOM_H in this case was conjectured by Bang-Jensen and Hell in [5] and proved recently in [8] using techniques from universal algebra.

Theorem 1.3 (Barto, Kozik, and Niven [8])**.** *Let H be a smooth digraph. If the core of H is a directed cycle, then H-colouring is in P. Otherwise H-colouring is NP-complete.*

The generalization of Theorem 1.2 can now be stated.

Theorem 1.4 (Bang-Jensen, MacGillivray and Swarts [7])**.** *Let T be a connected locally semicomplete digraph.*

- *If T is acyclic, then T-colouring is polynomial.*
- *If T is unicyclic and T is a directed cycle or T has the structure shown in Figure 1 with T_2 semicomplete and at least one of T_1 and T_3 is empty, then T-colouring is polynomial. Otherwise T-colouring is NP-complete.*
- *If T contains at least two cycles, then T-colouring is NP-complete.*

The third point above follows from the result of Barto, Kozik, and Niven [8]. Our goal in this paper is to prove the third point for *local tournaments* without having to appeal to other results (and in particular using only graph theoretic tools). We are only able to deal with local tournaments, since one of the tools we used (indicators equal to directed paths of length two—see Section 2.1) breaks down under the presence of two-cycles.

Given the fact that the third point in Theorem 1.4 has already been established, one might wonder why it would be interesting to find different NP-completeness proofs in the case when the local tournament T has at least two directed cycles. One reason is that the analysis makes careful use of the local tournament structure, and therefore helps pinpoint where the complexities arise. Another is that it showcases all of the techniques that have been built up over the last two decades in the study of the complexity of graph homomorphisms. Our paper's motivation is similar to the one by Hell and Rafiey [19]. In their paper Hell and Rafiey give a proof of Bulatov's dichotomy result [11] for list homomorphisms (also established using universal algebra). Their proof is a mixture of graph theory and universal algebra. To quote Paul Halmos [1]:

> *Combinatorics, the finite case, is where the genuine, deep insight is. Generalizing, making it infinite, is sometimes intricate and sometimes difficult, and I might even be willing to say that it's sometimes deep, but it is nowhere near as fundamental as seeing the finite structure.*

There is also a related, more significant, reason. The proof of the smooth digraph conjecture uses results from [12], [20] and [21] which imply that if a digraph H does not admit a weak near-unanimity function (see [8] for the definition), then HOM_H is NP-complete. Such functions have been conjectured by Bulatov, Jeavons and Krokhin [12] (although not in terms of weak near-unanimity functions) to be the dividing line between polynomial and NP-complete digraph homomorphism problems. Assuming that $\mathrm{P} \neq \mathrm{NP}$, no directed graph H such that HOM_H is NP-complete admits a weak near-unanimity function. In his Ph.D. thesis [24], Swarts developed a method for translating any NP-completeness proof that uses the three Hell-Nešetřil constructions into a proof that H has no weak near unanimity function, provided that the base cases—the ones where NP-completeness is proved directly using polynomial time reductions—can be handled. That is, Swarts' method allows the assumption that $\mathrm{P} \neq \mathrm{NP}$ to be removed. He also showed that all digraphs for which the results in [15] provide polynomial algorithms admit a weak near-unanimity function. We therefore think there is value in our long detailed argument since it uses the these three constructions and, if the base cases can be handled, makes it possible to directly determine precisely which local tournaments admit such a function.

For terms not defined in this paper, the reader may consult [4] for digraphs, [18] for homomorphisms and [13] for complexity theory.

2. Some Tools

Hell and Nešetřil [17] introduced a number of powerful tools for proving that a given digraph has an NP-complete homomorphism problem. The aim of this section is to introduce these and other tools that will be useful in our proof.

2.1. **The Indicator Construction.** Let I be a fixed digraph with two specified vertices i and j. The *indicator construction* (with respect to the *indicator* I, i, j) transforms a digraph H to the digraph H^* as follows. The vertex set of H^* is the same as that of H. Arcs are defined by the following rule: xy is an arc of H^* if and only if there exists a homomorphism from I to H mapping i to x and j to y. We then have the following result.

Lemma 2.1 (Hell and Nešetřil [17, 18])**.** *If the H^*-colouring problem is NP-complete, then the H-colouring problem is also NP-complete.*

2.2. **The (Vertex) Sub-indicator Construction.** Let J be a fixed digraph with specified vertices $k_1, k_2, \ldots, k_t$ and j. The *sub-indicator construction* (with respect to the *sub-indicator* $J, k_1, k_2, \ldots, k_t, j$) transforms a digraph H with specified vertices $x_1, x_2, \ldots, x_t$ to an induced subgraph H^+ defined as follows. Let W be the digraph obtained from a copy H and a copy of J by identifying each k_i with the corresponding x_i for $i = 1, 2, \ldots, t$. Then H^+ is the subgraph of H induced by those vertices u for which some retraction of W to H maps j to u.

Lemma 2.2 (Hell and Nešetřil [17, 18])**.** *Let H be a digraph that is a core. If the H^+-colouring problem is NP-complete, then the H-colouring problem is also NP-complete.*

Often, when using the sub-indicator construction, we take the vertices k_1, k_2, …, k_t above to be a set of isolated vertices in J. This has the effect that the digraph W above is $H \cup (J - \{k_1, k_2, \ldots, k_t\})$. In considering retractions of W to H, we see that we are actually considering homomorphisms of $J - \{k_1, k_2, \ldots, k_t\}$ to H.

2.3. **The Arc-sub-indicator Construction.** Let J be a fixed graph with a specified arc jj' and t specified vertices $k_1, k_2, \ldots, k_t$. The *arc-sub-indicator construction* (with respect to the *arc-sub-indicator* $J, k_1, k_2, \ldots, k_t, jj'$) transforms a digraph H with t specified vertices $x_1, x_2, \ldots, x_t$ into its subgraph H^- determined by the images of the arc jj' under retractions of W (defined as above) to H. This construction is therefore an arc version of the (vertex) sub-indicator outlined above.

Lemma 2.3 (Hell and Nešetřil [17])**.** *Let H be a core. If the H^--colouring problem is NP-complete, then so is the H-colouring problem.*

2.4. **Colouring by Wheels is NP-complete.** Let H be the wheel-graph shown below in Figure 2. H has vertices $\{0, 1, 2, \ldots, n\}$ and arcs $0i$, $i0$ for $i = 1, 2, \ldots, n$, $j(j+1)$ for $j = 1, 2, \ldots, n-1$ and $n1$.

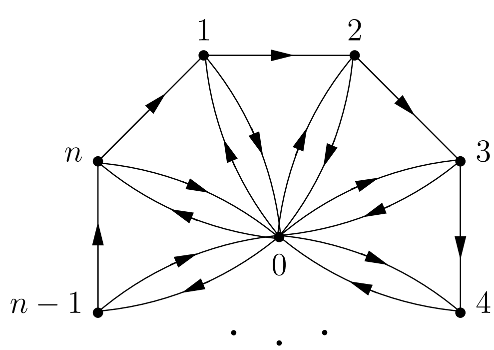

FIGURE 2. The target H.

Theorem 2.4. *H-colouring is NP-complete, even for acyclic inputs.*

Proof. The proof is via a reduction from not-all-equal 3-SAT without negated variables which is known to be NP-complete [23].

Throughout let $k \in \{1, 2, \ldots, n\}$ and define

$$k^+ = \begin{cases} k+1 & \text{if } 1 \le k < n, \\ 1 & \text{if } k = n, \end{cases}$$

and

$$k^- = \begin{cases} k-1 & \text{if } 1 < k \leq n, \\ n & \text{if } k = 1. \end{cases}$$

We only have the following transitive triples in H: $0kk^+$, $k0k^+$ and kk^+0. Let F be the digraph shown below in Figure 3

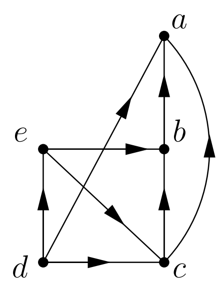

FIGURE 3. The gadget F.

If $F \to H$, then $c \not\mapsto 0$: if $c \mapsto 0$, then $b \mapsto k$ and $a \mapsto k^+$. This implies that $d \mapsto k$ and $e \mapsto k^+$. Therefore the arc eb is mapped to k^+k which is not an arc of H.

On the other hand there is a homomorphism $f : F \to H$ in which $f(c) = k$, where $k \in \{1, 2, \ldots, n\}$. This homomorphism is given by: $f(a) = 0$, $f(b) = k^+$, $f(c) = k$, $f(d) = k^-$ and $f(e) = 0$.

Let G be the digraph shown in Figure 4.

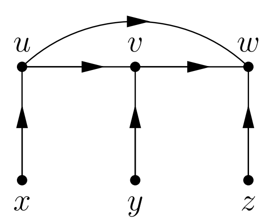

FIGURE 4. The gadget G.

If $G \to H$, then $(x, y, z) \neq (0, 0, 0)$ as this would force u, v and w to all map to nonzero vertices and no transitive triple on nonzero vertices alone exists. Also if $G \to H$, then $(x, y, z) \neq (k, k, k)$. If this was the case then u, v and w are forced to map to $\{0, k^+\}$ and no such transitive triple exists. On the other hand homomorphisms from G to H are shown in Table 1 where x, y and z have been pre-coloured with $\{0, k\}$ using a majority of 0's or a majority of k's.

We are now ready to exhibit the reduction. Let an instance of not-all-equal 3-SAT without negated variables be given by:

TABLE 1. Homomorphisms of the gadget G to H.

u	v	w	k	k^+	0	k	0	k^+	0	k	k^+
x	y	z	0	0	k	0	k	0	k	0	0
u	v	w	k^+	0	k^{++}	0	k	k^+	k	0	k^+
x	y	z	k	k	0	k	0	k	0	k	k

The variables: $X = \{x_1, x_2, \ldots, x_\ell\}$
The clauses: $C_1, C_2, \ldots, C_m$

Each clause $C_i = \{x_{i_1}, x_{i_2}, x_{i_3}\}$ with $x_{i_1}, x_{i_2}, x_{i_3} \in X$ and $i = 1, 2, \ldots, m$.

Construct a digraph D as follows: Take a copy of F and add vertices $x_1, x_2, \ldots, x_\ell$ to F as well as the arcs cx_i $i = 1, 2, \ldots, m$. For each clause $C_i = \{x_{i_1}, x_{i_2}, x_{i_3}\}$ take a copy of G and identify x, y, z in G with the corresponding $x_{i_1}, x_{i_2}, x_{i_3}$. This is illustrated in Figure 5.

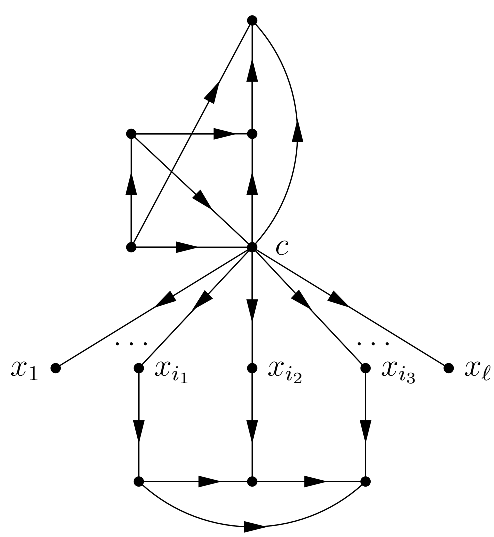

FIGURE 5. The digraph D.

If $D \to H$, then $c \mapsto k \in \{1, 2, \ldots, n\}$ which in turn implies that x_1, x_2, …, $x_\ell \to \{0, k^+\}$. The clause gadget G prevents all of the x's in the same clause from being mapped to the same vertex. This allows one to read off a satisfying truth assignment: $0 =$ "False" and $k^+ =$ "True".

If there exists a satisfying truth assignment we identify "True" with the vertex 1 in H and "False" with the vertex 0 in H. This produces a precolouring on the vertices $x_1, x_2, \ldots, x_\ell$ in D which can be extended to a homomorphism $D \to H$.

Therefore there exists a satisfying truth assignment for not-all-equal 3-SAT without negated variables if and only if $D \to H$. Thus H-colouring is NP-complete (even for acyclic inputs). □

2.5. **The Frobenius-Schur Index.** The result discussed in this section is a purely number theoretic result. It will help in choosing the correct lengths for directed paths that are to act as indicators and sub-indicators.

Given a set of relatively prime positive integers $B = \{a_1, a_2, \ldots, a_n\}$, a linear combination of these integers is an expression of the form

$$x_1a_1 + x_2a_2 + \cdots + x_na_n, \tag{2.1}$$

where each $x_i \in \{0, 1, 2, \ldots\}$. A natural question to ask is for the smallest integer ϕ such that each every integer $t \geq \phi$ can be represented as a linear combination of the form (2.1). The existence of such an integer ϕ is guaranteed by a result of Schur (see [10]).

Lemma 2.5. *Let S be a nonempty set of positive integers which is closed under addition. Let d be the greatest common divisor of the integers in S. Then there exists a positive integer N such that td is in S for every integer $t \geq N$.*

Since the set of all linear combinations of elements in B is closed under addition, Schur's result guarantees a threshold above which every integer is of the form (2.1).

Frobenius (according to [9]) then posed the problem of finding the smallest integer $\phi = \phi(a_1, a_2, \ldots, a_n)$ such that every integer $t \geq \phi$ is of the form (2.1) (or at least good bounds this number). The integer ϕ is known as the *Frobenius-Schur index* of the set B [10] or as the *conductor* of the set B [25]. Equivalently, one may ask for the largest integer not representable as (2.1). In general, this is a very hard problem with a rich literature [22]. The problem sometimes also goes by the name of *the money changing problem.* Given a fixed set of coins (the set B) what is the largest amount of money that cannot be changed using the coins in B? See [16] or [25] for more on this.

We are typically interested in finding the Frobenius-Schur index of the cycle lengths of a strong round local tournament D. By Lemma 3.6 we know that the cycle lengths of D is an interval of integers, $\{\ell, \ell+1, \ldots, n\}$, where ℓ is the girth of D and $n = |V(D)|$. Fortunately, in this case, the Frobenius-Schur index is known exactly.

Lemma 2.6 (Brauer [9])**.** *Let ℓ be a positive integer. Then*

$$\phi(\ell, \ell+1, \ldots, n) = \left\lfloor \frac{n-2}{n-\ell} \right\rfloor \ell.$$

We would like to give a short justification of this result as this may aid the reader later on when we apply this result to local tournaments.

Consider the following table. Row k of the table contains numbers that can be written as a linear combination using k numbers from $\{\ell, \ell+1, \ldots, n\}$.

$$\begin{array}{ccccccccc}
\ell, & \ell+1, & \ell+2, & \ldots, & n & = & \ell & + & (n-\ell). \\
2\ell, & 2\ell+1, & 2\ell+2, & \ldots, & 2n & = & 2\ell & + & 2(n-\ell). \\
3\ell, & 3\ell+1, & 3\ell+2, & \ldots, & 3n & = & 3\ell & + & 3(n-\ell). \\
 & & & & & \vdots & & & \\
k\ell, & k\ell+1, & k\ell+2, & \ldots, & kn & = & k\ell & + & k(n-\ell). \\
(k+1)\ell, & (k+1)\ell+1, & (k+1)\ell+2, & \ldots, & (k+1)n & = & (k+1)\ell & + & (k+1)(n-\ell). \\
 & & & & & \vdots & & &
\end{array}$$

Note that each row is a list of consecutive integers. Therefore each number that cannot be written as a linear combination has to occur somewhere between the rows of the table. Between the last entry of a row and the first entry of the next row there is a "gap" of integers that cannot be written as a linear combination. The size of the gap also decreases as one moves down the table so that it is inevitable that it eventually closes.

We now ask what is the largest integer that occurs somewhere between the rows of the table. This integer has to occur in a gap of size at least two (a gap of size one corresponds to two consecutive integers). So what is the largest integer k such that between rows k and $(k+1)$ there is still a gap of size at least two. That is,

$$(k+1)\ell - kn = (k+1)\ell - k(\ell + n - \ell) = \ell - k(n-\ell) \geq 2,$$

implying

$$k = \left\lfloor \frac{\ell - 2}{n - \ell} \right\rfloor .$$

So as soon as we move beyond this row, every integer can be written as a linear combination. Therefore the smallest integer ϕ defined above occurs at $(k+1)\ell$. That is,

$$\phi(\ell, \ell+1, \ell+2, \ldots, n) = \left(\left\lfloor \frac{\ell-2}{n-\ell} \right\rfloor + 1 \right) \ell = \left\lfloor \frac{n-2}{n-\ell} \right\rfloor \ell.$$

3. Local Tournaments

Since we aim to give a graph theoretic proof of the complexity of HOM_H where H is a local tournament, it comes as no surprise that the structure of local tournaments plays a central role in this proof. In this section we state the results on the structure of locally semi-complete digraphs (since local tournaments are a special case of these) that we need. Bang-Jensen and Gutin [4] is a standard reference on these matters.

A digraph D is said to be *strong* or *strongly connected* if for every pair of vertices x and y in D, there is directed path joining x to y and a directed path joining y to x.

A *strong component* of a digraph D is a maximal induced sub-digraph of D that is strong.

The strong components of a digraph D are disjoint and can be labeled as D_1, $D_2, \ldots D_t$ such that there is no arc from D_i to D_j unless $i < j$. We call such an ordering an *acyclic ordering* of the strong components of D. This ordering is unique for every connected local tournament (see e.g. [2, 4]).

We also use the following standard notation: out-neighbours (in-neighbours) of the vertex v: $N^+(v)$ ($N^-(v)$); maximum (minimum) in-degree: Δ^- (δ^-); maximum (minimum) out-degree: Δ^+ (δ^+).

Theorem 3.1 (Guo and Volkmann [4, 14])**.** *Let D be a connected locally semi-complete digraph that is not strong and let $D_1, D_2, \ldots, D_p$ be the acyclic ordering of the strong components of D. Then D can be decomposed into $r \geq 2$ induced subgraphs $D'_1, D'_2, \ldots, D'_r$ as follows:*

$$D'_1 = D_p, \quad \lambda_1 = p, \quad \lambda_{i+1} = \min\{j \mid N^+(D_j) \cap V(D'_i) \neq \varnothing\},$$

and

$$D'_{i+1} = D\big\langle V(D_{\lambda_{i+1}}) \cup V(D_{(\lambda_{i+1})+1}) \cup \cdots \cup V(D_{\lambda_i - 1})\big\rangle.$$

The sub-digraphs $D'_1, D'_2, \ldots, D'_r$ satisfy the properties below:

(a) *D'_i consists of some strong components of D and is semi-complete for $i = 1, 2, \ldots, r$.*
(b) *D'_{i+1} dominates the initial component of D'_i and there exists no arc from D'_i to D'_{i+1} for $i = 1, 2, \ldots, r-1$.*
(c) *If $r \geq 3$, then there is no arc between D'_i and D'_j for i, j satisfying $|j - i| \geq 2$.*

If D is a connected locally semi-complete digraph that is not strong, then the unique sequence $D'_1, D'_2, \ldots, D'_r$ defined in Theorem 3.1 is called the *semi-complete decomposition* of D.

A digraph on n vertices is said to be *round* if we can label its vertices $v_1, v_2, \ldots, v_n$ so that for each i, $N^+(v_i) = \{v_{i+1}, \ldots, v_{i+d^+(v_i)}\}$ and $N^-(v_i) = \{v_{i-d^-(v_i)}, \ldots, v_{i-1}\}$, where all subscripts are taken modulo n.

Let D be a digraph with vertex set $\{v_1, v_2, \ldots, v_n\}$ and let $G_1, G_2, \ldots, G_n$ be digraphs which are pairwise vertex disjoint. The *composition* $D[G_1, G_2, \ldots, G_n]$ is the digraph H with vertex set $V(G_1) \cup V(G_2) \cup \cdots \cup V(G_n)$ and arcs

$$\Big(\bigcup_{i=1}^{n} A(G_i)\Big) \cup \big\{g_i g_j \mid g_i \in V(G_i), g_j \in V(G_j), v_i v_j \in A(D)\big\}.$$

A locally semi-complete digraph D is *round decomposable* if there exists a round local tournament R on $r \geq 2$ vertices such that $D = R[S_1, \ldots, S_r]$, where each S_i is a strong semi-complete digraph.

For a strong digraph D a set $S \subseteq V(D)$ is a *separator* or *separating set* if $D - S$ is not strong.

Lemma 3.2 (Bang-Jensen, Guo, Gutin and Volkmann [3, 4])**.** *Let D be a strong locally semi-complete digraph which is not semi-complete. Then D is not round decomposable if and only if the following conditions are satisfied:*

(a) *There is a minimal separating set S such that $D - S$ is not semi-complete and for each such S, $D\langle S\rangle$ is semi-complete and the semi-complete decomposition of $D - S$ has exactly three components D'_1, D'_2, D'_3;*

(b) *There are integers α, β, μ, ν with $\lambda_2 \leq \alpha \leq \beta \leq p-1$ and $p+1 \leq \mu \leq \nu \leq p+q$ such that*

$$N^-(D_\alpha) \cap V(D_\mu) \neq \varnothing \quad \textit{and} \quad N^+(D_\alpha) \cap V(D_\nu) \neq \varnothing,$$

or

$$N^-(D_\mu) \cap V(D_\alpha) \neq \varnothing \quad \textit{and} \quad N^+(D_\mu) \cap V(D_\beta) \neq \varnothing,$$

where $D_1, D_2, \dots, D_p$ and $D_{p+1}, \dots, D_{p+q}$ are the acyclic orderings of the strong components of $D - S$ and $D\langle S\rangle$, respectively, and D_{λ_2} is the initial component of D'_2.

The structure described in Lemma 3.2 is illustrated in Figure 6.

Theorem 3.3 (Bang-Jensen, Guo, Gutin and Volkmann [3, 4])**.** *Let D be a connected locally semi-complete digraph. Then exactly one of the following possibilities holds.*

(a) *D is round decomposable with a unique round decomposition given by*

$$D = R[D_1, D_2, \dots, D_\alpha],$$

where R is a round local tournament on $\alpha \geq 2$ vertices and D_i is a strong semi-complete digraph for $i = 1, 2, \dots, \alpha$;

(b) *D is not round decomposable and not semi-complete and it has the structure described in Lemma 3.2;*

(c) *D is a semi-complete digraph which is not round decomposable.*

Proposition 3.4 (Bang-Jensen, Guo, Gutin and Volkmann [3, 4])**.** *Let D be a strong non-round decomposable locally semi-complete digraph and let S be a minimal separating set of D such that $D - S$ is not semi-complete. Let $D_1, D_2, \dots, D_p$ be the acyclic ordering of the strong components of $D - S$ and $D_{p+1}, D_{p+2}, \dots, D_{p+q}$ be the acyclic ordering of the strong components of $D\langle S\rangle$. Suppose that there is an arc $s \to v$ from S to D'_2 with $s \in V(D_i)$ and $v \in V(D_j)$, then*

$$D_i \cup D_{i+1} \cup \cdots \cup D_{p+q} \mapsto D'_3 \mapsto D_{\lambda_2} \cup \cdots \cup D_j.$$

Here, $A \mapsto B$, means that A dominates B and there are no arcs from B to A.

Lemma 3.5 (Bang-Jensen and Gutin [4])**.** *Let R be a strong round local tournament and let C be a shortest cycle of R and suppose C has $k \geq 3$ vertices. Then for every round labeling $v_0, v_1, \dots, v_{n-1}$ of R such that $v_0 \in V(C)$ there exist indices $0 < a_1 < a_2 < \cdots < a_{k-1} < n$ so that $C = v_0 v_{a_1} v_{a_2} \cdots v_{a_{k-1}} v_0$.*

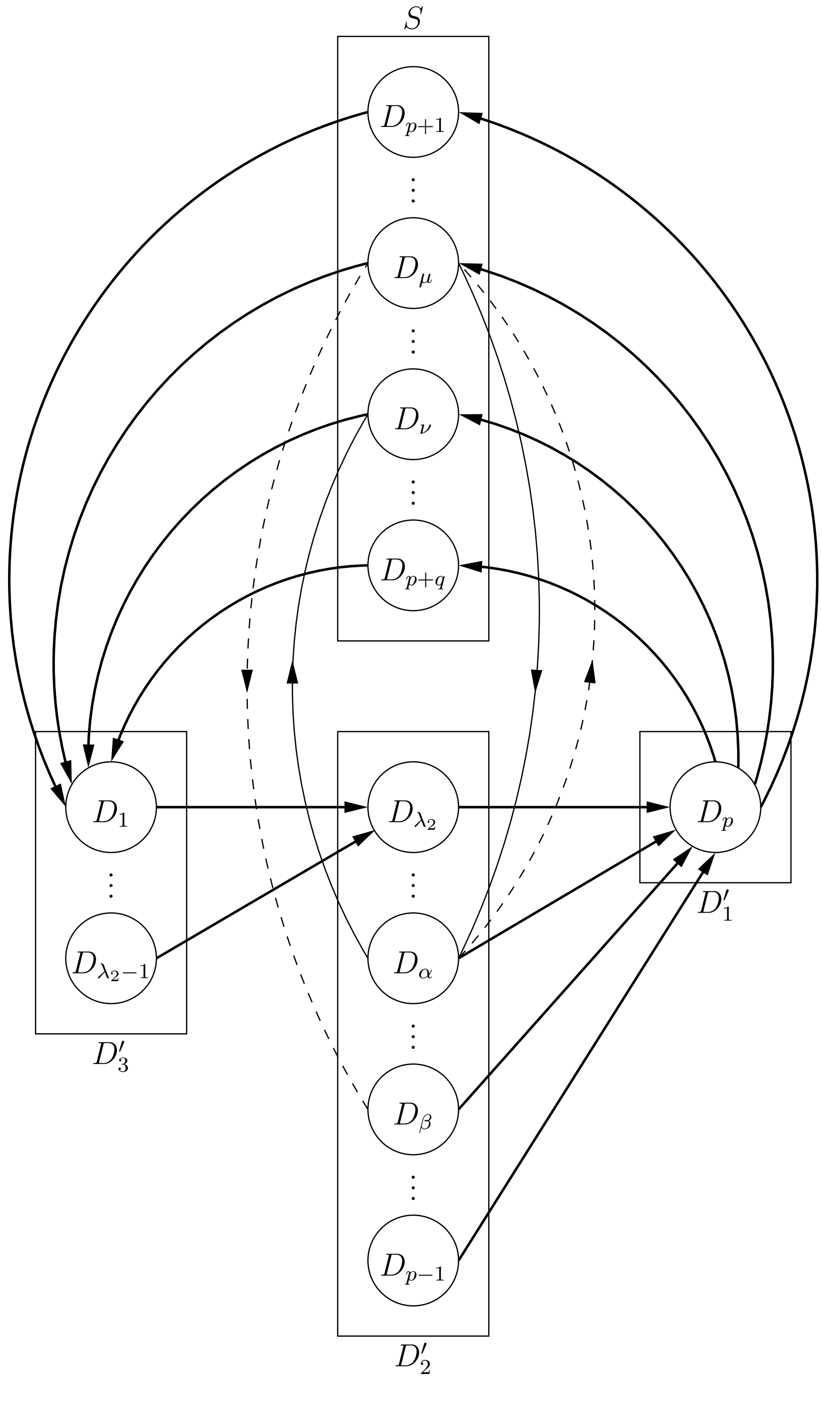

FIGURE 6. The structure of a strong locally semi-complete digraph that is not semi-complete and not round decomposable.

Lemma 3.6 (Bang-Jensen and Gutin [4]). *A strong round local tournament R on r vertices has cycles of length $k, k+1, \ldots, r$, where k is the girth of R.*

Lemma 3.7 (Bang-Jensen and Gutin [4]). *If a strong round local tournament with r vertices has a cycle of length k through a vertex v, then it has cycles of all lengths $k, k+1, \ldots, r$ through v.*

Strictly speaking a round digraph is also round decomposable (all $|D_i| = 1$). We prefer to distinguish between round and round decomposable digraphs. So when a local tournament is said to be round decomposable then at least one $|D_i| \geq 3$. Furthermore, since a connected locally semi-complete digraph that is not strongly connected is round decomposable (consider the unique acyclic ordering of its strong components), a round local tournament (in our sense) is strongly connected and therefore Hamiltonian (by Lemma 3.6).

In proving some of our results, we will use the sub-indicator construction (both the vertex and arc versions). A glance at Lemmas 2.2 and 2.3 confirms the fact that in order to apply a sub-indicator to a digraph H, one has to know that H is a core.

Proposition 3.8 (Bang-Jensen, MacGillivray and Swarts [7]). *A connected locally semi-complete digraph D is a core.*

4. Connected vs. Disconnected Local Tournaments

When H is a disconnected local tournament such that each component of H (all of which are local tournaments themselves) is polynomial time solvable, then HOM_H is also polynomial time solvable. NP-completeness results for disconnected local tournaments are much harder to obtain.

The NP-completeness results that follow are all for connected local tournaments since we only give polynomial time transformations from NP-complete problems to HOM_H when H is connected. A natural conjecture would be: if a disconnected local tournament H contains at least one component that is NP-complete, then HOM_H is NP-complete. The difficulty in proving this lies with constructing a polynomial transformation from some NP-complete problem. In general, it is hard to set up the transformation without forcing certain vertices of the transformed instance to map to a specific component of H. This may have the unintended consequence of restricting the images of one or more vertices of the transformed instance too severely.

On the other hand one can easily obtain a polynomial time Turing reduction. Here, an instance of some NP-complete problem Q is transformed into many different instances of HOM_H. The transformation has to run in polynomial time and furthermore an instance I of Q is a yes instance if and only if at least one the transformed instances is a yes instance of HOM_H. Let H be a disconnected local tournament that is also a core and let H' be a component of H such that $\mathrm{HOM}_{H'}$ is NP-complete. The polynomial time Turing reduction is from $\mathrm{HOM}_{H'}$ to HOM_H. Let G be an instance of $\mathrm{HOM}_{H'}$. We now form $|V(H')|$ instances of HOM_H as follows: take $|V(H')|$ copies each of G and H', let $v \in V(G)$ and $V(H') = \{h_1, h_2, \dots, h_n\}$. Denote by F_i the graph obtained by identifying the vertex v in G with the vertex h_i in H', $1 \leq i \leq n$.

It is now easy to see that $G \to H'$ if and only if there exists at least one $i \in \{1, 2, \dots, n\}$ such that $F_i \to H$.

This shows that for a disconnected local tournament with at least one NP-complete component, HOM_H is polynomial time solvable if and only if P=NP. Therefore solving HOM_H in polynomial time for a disconnected local tournament H with at least one NP-complete component, is highly unlikely.

From now on we assume that all local tournaments are connected.

5. Round Local Tournaments

Let D be a round local tournament containing at least two cycles, and let ℓ denote the length of the shortest cycle. The proof of NP-completeness proceeds as follows. If D has a unique cycle of length ℓ, the result follows. Otherwise, we show a minimum counterexample has $\delta^+ < \ell$, $\Delta^+ < \ell$, and $\phi = \ell$, where ϕ is the Frobenius-Schur index of the cycle lengths of D. From this it follows that no counterexample exists.

Lemma 5.1. *Let D be a round local tournament containing at least two cycles. If D has a unique cycle of shortest length, then D-colouring is NP-complete.*

Proof. Let $v_0, v_1, \dots, v_{n-1}$ be a round labeling of $V(D)$, C be a shortest cycle in D and ℓ be the length of C. Since D is strongly connected, by Lemma 3.5 there exist indices $0 < a_1 < a_2 < \cdots < a_{\ell-1} < n$ so that $C = v_0 v_{a_1} v_{a_2} \cdots v_{a_{\ell-1}} v_0$.

Since D has at least two cycles, there must be vertices of D not on C.

The proof is divided into two cases. Firstly, we consider the situation where there are three indices $i, i+1$ and $i+2 \pmod{\ell}$ such that $a_{i+2} - a_{i+1} \geq 2$ and $a_{i+1} - a_i \geq 2$. Therefore there are vertices of D, not on C, between a_i and a_{i+1} and between a_{i+1} and a_{i+2}. Once this has been dealt with, what then remains is the case where these vertices do not exist. This will imply the existence of four indices $i, i+1, i+2$ and $i+3 \pmod{\ell}$ such that $a_{i+3} - a_{i+2} = 1$, $a_{i+2} - a_{i+1} \geq 2$ and that $a_{i+1} - a_i = 1$. That is, $v_{a_{i+3}}$ and $v_{a_{i+2}}$ are consecutive, as are $v_{a_{i+1}}$ and v_{a_i}, and there exists a vertex of D not on C between $v_{a_{i+1}}$ and $v_{a_{i+2}}$.

Case 1: $a_{i+2} - a_{i+1} \geq 2$ and $a_{i+1} - a_i \geq 2$:

Let u be any vertex between a_i and a_{i+1} and let w be the successor of a_{i+1} on the outer n-cycle (w is between a_{i+2} and a_{i+1}). This is illustrated in Figure 7.

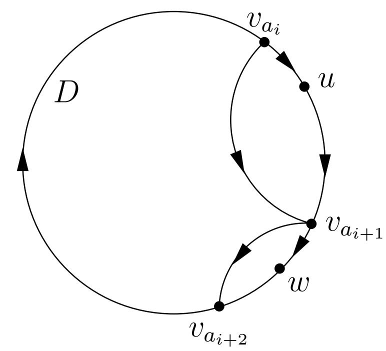

FIGURE 7. The first case where D has a unique cycle of shortest length

We handle this case by assuming the lemma is false and examining a minimum counterexample. That is assume that there exists a round local tournament D with at least two cycles and a unique cycle of shortest length that is of the form shown in Figure 7. In addition to this we assume that D has the minimum number of vertices possible. Also, D-colouring is not NP-complete.

We apply the sub-indicator J shown in Figure 8 to D. Note that the vertex on the left is identified with vertex a_{i+1} in D and that the vertex j of J is the vertex on the right. The sub-indicator is constructed by starting with a directed path of length $\ell(\ell-1)$. Skip the first two vertices and then attach ℓ directed ℓ-cycles to the next ℓ vertices, skip the next vertex, and then attach ℓ directed ℓ-cycles to the next ℓ vertices, and so on.

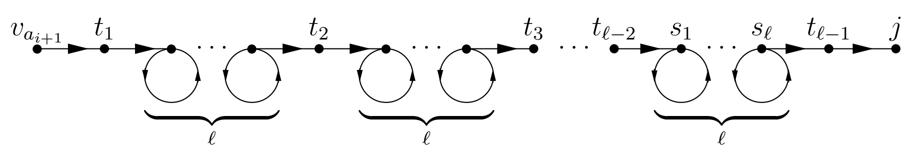

FIGURE 8. The sub-indicator for the first case.

We now consider all retractions of J to D, where the first vertex of J maps to $v_{a_{i+1}}$. The vertices of the path that are attached to the ℓ-cycles have to map to C. If we map all vertices of J from the first up to s_ℓ onto C, then $s_\ell \mapsto v_{a_{i-1}}$. Therefore j is able to map to u and $v_{a_{i+1}}$ (at least).

We claim that $j \not\mapsto w$. To see why this is the case, consider what happens when $j \mapsto w$. If this was to happen, then $s_\ell \mapsto v_{a_i}$ (if not, we either get a cycle of length less than ℓ or another ℓ cycle different from C). If $s_\ell \mapsto v_{a_i}$, then $s_1 \mapsto v_{a_{i+1}}$. Therefore the image of J

between the first vertex and vertex s_1 is a closed directed walk, W, of length $\ell(\ell-2)-1$. Denote by J' the subgraph of J induced by $V(J)-\{s_2, s_3, \ldots, s_\ell, t_{\ell-1}, j\}$.

The walk W can be decomposed into directed cycles. The length of each of these directed cycles is either ℓ or $\ell+1$. To see why this is the case consider any group of $\ell+3$ consecutive vertices on J'. Any such group will include at least $\ell+1$ of the vertices with ℓ cycles attached to them (also keep in mind that the first vertex is mapped onto $v_{a_{i+1}}$). These vertices can only map to $V(C)$ and so the images (of the vertices in J') can not be distinct. This implies that the longest cycle that vertices in J' can map to is at most $\ell+2$. In order for $\ell+2$ vertices in J' to map to an $\ell+2$ cycle in D, one needs $\ell+2$ vertices in J' that can have distinct images. Such a group of $\ell+2$ vertices only occurs between consecutive t_i's (see Figure 8). In order for the group to map to an $(\ell+2)$-cycle t_i and the vertex following t_{i+1} must have the same image. That means that t_i is mapped to a vertex on $V(C)$ and so the $\ell+2$ vertices do not have distinct images. This shows that the directed cycles in the decomposition of W can only be of length ℓ and $\ell+1$.

Since W can be decomposed into ℓ and $\ell+1$ cycles, we have that $\ell(\ell-2)-1 = k_1\ell + k_2(\ell+1) = (k_1+k_2)\ell + k_2$ where k_1 is the number ℓ-cycles and k_2 the number of $(\ell+1)$-cycles in such a decomposition. Note that $0 \leq k_1, k_2 \leq \ell-2$. This means that $\ell(\ell-1) = (k_1+k_2)\ell + (k_2+1)$ or that ℓ divides (k_2+1), but $1 \leq (k_2+1) \leq \ell-1$, so we have a contradiction. Therefore if the first vertex of J maps to $v_{a_{i+1}}$, then j cannot map to w.

We now also claim that j can map to every vertex of C. From before we already know that $j \mapsto v_{a_{i+1}}$. If we map t_1 to w and then all other vertices of J to C, we find that $j \mapsto v_{a_i}$: if $t_1 \mapsto w$, the vertex preceding t_2 maps to $v_{a_{i+1}}$ and the length of the path that remains in J is $\ell(\ell-2)-1$, a multiple of ℓ minus 1. In general by mapping $t_1, t_2, \ldots, t_k$ to w and all other vertices (from the first up to the vertex preceding t_{k+1}) to C, we find that the vertex preceding t_k maps to $v_{a_{i+1}}$. The length of the path remaining in J is $\ell(\ell-1)-(k\ell+k) = \ell(\ell-1-k)-k$ (we have used $k\ell+k+1$ vertices or $k\ell+k$ arcs up to this point). If we now map all the remaining vertices to C we see that $j \mapsto v_{a_{i+1-k}}$.

The result of this sub-indicator, D^+, contains (at least) the subgraph induced by $C \cup \{u\}$, but not the vertex w. Therefore D^+ contains at least two cycles and also has a unique shortest cycle of length ℓ. Since D has the minimum number of vertices for a counterexample, D^+-colouring is NP-complete, implying D-colouring is NP-complete, a contradiction.

Case 2: $a_{i+3}-a_{i+2} = 1$, $a_{i+2}-a_{i+1} \geq 2$ and that $a_{i+1}-a_i = 1$:

This case is illustrated in Figure 9.

The proof in this case is via a reduction from not all equal ℓ-SAT without negations.

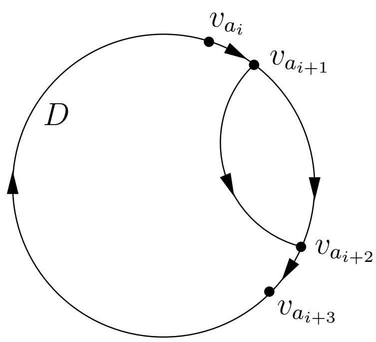

FIGURE 9. The second case where D has a unique cycle of shortest length

In carrying out the proof we need to construct the gadget Z shown in Figure 10. The gadget Z is constructed from a copy of D and two directed paths of length two, say $a'ua$ and $z'vz$. Identify a' and z', attach directed ℓ-cycles to vertices z', v and z and finally identify a with $v_{a_{i+2}}$.

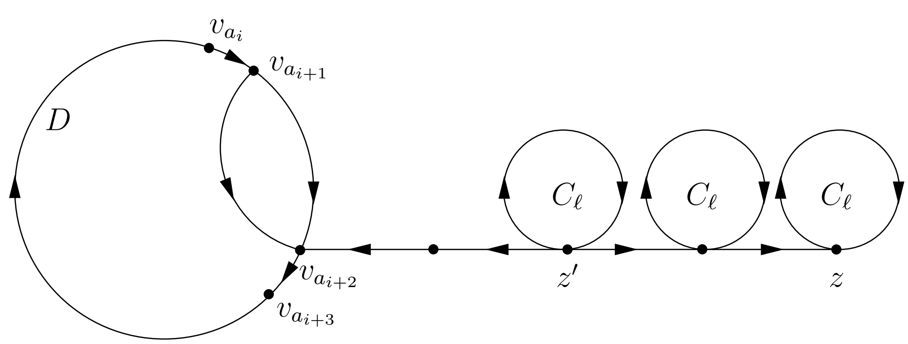

FIGURE 10. The gadget Z

In a retraction of Z to D, the vertex z' can only map to $v_{a_{i+1}}$ or v_{a_i}. This forces the pair of vertices (z', z) to map to the pair $(v_{a_i}, v_{a_{i+2}})$ or the pair $(v_{a_{i+1}}, v_{a_{i+3}})$.

We also need the gadget K shown in Figure 11 (it is the same as the sub-indicator used in the first case).

Let $f : K \to D$ be a homomorphism from K to D. If $f(r) = v_{a_i}$, then $f(s) \neq v_{a_{i+1}}$. If $f(s) = v_{a_{i+1}}$, then $f(t_{\ell-1}) = v_{a_i}$. This means that the image of K under f corresponds to a closed walk from v_{a_i} to v_{a_i} of length $\ell(\ell-1)-1$. Through a similar technique as in the previous case, one can show that this is impossible: the walk of length $\ell(\ell-1)-1$ can only be decomposed into ℓ and $\ell+1$ cycles (because of

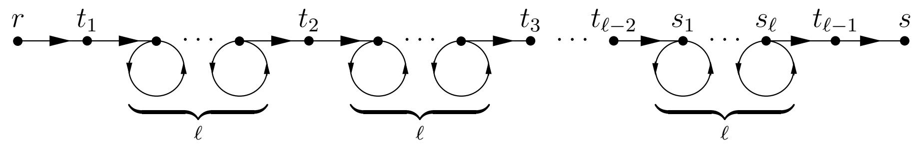

FIGURE 11. The gadget K

the placement of the ℓ-cycles) and this is impossible. As in case one, it is possible to show that if $f(r) = v_{a_i}$, then s can map to every vertex of C except $v_{a_{i+1}}$. This accomplished by mapping the t_i's of K "into" the gap between $v_{a_{i+1}}$ and $v_{a_{i+2}}$. On the other hand if $f(r) = v_{a_{i+1}}$, then s can map to any vertex of C (again the t_i's are mapped into the gap between $v_{a_{i+1}}$ and $v_{a_{i+2}}$). In a similar way it is possible to show that if $f(s) = v_{a_{i+3}}$, then $f(r)$ can map to any vertex of C except $v_{a_{i+2}}$ and that if $f(s) = v_{a_{i+2}}$, then r can map to any vertex of C.

Next, we construct a new gadget F from two copies of K, say with end-vertices r_1, s_1 and r_2, s_2 respectively. To construct F identify vertex s_1 with vertex r_2 and call this vertex c. This is shown in Figure 12.

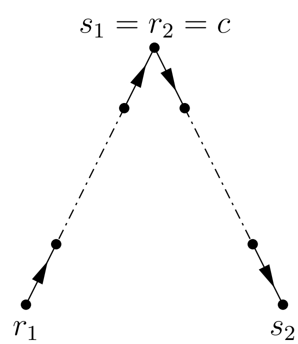

FIGURE 12. The gadget F

The gadget F has the property that if the pair (r_1, s_2) maps to the pair $(v_{a_i}, v_{a_{i+2}})$, under a homomorphism from F to D, then the vertex c can map to any vertex on C, except $v_{a_{i+1}}$. If the pair (r_1, s_2) maps to $(v_{a_{i+1}}, v_{a_{i+3}})$, then c can map to any vertex on C except $v_{a_{i+2}}$.

As mentioned earlier, the proof is via a reduction from not all equal ℓ-SAT without negated variables. So let an instance of not all equal ℓ-SAT without negated variables be given by:

$$\begin{array}{ll} \text{The variables:} & Z = \{z_1, z_2, \dots, z_a\}, \\ \text{The clauses:} & K_1, K_2, \dots, K_b. \end{array}$$

Each clause $K_j = \{z_{j_1}, z_{j_2}, \dots, z_{j_\ell}\}$ for each $j \in \{1, 2, \dots, b\}$.

We now construct an instance of D-colouring. For each variable z_j take a copy of the gadget Z. Identify all copies of D. Label the z and z'-vertices in the copies of Z as $z_1, z_2, \ldots, z_a, z'_1, z'_2, \ldots, z'_a$. These vertices correspond to variables in the instance of not all equal ℓ-SAT without negated variables. This produces the variable gadget.

For each clause $K_j = \{z_{j_0}, z_{j_1}, \ldots, z_{j_{\ell-1}}\}$ take ℓ copies of the gadget F, say F_{j_1}, F_{j_2}, $\ldots$, F_{j_ℓ} with end-vertices $(r_{j_{11}}, s_{j_{21}})$, $(r_{j_{12}}, s_{j_{22}})$, $\ldots$, $(r_{j_{1\ell}}, s_{j_{2\ell}})$ and top-vertices c_{j_1}, c_{j_2}, $\ldots$, c_{j_ℓ}, respectively. Now identify $r_{j_{1i}}$ with z'_{j_i} and $s_{j_{2i}}$ with z_{j_i} and form an ℓ-cycle through c_{j_1}, c_{j_2}, $\ldots$, c_{j_ℓ}. This produces the instance H of D-colouring shown in Figure 13.

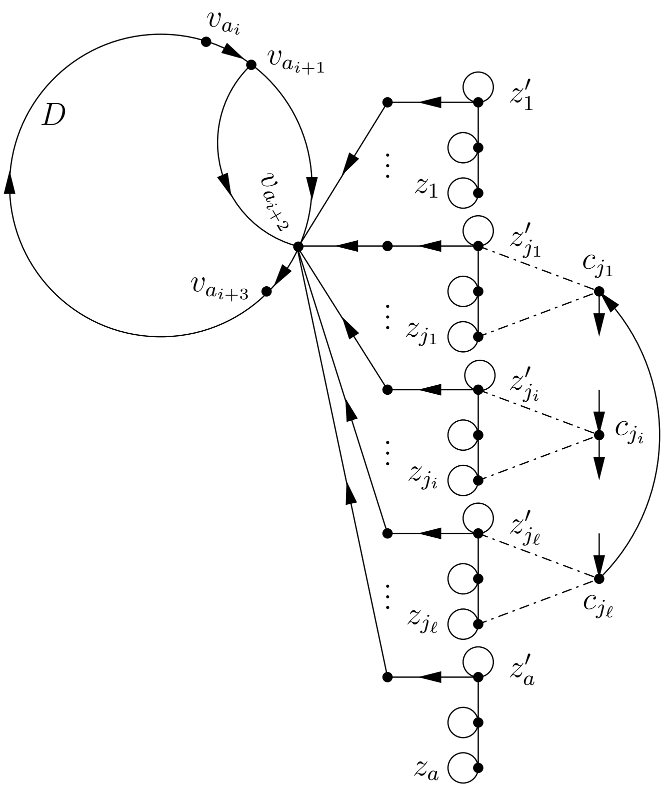

FIGURE 13. The instance H

We now show that $H \to D$ if and only if there exists a satisfying truth assignment for the instance of not all equal ℓ-SAT without negated variables. Define the pair $(v_{a_i}, v_{a_{i+2}})$ (in D) to be T and the pair $(v_{a_{i+1}}, v_{a_{i+3}})$ to be F.

If $H \to D$, then each pair (z'_i, z_i) maps to T or to F. Furthermore for a given clause $K_j = \{z_{j_1}, z_{j_2}, \ldots, z_{j_\ell}\}$ it is not the case that all pairs (z'_{j_i}, z_{j_i}), $1 \le i \le \ell$, map to T. If this was the case, none of the c_{j_i}s are allowed to map to $v_{a_{i+1}}$ and since there is a unique ℓ-cycle in D, we won't be able to complete the ℓ-cycle through the c_{j_i}s. In a similar way it is possible to show that the pairs (z'_{j_i}, z_{j_i}), $1 \le i \le \ell$ cannot all map to F (here we would be missing the vertex $v_{a_{i+2}}$ on the ℓ-cycle). Therefore in every clause there is at least one pair that maps to T and at least one pair that maps to F. A satisfying truth assignment can now be recovered by assigning "True" ("False") to variable z_i if (z'_i, z_i) maps to T (F).

Conversely, let a satisfying truth assignment be given. Pre-colour the pair (z'_i, z_i) by T (F) if z_i is assigned the value "True" ("False"). For the clause gadget corresponding to clause $K_j = \{z_{j_1}, z_{j_2}, \dots, z_{j_\ell}\}$ we extend the colouring as follows. Locate two consecutive vertices c_{j_k} and $c_{j_{k+1}}$ such that (z'_{j_k}, z_{j_k}) maps to F and $(z'_{j_{k+1}}, z_{j_{k+1}})$ maps to T (all subscripts are taken modulo ℓ). Map the corresponding F_{j_k} in such a way that c_{j_k} maps to $v_{a_{i+1}}$ and map $F_{j_{k+1}}$ in such a way that $c_{j_{k+1}}$ maps to $v_{a_{i+2}}$. Now map $F_{j_{k+2}}$ such that $c_{j_{k+2}}$ maps to $v_{a_{i+3}}$, $F_{j_{k+3}}$ such that $c_{j_{k+3}}$ maps to $v_{a_{i+4}}$, …, $F_{j_{k-1}}$ such that $c_{j_{k-1}}$ maps to v_{a_i}. By the properties of F and K discussed earlier, this is always possible. The remaining vertices of H map in an obvious way to D. Therefore $H \to D$. □

To complete the proof that colouring by a round local tournament is NP-complete, we assume that the result is false and examine a smallest counterexample (minimum number of vertices and minimum number of arcs). Being a counterexample, the corresponding colouring problem is not NP-complete. We derive additional properties of the counterexample to see that it cannot exist and in doing so we can then conclude that the colouring problem is in fact NP-complete.

The first property of this counterexample follows from Lemma 5.1.

Lemma 5.2. *Every vertex of a smallest counterexample is on a shortest cycle.*

Proof. Let D be a smallest counterexample. Since D-colouring is not NP-complete, by Lemma 5.1, there are two cycles of length ℓ. If D has vertices not on these shortest cycles, apply the sub-indicator C_ℓ to obtain an induced proper subgraph D'. Since D has at least two shortest cycles, D' has at least two cycles. Moreover D' is round as it is an induced subgraph of D. Thus D'-colouring is NP-complete implying D-colouring is NP-complete, a contradiction. □

The next property deals with the minimum out-degree of the smallest counterexample.

Lemma 5.3. *Let D be a smallest counterexample and denote by δ^+ the minimum out-degree of D. Then $\delta^+ < \ell$, where ℓ is the length of a shortest cycle in D.*

Proof. Assume that D has $\delta^+ \ge \ell \ge 3$. To prove the lemma we apply the arc-subindicator shown in Figure 14 (a transitive tournament on $\delta^+ + 1$ vertices with the two arcs spanning end-to-end as shown), with respect to the dashed arc, to D. Since every vertex in D has out-degree at least δ^+,

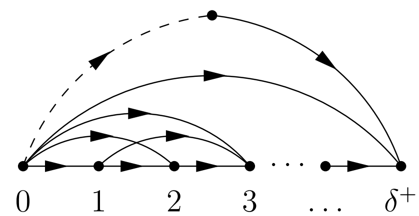

FIGURE 14. The arc-subindicator.

the result of this arc-subindicator is a digraph D' in which every vertex has out-degree at least 2 (the two vertices following the given vertex in the round enumeration). Every vertex also loses its out-neighbour furthest away from it in the round enumeration. This implies that D' has at least two cycles and is a round local tournament. Furthermore D' has at least one vertex with out-degree less than δ^+ (these are the vertices in D that have out-degree exactly δ^+) and so has fewer arcs than D. Since D has the minimum number of arcs possible for a counterexample, D' cannot be a counterexample and so D'-colouring is NP-complete. By the arc-subindicator construction this implies that D-colouring is also NP-complete, a contradiction. □

Next, we prove that a similar result holds for Δ^+.

Lemma 5.4. *Let D be a smallest counterexample and denote by Δ^+ the maximum out-degree of D. Then $\Delta^+ < \ell$, where ℓ is the length of a shortest cycle in D.*

Proof. Assume that D has $\Delta^+ \geq \ell$. By the previous result this implies that there exist two vertices x and y in D such that $d^+(x) < \ell$ and $d^+(y) \geq \ell$. Without loss of generality, $d^+(v_{n-1}) < \ell$ and $d^+(v_0) \geq \ell$.

Let's say that $d^+(v_0) = m \geq \ell$ and $d^+(v_{n-1}) = a < \ell$. We then use the sub-indicator formed by taking a copy of D and attaching a path of length ℓ to vertex v_0 and letting j be the final vertex on the path of length ℓ. This is shown below in Figure 15.

Every vertex is on a shortest cycle (an ℓ-cycle). In particular, v_0 is on one and moreover there is a shortest cycle containing v_0 that uses the arc v_0v_m. To see this consider any shortest cycle through v_0, if it is not using v_0v_m then there is at most one vertex of this cycle between v_0 and v_m (otherwise we can find a shorter cycle). Call this vertex u. The vertex u is adjacent to some v_i with $i > m$, otherwise there is a shorter cycle. This also means that v_m is adjacent to v_i. By replacing the arc v_0u with v_0v_m and then following the rest of the ℓ-cycle, we now have a shortest cycle that uses v_0v_m. Let C be such a shortest cycle through v_0. Label the vertices on C (starting with v_0) as $v_0 = u_0$, $v_m = u_1$, $\ldots$, $u_{\ell-1}$.

It is easy to see that vertex j maps to $v_\ell, v_{\ell+1}, \ldots, v_m$ by using the outer n-cycle together with the arcs $v_{\ell-1}v_i$ where $i \in \{\ell, \ell+1, \ldots, m\}$. Furthermore,

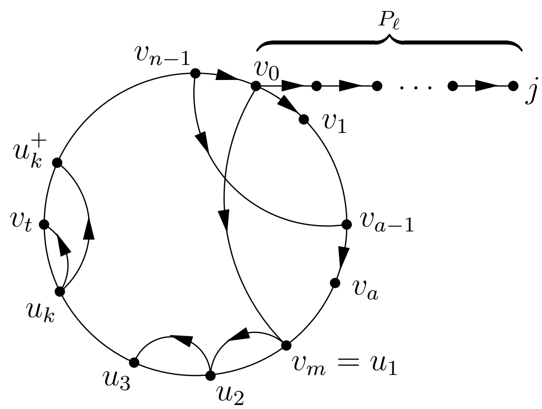

FIGURE 15. The sub-indicator.

j maps to any v_t with $m < t \leq n-1$: for a given v_t, let $s = \max\{x \mid x < t \text{ and } v_x \in C\}$. Since $v_s \in C$, $v_s = u_k$ for some $1 \leq k \leq \ell - 1$. The idea is that u_k is the first vertex on C, not including v_t, that is encountered when moving backwards along C. The successor u_k on C, $u_k^+ = v_i$, with $i \geq t$. This implies that $u_k v_t$ is an arc of D. In order to obtain a walk of length ℓ from $v_0 = u_0$ to v_t we first proceed along the n-cycle from v_0 to $v_{\ell-(k+1)}$ $(0 \leq \ell - (k+1) < m)$. Next, we use the arc $v_{\ell-(k+1)}v_m = v_{\ell-(k+1)}u_1$, then proceed along C to u_k and finally use the arc $u_k v_t$.

By using C, it is clear that j maps to v_0. The resulting digraph derived from using this sub-indicator, contains C. It therefore has at least one cycle.

We now show that vertex j does not map to v_a. Here we have that $N^-(v_a) \subseteq \{v_0, v_1, \ldots, v_{a-1}\}$. In order to obtain a contradiction assume that $W = x_0 x_1 \cdots x_{\ell-1}$ is walk of length ℓ from v_0 to v_a. That is,

$$x_0 = v_0,\ x_1 = v_{j_1},\ x_2 = v_{j_2},\ \ldots,\ x_{\ell-2} = v_{j_{\ell-2}}, \text{ and } x_{\ell-1} = v_a.$$

Let $k = \max\{0, j_1, j_2, \ldots, j_{\ell-2}, a\}$. Note that $k > a$, otherwise $j_i \leq a$ and W is a walk of length ℓ inside the transitive tournament $D[v_0, v_1, \ldots, v_a]$. Furthermore $v_k v_a$ is not an arc of D, for if it was then $d^+(v_{n-1}) \geq \ell$. Therefore $d(v_k, v_a) \geq 2$ and so the length of the sub-walk of W from v_0 to v_k is at most $\ell - 2$. In order to complete W, $v_k v_i \in A(D)$ for some $i \in \{0, 1, \ldots, a-1\}$. This implies that $v_k v_0 \in A(D)$ which means that there is a closed walk from v_0 to itself of length at most $\ell - 1$. This is clearly not possible and so j cannot map to v_a.

Now, if $d^+(v_0) > l$, then the result, say D', of this sub-indicator has at least two cycles, but does not contain v_a. Since it is smaller than D it is not a counterexample and so D'-colouring is NP-complete. Therefore D-colouring is NP-complete, a contradiction. Thus $d^+(v_0) = l$.

On the other hand, if there is a vertex $v_i \in \{v_m = v_\ell, v_{\ell+1}, \ldots, v_{n-2}\}$ with $d^+(v_i) \geq 2$, then the result of the sub-indicator will again have at least two cycles and as above this leads to D-colouring being NP-complete, a contradiction. This shows that $D[v_m = v_\ell, v_{\ell+1}, \ldots, v_{n-1}]$ is an induced path. Note that, in this case, v_{n-1} is the predecessor of v_0 on the ℓ-cycle C above.

If we now have that $d^+(v_{n-1}) \geq 2$, j will map to v_1 by using C. Therefore, the sub-indicator results in a digraph with at least two cycles and fewer vertices than D (j still does not map to v_a). Once again, we conclude that D-colouring is NP-complete, a contradiction. Hence, $d^+(v_{n-1}) = 1$.

We now apply the sub-indicator shown below in Figure 16. This is formed using a copy of D and attaching an oriented path as shown to v_0.

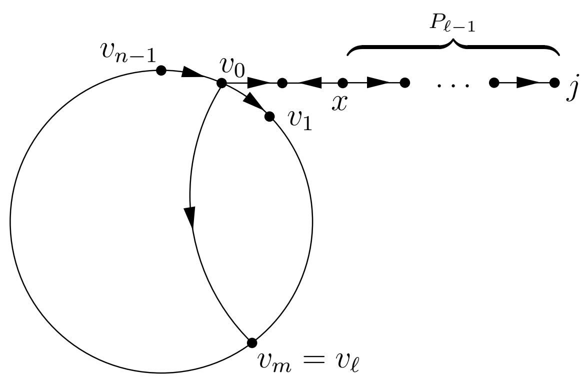

FIGURE 16. The new sub-indicator.

No vertex between v_0 and v_ℓ is adjacent to a vertex v_i with $i > \ell + 1$, otherwise v_ℓ has out-degree at least two. Therefore any vertex between v_0 and v_ℓ is adjacent to $v_{\ell+1}$ so that it can be on a cycle of length ℓ. Also note that the vertex x shown above can map to any of $\{v_0, v_1, \ldots, v_{\ell-1}\}$.

The distance between v_ℓ and v_0 is exactly $\ell - 1$, because of the ℓ-cycle C through v_0. This allows j to map to v_0: first map x to $v_i \in \{v_1, \ldots, v_{\ell-1}\}$, then use $v_i v_{\ell+1}$ together with the path of length $\ell - 2$ from $v_{\ell+1}$ to v_0. It is also possible to map j to $v_i \in \{v_{\ell-1}, v_\ell, \ldots, v_{n-1}\}$: first map x to $v_{i-(\ell-1)}$ $(0 \leq i - (\ell - 1) \leq n - 1 - (\ell - 1) = n - \ell = \ell - 1)$, then follow the outer n-cycle from $v_{i-(\ell-1)}$ to v_i.

This sub-indicator therefore guarantees that we have at least the ℓ-cycle C through v_0 and the vertex $v_{\ell-1}$. Thus the result has at least two cycles.

We now note that j does not map to v_1, otherwise there exists a directed walk of length $\ell - 1$ from $v_i \in \{v_0, v_1, \ldots, v_{\ell-1}\}$ (since x can only map to v_i) to v_1 and such a walk does not exist. Therefore the result has at least one fewer vertex and so its colouring problem is NP-complete, forcing D-colouring to be NP-complete, a contradiction. □

The next two results deal with the Frobenius-Schur index of the cycle lengths of a minimum counterexample. The first one may seem somewhat artificial, but it is needed to establish the second.

Lemma 5.5. *Let D be a round local tournament on the vertex set $\{0, 1, \ldots, n-1\}$ with $n = 2\ell - 2$, where ℓ is the length of a shortest cycle in D. If for every $v \in \{\ell-2, \ell-1, \ldots, n-2\}$, $N^+(v) = v+1 \pmod n$, then D cannot be a minimum counterexample to the D-colouring problem.*

Proof. Let $\phi(\ell, \ell+1, \ldots, n) = \phi$ be the Frobenius-Schur index of the cycle lengths of D. Recall that ϕ is the smallest integer such that every integer $x \geq \phi$ can be written as a linear combination of $\ell, \ell+1, \ldots, n$. By the minimality of ϕ, $\phi - 1$ cannot be written as such a linear combination. Therefore D does not possess a closed walk of length $\phi - 1$. If it did, this closed walk can be decomposed into arc disjoint cycles with lengths in the set $\{\ell, \ell+1, \ldots, n\}$. This is equivalent to writing $\phi - 1$ as a linear combination of $\ell, \ell+1, \ldots, n$ which is not possible.

We assume that D is a minimum counterexample and derive a contradiction. Since $n = 2\ell - 2$,

$$\phi = \left\lfloor \frac{n-2}{n-\ell} \right\rfloor \ell = 2\ell.$$

Therefore $\phi - 1 = 2\ell - 1 = n + 1$. Also, since 0 has to be on an ℓ-cycle, $0(\ell-1) \in A(D)$ and by the local tournament property $0a \in A(D)$ where $a \in \{1, 2, \ldots, \ell-1\}$. To derive the contradiction, we use a sub-indicator construction. This sub-indicator is constructed from a copy of D and by attaching a path of length $\phi - 1$ to vertex 0 and taking the end of this path to be the vertex j. Some of the possible images of j are: $1, 2, \ldots \ell - 1$. This is accomplished by going around the n-cycle once back to 0 and then using the arc $0a$, $a \in \{1, 2, \ldots, \ell-1\}$. To map j to vertex x, where $\ell \leq x \leq n-1$, we proceed as follows: use an $(n - (x - \ell + 1))$-cycle to go from 0 to 0 ($\ell \leq n-(x-\ell+1) \leq n-1$). Then use the remaining $\phi-1-(n-(x-\ell+1)) = x-\ell+2$ arcs ($2 \leq x - \ell + 2 \leq n - \ell + 1$) first to go from 0 to $\ell - 1$ and then from $\ell - 1$ to x along the n-cycle. Its impossible for j to map to 0 since this would correspond to a closed walk from 0 to 0 of length $\phi - 1$. Therefore the result of applying this sub-indicator is $D - \{0\}$. Furthermore, since $\ell - 2$ is not adjacent to ℓ, 1 is not adjacent to ℓ. In order for 1 to be on an ℓ-cycle, $(n-1)$ is adjacent to 1. Therefore $D - \{0\}$ has at least two cycles. Since D is a minimum counterexample, $(D - \{0\})$-colouring is NP-complete, and so D-colouring is NP-complete, a contradiction. $\square$

Lemma 5.6. *If D is a minimum counterexample, then the Frobenius-Schur index, ϕ, of the cycle lengths of D satisfies $\phi = \ell$, where ℓ is the length of a shortest cycle in D.*

Proof. Label the vertices of D as $\{0, 1, \ldots n-1\}$. By Lemma 2.6,

$$\phi = kl \text{ where } k = \left\lfloor \frac{n-2}{n-\ell} \right\rfloor.$$

We assume now that $\phi \geq 2\ell$, and derive a contradiction.

If $\phi \geq 2\ell$, then

$$k = \left\lfloor \frac{n-2}{n-\ell} \right\rfloor \geq 2,$$

and this happens if and only if $(n-2)/(n-\ell) \geq 2$ which is equivalent to $2\ell \geq n+2$. To obtain the sought after contradiction we employ as an indicator a directed path of length $\phi - 1$. This indicator has the property that its result will not contain loops by the fact that D has no closed walks of length $\phi - 1$.

Note that $\phi - 1 = (k-1)\ell + (\ell - 1)$. This implies that in applying the indicator to D, we obtain an arc from 0 to $\ell - 1$ (use $(k-1)$ ℓ-cycles to get from 0 to 0, and then use the remaining $(\ell - 1)$ arcs to get from 0 to $\ell - 1$). Also, $\phi - 1 = (k-2)\ell + (\ell+1) + (\ell-2)$. This produces an arc from 0 to $\ell - 2$ when applying the indicator.

We now show that it is also possible to have arcs from $\ell - 1$ to 0 and from $\ell - 2$ to 0. The number of arcs on the outer n-cycle from $\ell - 1$ to 0 is $n - \ell + 1$. Consider now $\phi - 1 - (n - \ell + 1) = \phi - (n - \ell + 2)$. If it is possible to write $\phi - (n - \ell + 2)$ as a linear combination of cycle lengths, then there will a closed walk from any vertex back to itself of this length. In particular, there will be a walk of this length from $\ell - 1$ to $\ell - 1$. A further $n - \ell + 1$ arcs will then carry the walk from $\ell - 1$ to 0 along the outer n-cycle. Therefore, applying the indicator above will produce an arc from $\ell - 1$ to 0 provided that $\phi - (n - \ell + 2)$ is a sum of cycle lengths. This is indeed the case: We know that all $x \geq \phi = k\ell$ is a sum of cycle lengths. Furthermore $(k-1)\ell, (k-1)\ell + 1, \ldots, (k-1)n$ are also linear combinations (of $k-1$) of the cycle lengths. Let $t = k\ell - (k-1)n$, this represents the length of a "gap" of integers that cannot be written as a linear combination of the cycle lengths. We see that $k(\ell - n) = t - n$ so that

$$k = \frac{t-n}{\ell - n} = \frac{n-t}{n-\ell} = \frac{n-2}{n-\ell} - \frac{t-2}{n-\ell}.$$

Since

$$k = \left\lfloor \frac{n-2}{n-\ell} \right\rfloor,$$

we get that $(t-2)/(n-\ell) < 1$. Therefore $t < n - \ell + 2$ or $t \leq n - \ell + 1$. Further from $2\ell \geq n+2$ it follows that $n - \ell + 2 \leq \ell$. Thus

$$\phi - (n - \ell + 2) \in \{(k-1)\ell = \phi - \ell, (k-1)\ell + 1, \ldots, (k-1)n = \phi - t\}$$

and so $\phi - (n - \ell + 2)$ is a sum of cycle lengths. To obtain an arc from $\ell - 2$ to 0, we are searching for a walk of length $\phi - 1$ from $\ell - 2$ to 0. The number of arcs on the outer n-cycle from $\ell - 2$ to 0 is $n - \ell + 2$. Here $\phi - 1 - (n - \ell + 2) = \phi - (n - \ell + 3)$. Also $\phi - (n - \ell + 3) > \phi - (n - \ell + 2) > t$ and $\phi - (n - \ell + 3) \leq \ell + 1$. If $\phi - (n - \ell + 3) \leq \ell$, then

$$\phi - (n - \ell + 3) \in \{(k-1)\ell = \phi - \ell, (k-1)\ell + 1, \ldots, (k-1)n = \phi - t\}.$$

This enables a closed walk from $\ell - 2$ to $\ell - 2$ of length $\phi - (n - \ell + 3)$ to be formed. The remaining $n - \ell + 2$ arcs on the walk of length $\phi - 1$ then goes from $\ell - 2$ to 0. What remains at this point is the case $n - \ell + 3 = \ell + 1$, or $n - \ell + 2 = \ell$. This implies that the number of arcs on the n-cycle from $\ell - 2$ to 0 is ℓ. If there exists a vertex in $\{\ell - 2, \ell - 1, \ldots, n - 2\}$ with out-degree at least two, then there exists a path of length $\ell - 1$ from $\ell - 2$ to 0. This path together with a closed walk of length $\phi - (n - \ell + 2)$ provides us with a walk of length $\phi - 1$ from $\ell - 2$ to 0. If there does not exist a vertex in $\{\ell - 2, \ell - 1, \ldots, n - 2\}$ with out-degree at least two, then $D[\ell - 2, \ell - 1, \ldots, n - 1, 0]$ is an induced path. This is not possible by Lemma 5.5.

We now see that using a path of length $\phi - 1$ as an indicator, produces the arcs $0(\ell - 1)$, $(\ell - 1)0$, $(\ell - 2)0$ and $0(\ell - 2)$. The same, of course, applies to any other vertex in D. Starting at vertex a, we have symmetric arcs between a and $a + (\ell - 1)$ as well as between a and $a + (\ell - 2)$, where addition is done modulo n. Again, if it is not possible to obtain a path of length $\ell - 1$ from $a + (\ell - 2)$ to a, then we are in the case dealt with by Lemma 5.5. If we now apply as arc-subindicator the digraph with vertices $\{i, j\}$ and arcs ij and ji, with respect to the arc ij, the end result will be only the symmetric arcs or in other words the undirected part of the indicator construction. This undirected part contains the circulant on the vertices $\{0, 1, \ldots, n-1\}$ and the edges $a(a + (\ell - 1))$ and $a(a + (\ell - 2))$. Such a circulant is not bipartite and so colouring by the undirected portion of the indicator construction (that was obtained through the sub-indicator construction) is NP-complete, and so colouring by the whole result of the indicator construction is NP-complete. This implies that D-colouring is NP-complete, which is a contradiction. □

The final contradiction is now obtained by noting that if D is a minimum counterexample to the D-colouring problem, then D has $\Delta^+ < \ell$ and $\phi = \ell$. Here ℓ is the length of a shortest cycle in D and ϕ is the Frobenius-Schur index of the cycle lengths of D. We show next that these two conditions on D are incompatible, and so D does not exist.

Lemma 5.7. *Let D be a round local tournament with the Frobenius-Schur index of its cycle lengths $\phi(\ell, \ell + 1, \ldots, n) = \phi = \ell$ and $\Delta^+(D) < \ell$, where ℓ is the length of a shortest cycle in D. Then D cannot be a minimum counterexample to the D-colouring problem.*

Proof. We assume that D is a minimum counterexample and then derive a contradiction.

Label the vertices of D as $\{0, 1, \ldots, n - 1\}$. Since

$$\phi = \left\lfloor \frac{n-2}{n-\ell} \right\rfloor \ell = \ell,$$

$(n - 2)/(n - \ell) < 2$, that is $2\ell \leq n + 1$ or $n - \ell + 1 \geq \ell$. Also $\Delta^+ < \ell$ and so every vertex has out-degree at most $\ell - 1$.

To obtain the contradiction we use an indicator that is equal to a path of length $\ell - 1$ with the two end-vertices of the path as the distinguished vertices of the indicator. We see that the result of this indicator, D^*, has an arc from 0 to $\ell-1$ (using the n-cycle from 0 to $\ell-1$). Also, there is an arc from 0 to ℓ in D^*: If there is at least one vertex in $\{0,1,\ldots,\ell-2\}$ with out-degree at least two, then there exists a path of length $\ell-1$ from 0 to ℓ. We show that such a vertex exists. Assume that all vertices in $\{0,1,\ldots,\ell-2\}$ have out-degree one. Then $D[0,1,\ldots,\ell-1]$ is an induced path of length $\ell-1$. In order for 0 to be on a cycle of length ℓ, $(\ell-1)0 \in A(D)$. This implies that $d^+(\ell-1) = n-(\ell-1) \geq \ell$, which contradicts $\Delta^+ < \ell$. Therefore, the required vertex exists and there is a path of length $\ell-1$ from 0 to ℓ.

Next, we show that there are also arcs from $\ell-1$ to 0 and from ℓ to 0.

Let $x_0, x_1, \ldots, x_{\ell-1}, x_0$ be an ℓ-cycle starting at $x_0 = \ell-1$. Since $\Delta^+ < \ell$, $N^-(\ell-1) \subseteq \{0,1,\ldots,\ell-2\}$. Let x_k be the maximum of $\{x_0,x_1,\ldots,x_{\ell-1}\}$. By the local tournament property $x_0,x_1,\ldots,x_k,0$ is a path, say P, from $\ell-1$ to 0 of length at most $\ell-1$. If P has length $\ell-1$ we are done. Otherwise, recall $n-(\ell-1) = n-\ell+1 \geq \ell$. Thus, there is at least one vertex, say $y \in \{\ell,\ell+1,\ldots,n-1\} \setminus V(P)$, between successive vertices of P, say x' and x''. (Note $x' = x_j$, $x'' = x_{j+1}$ or $x' = x_k$, $x'' = 0$.) By the local tournament property $x'y$ and yx'' are arcs. Thus, P may be augmented with y to obtain a path from $\ell-1$ to 0 of length $\ell(P)+1$. Continuing in this manner, one obtains a path of length $\ell-1$ from vertex $\ell-1$ to 0.

To see that we will obtain an arc from vertex ℓ to vertex 0 we proceed in a similar way as before. The length of the path from vertex ℓ to vertex 0 on the n-cycle is $n-\ell \geq \ell-1$. As above, using an ℓ-cycle through vertex ℓ, one can construct a path of length $\ell-1$ from vertex ℓ to vertex 0.

The above, of course, also applies to any vertex a: we obtain symmetric arcs between a and $a+(\ell-1)$ and between a and $a+\ell$, where addition is done modulo n. This implies that the result of the indicator construction, D^*, contains the undirected circulant with edges $a(a+\ell-1)$ and $a(a+\ell)$. This circulant is not bipartite. If we now apply the arc sub-indicator with vertices $\{i,j\}$ and arcs ij and ji to D^*, this sub-indicator results in the undirected portion, say D^{**}, of D^* (which contains the circulant). D^{**}-colouring is NP-complete, therefore D^*-colouring is NP-complete. Thus D-colouring is NP-complete, a contradiction. $\square$

Theorem 5.8. *If D is a round local tournament, containing at least two cycles, then D-colouring is NP-complete.*

Proof. Since a minimum counterexample was shown not to exist, we conclude that the theorem is indeed true. $\square$

6. Round Decomposable Local Tournaments

Let $D = R[D_0, D_1, \ldots, D_{n-1}]$ be a round decomposable local tournament with at least two directed cycles. Here R is a round local tournament on $n \geq 2$ vertices and each D_i is a strongly connected tournament. The proof of NP-completeness in this case will proceed as follows:

- D-colouring is NP-complete if there exists at least one D_i with $|D_i| \geq 4$.

- Therefore $|D_i| = 1, 3$. If R is acyclic, D-colouring is NP-complete.
- If R contains a cycle and $|D_i| = 1, 3$, then
  - D-colouring is NP-complete if $|D_i| = 3$ for at least two D_i's.
  - Now $|D_i| = 3$ for exactly one i.
    * D-colouring is NP-complete if $R = \overrightarrow{C}_n$.
    * D-colouring is NP-complete if $R \neq \overrightarrow{C}_n$.

Lemma 6.1. *Let $D = R[D_0, D_1, \ldots, D_{n-1}]$ be a round decomposable local tournament with $|D_i| \geq 4$ for at least one $i \in \{0, 1, \ldots, n-1\}$. Then the D-colouring problem is NP-complete.*

Proof. To prove this result we use the sub-indicator shown below in Figure 17. This is constructed using a copy of D_{i-1} and D_{i+1}, provided that both D_{i-1} and D_{i+1} exist. If only one of D_{i-1} and D_{i+1} exists (at least one exists), use only the one that exists. Add to this a vertex j such that $V(D_{i-1})$ dominates j and j dominates $V(D_{i+1})$. We take the vertices $k_1, k_2, \ldots, k_t$ of J to be exactly $V(D_{i-1}) \cup V(D_{i+1})$.

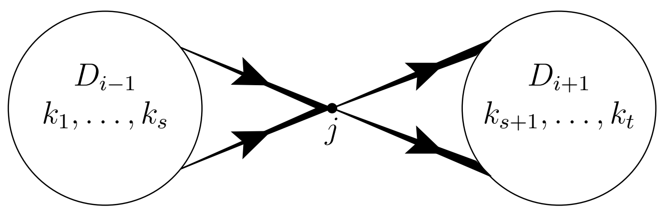

FIGURE 17. The sub-indicator for the first round decomposable case.

Furthermore we take the vertices $x_1, x_2, \ldots, x_t$ of D (required for the sub-indicator construction) also to be $V(D_{i-1}) \cup V(D_{i+1})$. In this way when we perform the sub-indicator construction we force the copy of D_{i-1} (D_{i+1}) in J to map to D_{i-1} (D_{i+1}) in D. Since j retracts to every vertex of D_i, the result of the sub-indicator construction, D^+, is exactly D_i. Since D_i is a strong tournament on at least 4 vertices, D_i-colouring is NP-complete. Therefore D-colouring is NP-complete. □

We have now reduced the problem to that of considering round decomposable local tournaments $D = R[D_0, D_1, \ldots, D_{n-1}]$ (with at least two directed cycles) where each $|D_i| = 1, 3$.

Lemma 6.2. *Let $D = R[D_0, D_1, \ldots, D_{n-1}]$ be a round decomposable local tournament containing at least two directed cycles, and with each $|D_i| = 1, 3$. If R is acyclic, then D-colouring is NP-complete.*

Proof. Since D contains at least two directed cycles and R is acyclic, there are at least two D_i's with $|D_i| = 3$ (each such D_i is a directed triangle). For a given D_i, let v_i be the corresponding vertex in R. Let $d = \max\{d(v_i, v_j) \mid |D_i| = |D_j| = 3\}$ and define

$$k = \begin{cases} d & \text{if } d \not\equiv 0 \pmod 3, \\ d+1 & \text{if } d \equiv 0 \pmod 3. \end{cases}$$

We use as an indicator two directed C_3s with a path of length k joining the two C_3s. Take as vertex a the first vertex of the path and as vertex b the last vertex of the path. This is shown in Figure 18.

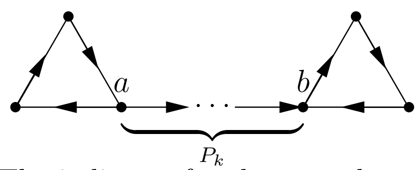

FIGURE 18. The indicator for the second round decomposable case.

Both vertex a and vertex b map to C_3s in D. Furthermore by the choice of k any vertex in a D_i with $|D_i| = 3$ is within a distance k of any other vertex in a D_j with $|D_j| = 3$ where $j \geq i$. Also, a and b cannot map to the same vertex. Each C_3 in D results in a C_3 in D^*, either the C_3 is preserved or its direction is reversed, depending on whether k is congruent to 1 or 2 (mod 3), respectively. Therefore, if D contains $t \geq 2$ C_3s, $D^* = T_t[C_3, C_3, \ldots, C_3]$, where T_t is a transitive tournament on t vertices. Hence D^*-colouring is NP-complete, and so D-colouring is NP-complete. □

We now consider the case where $D = R[D_0, D_1, \ldots, D_{n-1}]$ is a round decomposable local tournament where each $|D_i| = 1, 3$ and R contains at least one cycle. The proof is much the same as the case where R is acyclic except that more care is needed to ensure that one does not introduce loops when using the indicator construction.

Lemma 6.3. *Let $D = R[D_0, D_1, \ldots, D_{n-1}]$ be a round decomposable local tournament such that R contains at least one cycle and $|D_i| = |D_j| = 3$ for $i \neq j$ and $i, j \in \{0, 1, \ldots, n-1\}$. Then D-colouring is NP-complete.*

Proof. Let ℓ be the length of a shortest cycle in R. Then for any $u, v \in V(R)$, u and v are within a distance $\lfloor \ell/2 \rfloor + 1$ of each other. This is accomplished by using the arcs of a shortest cycle as well as the local tournament property. Note also that $\ell - 1 \geq \lfloor \ell/2 \rfloor + 1$. Here we use one of the two indicators shown in Figure 19, depending on whether $\ell - 1 \not\equiv 0 \pmod 3$ or $\ell - 1 \equiv 0 \pmod 3$.

The two vertices a and b, in either case, map to a 3-cycle in D. Also, a and b never map to the same vertex: If the whole indicator maps to the same C_3 in D, then clearly a and b have different images (in both cases). Therefore, in order for a and b to possibly map to the same vertex in D, the image of the indicator has to involve vertices in D that are not all restricted

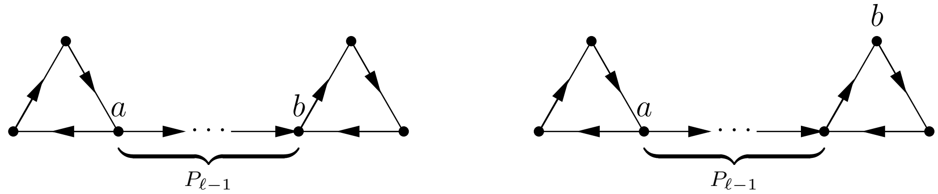

FIGURE 19. The two indicators for the third round decomposable case. The one on the left is used when $\ell - 1 \not\equiv 0 \pmod 3$, the one on the right when $\ell - 1 \equiv 0 \pmod 3$.

to the same C_3. This would imply though that there exists a closed walk in R of length $\ell - 1$, which is not possible.

By mapping the whole indicator (in either case) to the same C_3 in D, we either preserve the C_3, or reverse its orientation. Furthermore any two vertices on two distinct C_3s in D are within distance $\lfloor \ell/2 \rfloor + 1$ of each other. Since $\ell - 1 \geq \lfloor \ell/2 \rfloor + 1$, the result of applying the appropriate indicator, D^*, will be a semi-complete digraph with at least two C_3s. This means that D^*-colouring is NP-complete, and so D-colouring is NP-complete. $\square$

At this point we are left with a round decomposable local tournament $D = R[D_0, D_1, \ldots, D_{n-1}]$, where exactly one $|D_i| = 3$. This is dealt with in two ways: first when R is a directed n-cycle, and secondly when R is not a directed n-cycle.

Lemma 6.4. *Let $D = R[D_0, D_1, \ldots, D_{n-1}]$ be a round decomposable local tournament with exactly one $|D_i| = 3$ and $R = C_n$. Then D-colouring is NP-complete.*

Proof. The proof is by induction on n. Without loss of generality we may assume that $|D_0| = 3$. That is, $D_0 = C_3$ and $D_j = K_1$ for $j \in \{1, 2, \ldots, n-1\}$. Also, label the vertices of $R = C_n$ as $v_0, v_1, \ldots, v_{n-1}$, so that v_i corresponds to D_i for $i = 0, 1, \ldots, n-1$. The first base case ($n = 3$) is actually a tournament and is shown below in Figure 20. Therefore in this case D-colouring is NP-complete. The second base case ($n = 4$) is shown in Figure 21.

Here we first apply an indicator equal to a directed path of length 2, where the vertices i and j of the indicator are taken to be the initial vertex of the path and the terminal vertex of the path, respectively. This produces the result, D^*, shown in Figure 22. Note that the orientation of the C_3 changed, and that there are symmetric arcs between v_2 and the triangle and between v_1 and v_3. These symmetric arcs are drawn as undirected edges. D^* is also a core.

At this point we apply a sub-indicator, J, that is also equal to a directed path of length 2. We let j be the terminal vertex of the path and k_1 the initial vertex of the path. We also take $x_1 = v_2$. That is we identify k_1 in J with v_2 in D^* and consider all retractions to D^*, keeping track of the

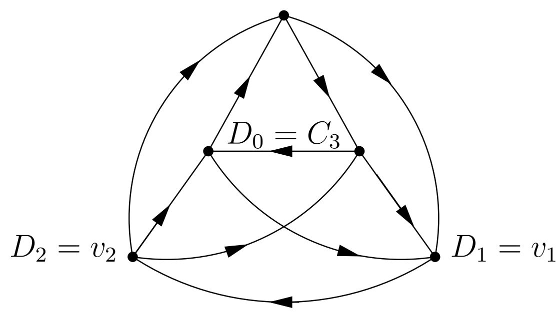

FIGURE 20. The first base case for the fourth round decomposable case.

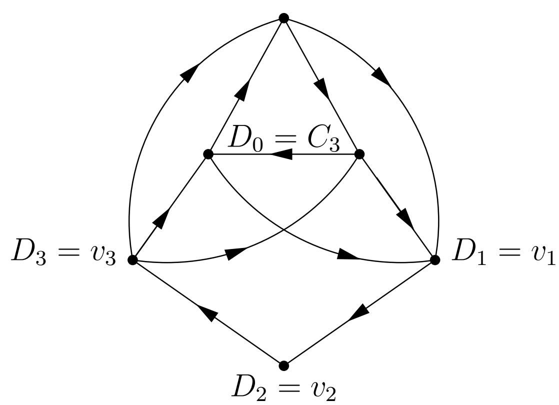

FIGURE 21. The second base case for the fourth round decomposable case.

possible images of j. The result, D^{*+}, is equal to $D^* - v_3$, the core of which is a wheel with three spokes (or a semi-complete digraph on four vertices with at least two cycles). This sequence of digraphs is shown in Figure 22.

Since the wheel-colouring problem is NP-complete, we find that D^*-colouring is NP-complete, and that ultimately, D-colouring is NP-complete.

This completes the two base cases. The rest of the proof proceeds in exactly the same way as the second base case. The only exception is that the end-result is not a wheel, but a smaller instance of the same problem.

Let $D = C_n[C_3, K_1, \ldots, K_1]$ and $n \geq 5$. Assume that D'-colouring is NP-complete for every $D' = C_m[C_3, K_1, \ldots, K_1]$ with $m < n$. As with the second base case, we first apply an indicator, I, equal to a path of length 2, with i and j equal to the initial and terminal vertices of the path, respectively. The result, D^*, is shown in Figure 23. Note that, as before, the orientation of the C_3 (shown in the figure as the small triangle) changes. We also have

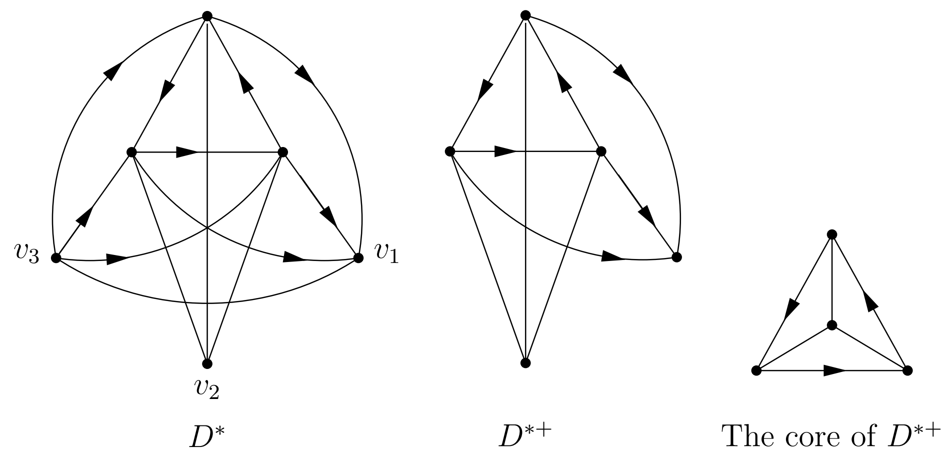

FIGURE 22. The sequence of digraphs obtained for the second base case.

the arcs v_0v_1, v_0v_2, $v_{n-2}v_0$, $v_{n-1}v_0$ and $v_{n-1}v_1$. In addition to these we also have the arcs v_iv_{i+2} for $i = 1, 2, \ldots, n-3$.

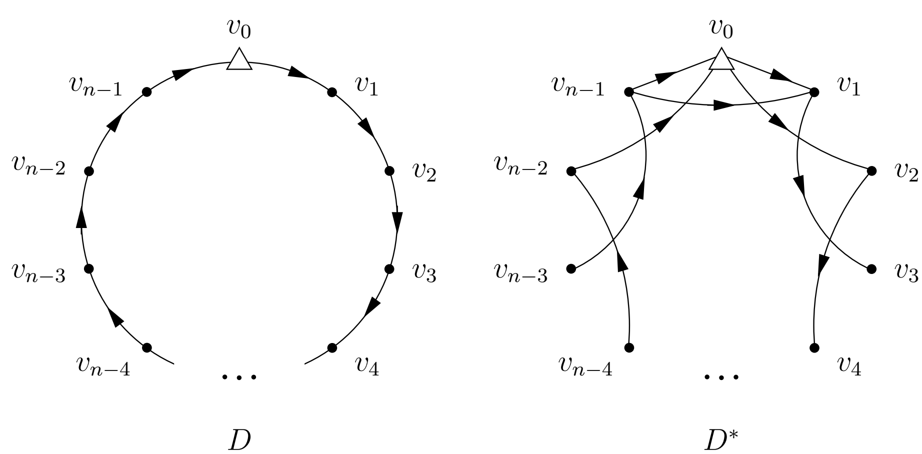

FIGURE 23. Applying the indicator I to D yields D^*.

We now apply the sub-indicator, J, that is equal to a path of length $n-2$, with the terminal vertex equal to j and the initial vertex equal to k_1. We also take $x_1 = v_2$ in D^*. Therefore we identify k_1 and v_2 and consider all retractions to D^*, recording the images of j in the process. We now complete the proof based on the parity of n.

Let n be even. Then v_0 and v_2 are on an $(n/2)$-cycle in D^*: $v_0v_2v_4 \cdots v_{n-4}v_{n-2}$. This means that there exists a path of length $n/2 - 1$ from v_2

to v_0. Since $n \geq 5$, $n-2 > n/2-1$, and so j maps to every vertex on the C_3 (by varying where we enter the C_3 and then wrapping around it). Furthermore, v_0 is on an $(n/2)$-cycle with all the v_i's where i is even and on an $(n/2+1)$-cycle with all the v_js, j odd. Using the first $n/2-1$ arcs of the P_{n-2} sub-indicator, we move from v_2 to v_0, this means that the remaining $(n-2)-(n/2-1) = n/2-1$ arcs can be used to go from v_0 to every v_i, (i even) and from v_0 to every v_j (j odd) except v_{n-1}. This is done by wrapping around the C_3 first, if needed. To see that j cannot map to v_{n-1}, note that the directed distance from v_2 to v_{n-1} is $n-1$. The core of this can be found by wrapping the path $v_0v_1v_3v_5\cdots v_{n-3}$ around the C_3. The result of this sub-indicator, D^{*+}, has a core equal to $C_{n/2}[C_3, K_1, \ldots, K_1]$. By the induction hypothesis, D^{*+}-colouring is NP-complete, so that D^*-colouring is NP-complete, implying that D-colouring is NP-complete in this case.

Let n be odd. Then v_0 and v_2 are on an $((n+1)/2)$-cycle in D^* given by $v_0v_2v_4\cdots v_{n-3}v_{n-1}$. Therefore we have a path of length $(n+1)/2-1 = (n-1)/2$ from v_2 to v_0. As before j maps to every vertex on the C_3. Now v_0 is on an $((n+1)/2)$-cycle with the v_i's (i even). Also, v_0 is on an $((n+1)/2)$-cycle with the v_js (j odd). Here we can use the first $(n-1)/2$ arcs of the P_{n-2} sub-indicator to go from v_2 to v_0. Then use the remaining $(n-2)-(n-1)/2 = (n-3)/2$ arcs to map j to $v_1, v_2, \ldots, v_{n-3}$ (again one may have to wrap the path around the C_3 first). It is also possible to map j to v_{n-2}, first go from v_2 to v_{n-1} using the $(n+1)/2-2 = (n-3)/2$ arcs on the cycle through the v_i's with i even. Now use the remaining $(n-2)-(n-3)/2 = (n-1)/2$ arcs on the P_{n-2} for the following walk: $v_{n-1}v_1v_3v_5\cdots v_{n-2}$. It is not possible to map j to v_{n-1}, first note that the shortest cycle through v_{n-1} has length $(n+1)/2$ ($v_{n-1}v_0v_2v_4\cdots v_{n-3}$). Secondly, starting at v_2, one is forced to use the arcs $v_{2i}v_{2i+2}$ for $i = 1, 2, \ldots, (n-3)/2$ (a total of $(n-3)/2$ arcs). This takes us to v_{n-1}. At this point, if j is going to map to v_{n-1}, the remaining $(n-2)-(n-3)/2 = (n-1)/2$ arcs will have to form a closed walk from v_{n-1} back to itself, which is not possible. Thus the result of this sub-indicator, D^{*+}, is $D^* - v_{n-1}$. The core of this may be obtained by wrapping the path $v_0v_2v_4\cdots v_{n-3}$ around the C_3. This core is equal to $C_{(n+1)/2}[C_3, K_1, \ldots, K_1]$. Therefore by the induction hypothesis, D^{*+}-colouring is NP-complete, implying that D^*-colouring is NP-complete. It now follows that D-colouring is NP-complete in this case.

This completes the proof. $\square$

We have now reached the final case for the round decomposable local tournaments.

Lemma 6.5. *Let $D = R[D_0, D_1, \ldots, D_{n-1}]$ be a round decomposable local tournament with exactly one $|D_i| = 3$ and $R \neq C_n$, but R contains at least one cycle. Then D-colouring is NP-complete.*

Proof. Label the vertices of R as $v_0, v_1, \ldots, v_{n-1}$. Then to each vertex v_i of R there corresponds a component D_i of the round decomposition. As in the previous proof, we assume that $D_0 = C_3$ and that $D_i = K_1 = v_i$ for $i = 1, 2, \ldots, n-1$.

Let P be the set of predecessors of v_0 on the shortest cycles through v_0 and define $j = \max\{k \mid v_k \in P\}$. Then v_j is the "closest" predecessor of v_0 on all the shortest cycles through v_0. Denote the length of a shortest cycle through v_0 by ℓ.

If there exists a vertex v_i with $0 \le i \le j-1$ such that $d^+(v_i) \ge 2$ (in R), we use the following sub-indicator: a path of length $\ell - 1$ with the terminal vertex equal to j and the initial vertex equal to k_1. Also, let any vertex of $D_0 = C_3$ be x_1. Therefore we identify k_1 with x_1 and consider all retractions of the path to D. The possible images of j are: exactly one vertex of D_0 (by wrapping around the 3-cycle in D_0) and all vertices v_t with $t = 1, 2, \ldots, j$ (by wrapping around D_0, if necessary, and then using one of the shortest cycles through v_0). This sub-indicator is illustrated in Figure 24.

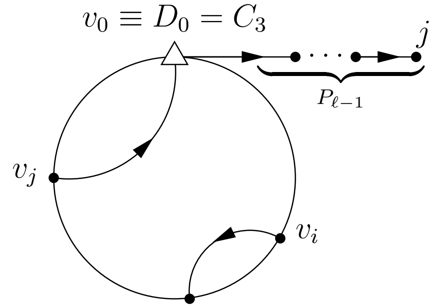

FIGURE 24. An illustration of the first sub-indicator construction in the fifth round decomposable case.

If $0 \le i \le j-2$, then since $j \mapsto v_i$, the result of this sub-indicator, D^+, is a round local tournament with at least two cycles. If $i = j-1$ (that is no vertex v_t with $0 \le t \le j-2$ has $d^+(v_t) \ge 2$ and $d^+(v_{j-1}) \ge 2$), then v_{j-1} is the predecessor of v_j on a shortest cycle through v_0 and v_j. This implies that there exists a walk of length $\ell - 2$ from v_0 to v_{j-1}. Also, v_{j-1} is not adjacent to v_0 as this would result in a shorter cycle through v_0. Therefore v_{j-1} has an out-neighbour (say v') between v_j and v_0. By following the walk of length $\ell - 2$ by the arc $v_{j-1}v'$, we see that $j \mapsto v'$. This again implies that the result of this sub-indicator, D^+, is a round local tournament with at least two cycles. Therefore D^+-colouring is NP-complete, implying that D-colouring is NP-complete.

We are now left with the case where there does not exist a vertex v_i with $0 \le i \le j-1$ such that $d^+(v_i) \ge 2$ (in R). This would imply that $v_0 v_1 \ldots v_j$ is an induced path (in R).

- If $v_j = v_{n-1}$, then $d^+(v_j) = d^+(v_{n-1}) \ge 2$ since $R \ne C_n$. Here we apply the same sub-indicator as above and the result is again a round local tournament with at least two cycles.

- If $v_j \neq v_{n-1}$, then $v_{j+1} \neq v_0$. Denote the length of a shortest cycle through v_{j+1} by s. Use as a sub-indicator a path of length $s-1$, with the terminal vertex equal to j and the initial vertex equal to k_1. Let x_1 be equal to v_{j+1}, so that we attach the path of length $s-1$ to v_{j+1} and consider all retractions to D. Note that $N^-(v_{j+1}) = \{v_j\}$ and since $v_j v_0 \in A(R)$, we also have $v_{j+1}v_0 \in A(R)$. Therefore the possible images of j are: every vertex in D_0 as well as each v_t with $t = 1, 2, \ldots, j$. This sub-indicator is illustrated in Figure 25.

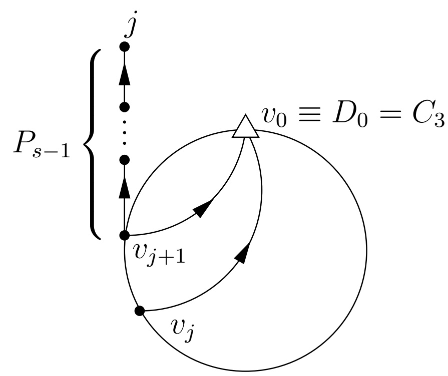

FIGURE 25. An illustration of the second sub-indicator construction in the fifth round decomposable case.

So the result of this sub-indicator, $D^+ = C_{j+1}[C_3, K_1, K_1, \ldots, K_1]$. We know that D^+-colouring is NP-complete, therefore D-colouring is NP-complete.

□

7. Non-Round Decomposable Local Tournaments

In discussing the complexity of non-round decomposable local tournaments, it's not too surprising that the structural result in Lemma 3.2 plays a central role. Note that a non-round decomposable local tournament has at least two directed cycles, since a local tournament with at most one directed cycle is round decomposable. The proof that colouring by a non-round decomposable local tournament is NP-complete proceeds along a familiar line. We assume that the result is false and examine a smallest counterexample (one that has the minimum number of vertices). Such a counterexample is then shown not to exist, and the result follows.

The counterexample will have the structure described in Lemma 3.2, since it is non-round decomposable. Let S, D'_1, D'_2 and D'_3 be the subsets of vertices of D defined in Lemma 3.2. The first lemma deals with the possibilities for in-and out-neighbours of the vertices in S.

Lemma 7.1. *Let D be a non-round decomposable local tournament that is also a minimum counterexample to the D-colouring problem. Then every* *vertex in S can only have in-neighbours or out-neighbours in D'_2, but not both.*

Proof. Since D is a counterexample, D-colouring is not NP-complete.

D is not round decomposable so it has the structure described in Lemma 3.2. S, D'_1, D'_2 and D'_3 are the subsets of vertices of D defined in Lemma 3.2.

Let $s \in S$ and $a, a' \in D'_2$ such that $as, sa' \in A(D)$. D does not have symmetric arcs, so $a \neq a'$. By Lemma 3.2 there is a vertex $y \in D'_1$ with $ay, a'y, ys \in A(D)$. Lemma 3.2 and Proposition 3.4 imply that there is a vertex $x \in D'_3$ such that $sx, xa' \in A(D)$. This is illustrated in Figure 26.

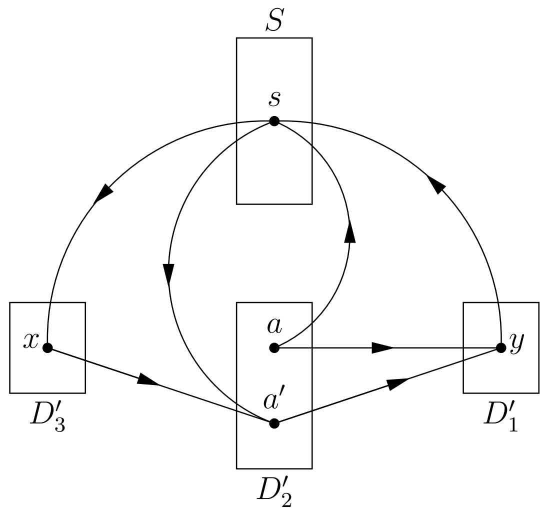

FIGURE 26. Proving that no vertex in S can have both in- and out-neighbours in D'_2.

By Theorem 3.1 D'_2 is a tournament. Therefore a and a' are adjacent.

- If $aa' \in A(D)$, we use a sub-indicator equal to a directed path of length 2, where the terminal vertex is equal to j and the initial vertex is identified with vertex a in D. That is we consider all retractions of this path of length 2 to D keeping track of the images of j and identifying its initial vertex with vertex a in D. The vertex j maps to (at least) the following vertices: s, a', x, y but not the vertex a. The result of this sub-indicator, D^+, will have fewer vertices, but will still contain at least two cycles ($sxa'y$ and $sa'y$). Since D^+ is smaller than D it cannot be a counterexample, and so D^+-colouring is NP-complete, but this would imply that D-colouring is NP-complete, a contradiction.
- If $a'a \in A(D)$, we apply an indicator equal to a directed path of length 2 to D. This will produce the following symmetric arcs: $a's$

($a'ys$ and sxa'), sa ($sa'a$ and ays), ax (asx and $xa'a$), xy ($xa'y$ and ysx) and ya' (ysa' and $a'ay$). Therefore the result of the indicator, D^*, contains an undirected 5-cycle, $a'saxy$, and so it is not bipartite. The undirected portion of D^* may be extracted using an arc sub-indicator equal to a pair of symmetric arcs. This produces an undirected, non-bipartite graph D^{*+}. D^{*+}-colouring is NP-complete, implying that D^*-colouring is NP-complete, which in turn implies that D-colouring is NP-complete, a contradiction.

□

Theorem 7.2. *Let D be a non-round decomposable local tournament, then D-colouring is NP-complete.*

Proof. Assume that the theorem is false. Among all counterexamples to the theorem, let D be one with the minimum number of vertices.

Therefore D is a non-round decomposable local tournament such that D-colouring is not NP-complete. Furthermore, D has the minimum number of vertices possible.

Hence D has the structure described in Lemma 3.2. As before we let S, D_1', D_2' and D_3' be the subsets of vertices of D defined in Lemma 3.2.

By Lemma 3.2 there are arcs between S and D_2', oriented in opposite directions and with their vertices in specific locations with respect to one another. That is there are vertices $s, s' \in S$ and $a, a' \in D_2'$ such that $sa, a's' \in A(D)$. By the previous lemma $s \neq s'$, but a may be equal to a'. Furthermore, Lemma 3.2 implies the existence of a vertex $y \in D_1'$ such that $ay, a'y, ys, ys' \in A(D)$. Also, by Lemma 3.2 and Proposition 3.4, there exists a vertex $x \in D_3'$ together with the arcs $sx, s'x, xa, xa'$. We now distinguish two cases: $ss' \in A(D)$ and $s's \in A(D)$ (by Lemma 3.2 S is semi-complete).

- Let $ss' \in A(D)$. Then $s', a \in N^+(s)$ and since D is a local tournament, s' and a have to be adjacent. The previous lemma forbids the arc $s'a$ and so $as' \in A(D)$. This is illustrated in Figure 27.

  Apply a sub-indicator equal to a directed path of length 2, where the terminal vertex is equal to j and the initial vertex is identified with the vertex s in D. We retract the path to D and determine the images of j. The images of j include the following vertices: s', x, y, a but not s. The result of this sub-indicator, D^+, has fewer vertices than D, but still contains at least two cycles ($s'xay$ and $s'xa$). Since its smaller, D^+-colouring is NP-complete, implying that D-colouring is NP-complete, a contradiction.
- Let $s's \in A(D)$. This is shown in Figure 28.

  Here we apply an indicator equal to a directed path of length 2. This produces the following symmetric arcs: $s'x$ ($s'sx$ and $xa's'$), xy (xay and ysx), ys ($ys's$ and say), sa (sxa and ays) and as' (ays' and $s'xa$). The result of this indicator, D^*, will therefore contain an undirected 5-cycle, $s'xysa$. Thus the undirected portion of D^* is not bipartite. As before, we may extract the undirected portion with

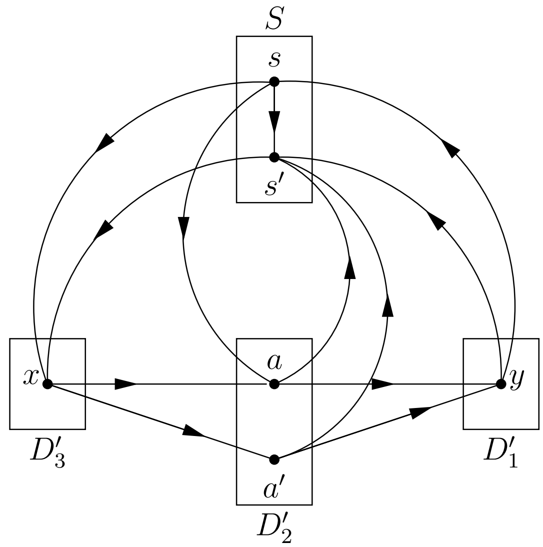

FIGURE 27. The case $ss' \in A(D)$.

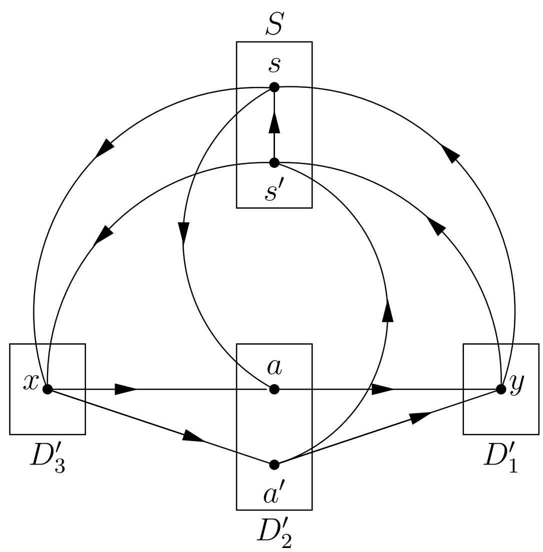

FIGURE 28. The case $s's \in A(D)$.

an arc sub-indicator equal to a pair of symmetric arcs. All of this implies that D^*-colouring is NP-complete which in turn says that D-colouring is NP-complete, a contradiction.

This shows that the counterexample does not exist. Therefore the theorem is true. $\square$

8. The Complexity of Colouring by Connected Local Tournaments With At Least Two Directed Cycles

Theorem 3.3 guarantees that a local tournament with at least two directed cycles falls into one of three categories:

(i) Round local tournaments (Section 5),
(ii) Round decomposable local tournaments (Section 6), or
(iii) Non-round decomposable local tournaments (Section 7).

Since these have all been covered above, we now have the following theorem.

Theorem 8.1. *Let T be a connected local tournament with at least two directed cycles, then HOM_T is NP-complete.*

Proof. A minimal counterexample was shown not to exist, therefore the theorem is true. □

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