



CARDS, PERMUTATIONS, AND SUMS

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ABSTRACT. We are given four cards, each containing four nonnegative real numbers, written one below the other, so that the sum of the numbers on each card is 1. We are allowed to put the cards in any order we like, then we write down the first number from the first card, the second number from the second card, the third number from the third card, and the fourth number from the fourth card, and we add these four numbers together. We wish to find real intervals $[a, b]$ with the following property: there is always an ordering of the four cards so that the above sum lies in $[a, b]$. We prove that the intervals $[1, 2]$ and $[2/3, 5/3]$ are solutions to this problem. It follows that

$$[0, 1], [1/3, 4/3], [2/3, 5/3], \text{ and } [1, 2]$$

are the only minimal intervals which are solutions. We also discuss a generalization to n cards, and give an equivalent formulation of our results in matrix terms.

1. INTRODUCTION

The following problem (slightly paraphrased) appeared in Lenza and Sands [2]:

The n -card problem. Let $n \geq 4$ be an integer, and suppose you are given n cards, each containing n nonnegative real numbers, written one below the other, so that the sum of the numbers on each card is 1. You are allowed to put the cards in any order you like, then you write down the first number from the first card, the second number from the second card, \dots , and the n th number from the n th card, and you add these n numbers together. What are the minimal intervals $[a, b]$ with the property that, no matter which cards you are given, there is always an ordering of the cards so that the sum will lie in $[a, b]$?

Note that every such interval $[a, b]$ must contain 1, since if we take any n identical cards, the sum is 1 no matter what order we use.

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If, for some integer n , an interval $[a, b]$ is a solution to the n -card problem, we say that $[a, b]$ has the n -card property. In [2], Lenza and Sands proved that the interval $[1/3, 4/3]$ has the 4-card property, and asked whether the intervals $[2/3, 5/3]$ and $[1, 2]$ also have the 4-card property. In this note we settle these two cases in the affirmative.

Theorem 1.1. *The interval $[1, 2]$ has the 4-card property.*

Theorem 1.2. *The interval $[2/3, 5/3]$ has the 4-card property.*

Both of these intervals are best possible. As a result, we will see that the list of all minimal intervals with the 4-card property is

$$[0, 1], [1/3, 4/3], [2/3, 5/3], \text{ and } [1, 2].$$

2. HISTORY AND PRELIMINARIES

This problem began as a weak form of the 3-card problem in the 2000 Calgary Junior Mathematics Contest, an annual contest for students from the Calgary area in Grade 9 or under.¹ A slightly stronger version was then submitted to the Canadian problems journal *CruX Mathematicorum with Mathematical Mayhem*, where it was published in 2001 as Problem 2620 [4]:

You are given three cards, each containing three nonnegative real numbers, written one below the other, so that the sum of the numbers on each card is 1. You are allowed to put the cards in any order you like, then you write down the first number from the first card, the second number from the second card, and the third number from the third card, and you add these three numbers together. Show that, no matter which cards you are given, there is always an ordering of the cards so that the sum will lie in $[1/2, 3/2]$.

A solution to this problem appeared in *CruX* in March, 2002 [4].

Note: For the rest of this paper, when referring to a set of cards satisfying the conditions of the n -card problem, for simplicity we shall say something like “a set of n cards,” taking it to be understood that each card has n nonnegative numbers on it adding to 1.

In 2003, Eric Lenza, an undergraduate mathematics student at the University of Calgary, was given the general problem as an NSERC summer research project. Lenza found a new proof that $[1/2, 3/2]$ is a solution for the 3-card problem. Lenza’s solution to the 3-card problem appears in [2], and uses the facts that, for any set of three cards,

- at most 2 of the $3! = 6$ sums are greater than $3/2$ and
- at most 3 of the sums are less than $1/2$.

¹The contest website is <http://math.ucalgary.ca/community/outreach/junior-math-contest>.

Thus, at least one of the six sums must lie in $[1/2, 3/2]$.

Here is an example. Suppose the three cards are

$$\begin{array}{|c|} \hline 1 \\ \hline 0 \\ \hline 0 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 1 \\ \hline 0 \\ \hline 0 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 0 \\ \hline 1/2 \\ \hline 1/2 \\ \hline \end{array} .$$

With the cards ordered this way, the sum is $3/2$. But if the cards are ordered

$$\begin{array}{|c|} \hline 0 \\ \hline 1/2 \\ \hline 1/2 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 1 \\ \hline 0 \\ \hline 0 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 1 \\ \hline 0 \\ \hline 0 \\ \hline \end{array} ,$$

the sum is 0. In fact, the six orderings of these three cards all give sums of $3/2$ or 0.

On the other hand, for the three cards

$$\begin{array}{|c|} \hline 1 \\ \hline 0 \\ \hline 0 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 0 \\ \hline 1/2 \\ \hline 1/2 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 0 \\ \hline 1/2 \\ \hline 1/2 \\ \hline \end{array} ,$$

all sums are either $1/2$ or 2.

These two examples show that the interval $[1/2, 3/2]$ in the above *Crux* problem is best possible, that is, cannot be replaced by any proper subinterval.

More generally, Lenza and Sands [2] found all minimal intervals $[a, b]$ with the 3-card property. They are:

$$[0, 1], [1/2, 3/2], \text{ and } [1, 2].$$

The case of $[0, 1]$ is easy, in fact for arbitrary n (see Lemma 2.1 below). To do $[1, 2]$, show that for any three cards, at most four of the six sums can be less than 1, and at most one sum can be greater than 2. This was left to the reader in [2], and we shall do the same here.

Lenza and Sands also began serious study of the 4-card problem by proving that $[1/3, 4/3]$ has the 4-card property [2]. Their method of proof was similar to Lenza's 3-card solution: they showed that, for any set of four cards,

- at most 12 of the $4! = 24$ sums are greater than $4/3$ and
- at most 11 of the 24 sums are less than $1/3$;

thus at least one of the 24 sums must lie in $[1/3, 4/3]$. This interval is again best possible, and moreover, both the numbers 12 and 11 in the proof are best possible: the cards

$$\begin{array}{|c|} \hline 1 \\ \hline 0 \\ \hline 0 \\ \hline 0 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 1 \\ \hline 0 \\ \hline 0 \\ \hline 0 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 0 \\ \hline 1/3 \\ \hline 1/3 \\ \hline 1/3 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 0 \\ \hline 1/3 \\ \hline 1/3 \\ \hline 1/3 \\ \hline \end{array}$$

have 12 sums of $5/3$ (and 12 sums of $1/3$), and the cards

1	0	0	$1/4$
0	1	0	$1/4$
0	0	1	$1/4$
0	0	0	$1/4$

have 11 sums of $1/4$ (and all other sums > 1).

We next give a simple proof that the interval $[0, 1]$ has the n -card property for every integer $n \geq 2$, and is minimal.

Lemma 2.1. $[a, b] = [0, 1]$ is a solution to the n -card problem for every $n \geq 2$.

Proof. Since the sum of the numbers on every card is 1, the average of all $n!$ permutation sums arising from ordering the cards must be 1. Thus at least one permutation sum must lie in $[0, 1]$. Any set of n identical cards shows that the upper bound 1 cannot be reduced, since every sum for such a set would be exactly 1, and the set of n cards

$$\left\{ \begin{array}{l} \left(1, 0, 0, \dots, 0 \right) \quad (n-1 \text{ cards}), \\ \left(0, \frac{1}{n-1}, \frac{1}{n-1}, \dots, \frac{1}{n-1} \right) \quad (1 \text{ card}) \end{array} \right.$$

shows that the lower limit 0 cannot be increased, since all sums for these n cards are either 0 or greater than 1. Thus $[0, 1]$ is a minimal solution to the n -card problem. \square

Theorem 1 is proved in the next section, and Theorem 2 is handled in Section 4. In the final section we give some open problems, and point out how to rephrase our problems and results in matrix terms.

3. PROOF OF THEOREM 1.1

In this section we prove that $[1, 2]$ has the 4-card property. Proceeding as in [2], we prove this theorem by finding upper bounds on how many sums can be less than 1, and how many can be greater than 2. The first part is easy. We have the following general result.

Lemma 3.1. *Let $n \geq 2$ be an integer. For a set of n cards as in the n -card problem, consider all the $n!$ sums arising from the $n!$ arrangements of these cards. Then at most $n! - (n-1)!$ of these sums can be less than 1.*

Proof. Denote the cards by

$$\begin{aligned} C_1 &= (c_{11}, c_{12}, \dots, c_{1n}), \\ C_2 &= (c_{21}, c_{22}, \dots, c_{2n}), \\ &\vdots \\ C_n &= (c_{n1}, c_{n2}, \dots, c_{nn}). \end{aligned}$$

Then an arrangement of the cards will result in a sum of the form

$$c_{1\pi(1)} + c_{2\pi(2)} + \dots + c_{n\pi(n)},$$

where $\pi = (\pi(1)\pi(2)\dots\pi(n))$ is a permutation of $\{1, 2, \dots, n\}$. Here, for each i , the $\pi(i)$ th number on card i has been chosen. In what follows, we shall frequently refer to this permutation instead of to the sum itself.

The $n!$ permutations can be arranged into $(n-1)!$ sets of n permutations each, where each set contains a permutation and its *cyclic shifts*:

$$\begin{aligned} &(\pi(1)\pi(2)\dots\pi(n)), \\ &(\pi(2)\pi(3)\dots\pi(n)\pi(1)), \\ &\vdots \\ &(\pi(n)\pi(1)\pi(2)\dots\pi(n-1)). \end{aligned}$$

We call such a set of n permutations a *cyclic n -tuple*. The corresponding sums for these cards are

$$\begin{aligned} &c_{1\pi(1)} + c_{2\pi(2)} + \dots + c_{n\pi(n)}, \\ &c_{1\pi(2)} + c_{2\pi(3)} + \dots + c_{n\pi(1)}, \\ &\vdots \\ &c_{1\pi(n)} + c_{2\pi(1)} + \dots + c_{n\pi(n-1)}. \end{aligned}$$

Thus the sum of all these numbers is just the sum of all numbers on all the cards, which is $1 + 1 + \dots + 1 = n$.

Now suppose that more than $n! - (n-1)!$ of the permutation sums are less than 1. Then the corresponding permutations must include one complete set of n cyclic shifts, so the sum of all numbers involved in the corresponding n sums must be exactly n . But this contradicts the assumption that each permutation sum in this set is strictly less than 1. Thus at most $n! - (n-1)!$ permutation sums less than 1 are possible. \square

Note that Lemma 3.1 is best possible, in that $n! - (n-1)!$ permutation sums less than 1 can be attained. Consider the set of n cards consisting of $n-1$ cards of the form $(1/n, 1/n, \dots, 1/n)$ and one card $(1, 0, 0, \dots, 0)$. Then all permutation sums are either $(n-1)/n = 1 - 1/n$ or $2 - 1/n$, depending on whether 0 or 1 in the last card is chosen. The number of permutations in which a 0 is chosen is exactly $n! - (n-1)!$.

In the case $n = 4$, Lemma 3.1 says that, for any four cards, at most 18 of the 24 permutation sums can be less than 1. One more lemma will complete the proof of Theorem 1.1.

As the rest of this paper deals almost exclusively with the case $n = 4$ (that is, sets of four cards), we change notation and from now on will denote the cards by A, B, C, D , where for example the four numbers on card A are a_1, a_2, a_3, a_4 in that order. Thus a permutation $\pi = (\pi(1)\pi(2)\pi(3)\pi(4))$ of $\{1, 2, 3, 4\}$ will correspond to the sum $a_{\pi(1)} + b_{\pi(2)} + c_{\pi(3)} + d_{\pi(4)}$.

Lemma 3.2. *For any four cards, there are at most four sums greater than 2.*

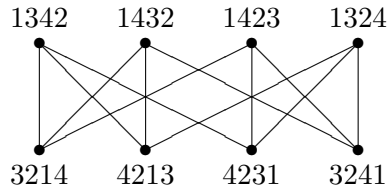
Proof. Suppose for some set of four cards there are five sums greater than 2, and consider the corresponding five permutations. First observe that no two of these permutations can be disjoint, that is, we cannot have two permutations π and τ among these five permutations so that for each i , $\pi(i) \neq \tau(i)$. For if this occurred, then the corresponding sums $a_{\pi(1)} + b_{\pi(2)} + c_{\pi(3)} + d_{\pi(4)}$ and $a_{\tau(1)} + b_{\tau(2)} + c_{\tau(3)} + d_{\tau(4)}$ would contain eight different entries from the four cards, so their sum would be at most the sum of all the entries on the cards, which is 4, contradicting the assumption that each sum is greater than 2.

Thus every pair of the five permutations must have at least one entry in common. Suppose that each pair of these permutations has exactly one entry in common: how many permutations are possible? We may assume that (1234) and (1342) are two of the permutations. It can be easily checked that the only permutations that can be added which agree with each of these two permutations in exactly one place are (1423), (2314), (3241) and (4132), and since these four permutations are pairwise disjoint, only one of them can be added to the list, so the list cannot be extended beyond three permutations. Thus if we are to obtain five permutation sums greater than 2, we certainly need two permutations which agree in two places.

So we may assume that two of the five permutations are (1234) and (1243). Now the only permutations which agree in at least one place with each of these are

(1342), (1423), (1432), (1324), (3241), (4213), (4231), and (3214).

These can be arranged as the vertices of the following bipartite graph:



Here the edges denote that the two permutations are disjoint. Thus we need to find three of these permutations which form an independent set in this graph. The only such sets are contained in either the bottom or top level of this graph. But group the four vertices in each of these levels into pairs,

with the pairs (1342), (1423) and (1432), (1324) in the top level and the pairs (3241), (4213) and (4231), (3214) in the bottom level. Then if we choose three vertices from one of these levels, we must choose both entries in one of these pairs. However, this is impossible; none of these pairs can be combined with both (1234) and (1243). For example, the pair (1342), (1423) cannot be combined with (1234), because this would say that

$$\begin{aligned} a_1 + b_3 + c_4 + d_2 &> 2, \\ a_1 + b_4 + c_2 + d_3 &> 2, \end{aligned}$$

and

$$a_1 + b_2 + c_3 + d_4 > 2,$$

and thus

$$\begin{aligned} 6 &< 3a_1 + (b_2 + b_3 + b_4) + (c_2 + c_3 + c_4) + (d_2 + d_3 + d_4) \\ &\leq 3a_1 + 1 + 1 + 1, \end{aligned}$$

which implies that $a_1 > 1$, a contradiction. Note that the three permutations (1234), (1342), (1423) all agree in one coordinate and pairwise differ in all other coordinates (we call such a triple of permutations a *cyclic triple*). The above proof shows that a cyclic triple cannot all give sums greater than 2. Cyclic triples similarly occur with the other three pairs above.

Therefore, at most four sums can be greater than 2, as claimed. \square

From Lemma 3.1, at most 18 sums can be less than 1. This and Lemma 3.2 account for only 22 of the 24 permutations of the four cards, so at least two of the permutations must give a sum lying in $[1, 2]$. This completes the proof of Theorem 1.1.

(Intersecting permutations, which occur in the proof of Lemma 3.2, have been studied in numerous papers, for example [1].)

Note that Lemma 3.2 is best possible, in that four sums greater than 2 can be attained: use two cards $(1, 0, 0, 0)$, one card $(0, 1, 0, 0)$, and one card $(1/4, 1/4, 1/4, 1/4)$; then these four cards have four sums equal to $9/4$ and all other sums less than 2.

We generalize this example to arbitrary integers $n > 4$. If n is odd, choose

$$\begin{cases} (1, 0, \dots, 0) & (n-1)/2 \text{ cards,} \\ (0, 1, 0, \dots, 0) & (n-1)/2 \text{ cards,} \\ (1/n, 1/n, \dots, 1/n) & \text{one card.} \end{cases}$$

Then there are

$$\binom{n-1}{2} \binom{n-1}{2} (n-2)! = \frac{(n-1)(n-1)!}{4}$$

permutations giving a sum of $2 + \frac{1}{n}$. If n is even, similarly choose

$$\begin{cases} (1, 0, \dots, 0) & n/2 \text{ cards,} \\ (0, 1, 0, \dots, 0) & (n-2)/2 \text{ cards,} \\ (1/n, 1/n, \dots, 1/n) & \text{one card.} \end{cases}$$

Then there are

$$\binom{n}{2} \binom{n-2}{2} (n-2)! = \frac{n(n-2)(n-2)!}{4}$$

permutations giving a sum of $2 + 1/n$. If $n \geq 5$, then

$$\begin{cases} \frac{(n-1)(n-1)!}{4} \geq (n-1)! & (n \text{ odd}), \\ \frac{n(n-2)(n-2)!}{4} \geq (n-1)! & (n \text{ even}). \end{cases}$$

Note that the number of sums that are greater than 2 in the above example (at least $(n-1)!$), plus the number of permutation sums that are less than 1 in the example following Lemma 3.1 ($n! - (n-1)!$), totals to at least $n!$. This means that, if we are to prove that the interval $[1, 2]$ has the n -card property for any $n \geq 5$, we will not be able to use the same technique as for Theorem 1.1.

4. PROOF OF THEOREM 1.2

Now we look at the interval $[2/3, 5/3]$. This will be harder to settle, although our proof will again use the same method as in Theorem 1.1.

Recall that, given four cards A, B, C and D , we may consider the permutation $\pi = (\pi(1)\pi(2)\pi(3)\pi(4))$ of $\{1, 2, 3, 4\}$ instead of the sum $a_{\pi(1)} + b_{\pi(2)} + c_{\pi(3)} + d_{\pi(4)}$ that arises from the permutation. Moreover, we shall think of a set of k such permutations as if they were arranged into a k by 4 matrix, with each permutation forming a row of the matrix. Thus we shall speak of the ‘‘columns’’ of a set of permutations. For example, column 1 of such a set would consist of the first entries in each of the permutations.

With this understanding, we proceed to the upper bound $5/3$ of our interval.

Proposition 4.1. *Given any four cards, at most eight of the 24 permutation sums can be strictly greater than $5/3$.*

Proof. For a given set of four cards, let \mathcal{S} be the set of all permutations giving sums greater than $5/3$.

Step 1: First we note that no three permutations in \mathcal{S} can together have at most one repeated entry; we call such a triple of permutations *almost disjoint*. We prove this by contradiction.

Case (i): Suppose without loss of generality that there are three permutations in \mathcal{S} so that 1 appears twice and 2 appears once in the first column, while the other three columns contain no repeated entries. Then the total of the corresponding three sums would be greater than $3(5/3) = 5$, the sum corresponding to the first column (that is, the sum of the entries from card A used in the sums) would be $2a_1 + a_2$, and the sum of every other column would be at most 1. This yields $2a_1 + a_2 + 3 > 5$, so $2 < 2a_1 + a_2 \leq 2(a_1 + a_2) \leq 2$, a contradiction.

Case (ii): If there are three permutations in \mathcal{S} which are pairwise disjoint, then the total of all entries in these three permutations is again greater than 5, but this time the sum of each column is at most 1, so the sum of all entries is at most 4, which is again a contradiction.

Step 2: Next we show that there cannot be 7 permutations in \mathcal{S} where some entry in some column (say 1 in the first column) occurs six times. For then the seventh permutation (say 2134) together with two of the permutations containing 1 (in this case 1342 and 1423) would form an almost disjoint triple in \mathcal{S} , which is impossible by Step 1. All other cases are handled by permuting the last three columns.

Step 3: Now suppose that \mathcal{S} contains nine permutations. From Step 2, no entry can appear six times in any column.

Suppose that, among the nine permutations in \mathcal{S} , some entry (say 1 in column 1) appears exactly five times; without loss of generality, the five permutations 1234, 1243, 1324, 1342, and 1423 are in \mathcal{S} , while 1432 is not. Of the remaining 18 permutations, just six of them can be added to these five without forming an almost disjoint triple: 2143, 2314, 3124, 3241, 4213, and 4321. So the remaining four permutations in \mathcal{S} must come from this set of six. However, we can partition these six permutations into three sets of two, add one of the original five permutations to each pair and obtain the following almost disjoint (in fact disjoint) triples:

$$\{1234, 2143, 4321\}, \{1342, 3124, 4213\}, \{1423, 2314, 3241\}.$$

Thus at most three of the permutations 2143, 2314, 3124, 3241, 4213, and 4321 can be added to the five 1234, 1243, 1324, 1342, 1423; nine permutations in \mathcal{S} are impossible this way.

So, in \mathcal{S} , every entry can appear at most four times in every column. If no entry appears four times in any column, then the sum corresponding to each column will be at most 3, so the sum of all the entries of the corresponding sums will be at most 12, while the sum of all these entries must be greater than $9(5/3) = 15$, a contradiction.

Thus some entry (say 1 in column 1) must appear exactly four times. By symmetry, there are two choices for the four permutations in \mathcal{S} in which this entry appears:

$$1234, 1243, 1324, 1342 \quad \text{and} \quad 1234, 1243, 1342, 1423.$$

First assume the four permutations 1234, 1243, 1324, 1342 are in \mathcal{S} . To these, *none* of the permutations 2413, 2431, 3412, 3421, 4123, 4132 can be added because each would create an almost disjoint triple in \mathcal{S} . This leaves 12 remaining permutations, from which we wish to choose five to be in \mathcal{S} . Partition these 12 into four sets of three, as follows:

$$\{2134, 3241, 4213\}, \{2143, 3214, 4231\}, \{2341, 3142, 4312\}, \text{ and} \\ \{2314, 3124, 4321\}.$$

For each of these triples, any two of its permutations together with one of the permutations 1234, 1243, 1324, 1342 already in \mathcal{S} will form an almost disjoint triple:

- With $\{2134, 3241, 4213\}$, use 1324.
- With $\{2143, 3214, 4231\}$, use 1342.
- With $\{2341, 3142, 4312\}$, use 1234.
- With $\{2314, 3124, 4321\}$, use 1243.

Thus we cannot add more than four permutations to the above four, and \mathcal{S} contains at most eight permutations, as claimed.

So assume instead that \mathcal{S} contains the four permutations 1234, 1243, 1342, 1423. This time each of the following nine permutations would create an almost disjoint triple:

$$2134, 2341, 2413, 3142, 3214, 3421, 4123, 4231, 4312.$$

So none of these can be in \mathcal{S} . The remaining nine permutations can be grouped into the following sets of three:

$$\{2314, 3241, 4132\}, \{2143, 3412, 4321\}, \text{ and } \{2431, 3124, 4213\}.$$

No more than one permutation from each of these groups can be added to \mathcal{S} , because any two of each group forms an almost disjoint (in fact disjoint) triple, as follows:

- With $\{2314, 3241, 4132\}$, use 1423.
- With $\{2143, 3412, 4321\}$, use 1234.
- With $\{2431, 3124, 4213\}$, use 1342.

So we cannot add more than three permutations to \mathcal{S} , and there are at most seven permutations in \mathcal{S} this time.

Thus, for any four cards, the maximum number of permutation sums greater than $5/3$ is eight, as claimed. \square

Proposition 4.1 is best possible; the four-card sets

$$(1, 0, 0, 0), (1, 0, 0, 0), (0, 1/3, 1/3, 1/3), (0, 0, 1/2, 1/2)$$

and

$$(1, 0, 0, 0), (1, 0, 0, 0), (0, 1, 0, 0), (0, 1, 0, 0)$$

each have eight permutation sums greater than $5/3$.

Next we will handle the lower bound $2/3$. For this we will need the following observations.

Lemma 4.2. *For any set of four cards, the following sets of permutations cannot all correspond to sums less than $2/3$:*

- (i) *Six (or fewer) permutations with all four columns complete: that is, x permutations of $\{1, 2, 3, 4\}$, where $x \in \{4, 5, 6\}$, so that, when arranged in matrix form, each of the four columns contains each of 1, 2, 3 and 4 at least once.*
- (ii) *Nine permutations with three columns doubly complete: that is, nine permutations of $\{1, 2, 3, 4\}$ so that three of the four columns contain each of 1, 2, 3 and 4 at least twice.*

Proof. (i) If column i contains all of 1, 2, 3, 4 at least once, it means that each entry of card i appears at least once in the corresponding sums, so the sum of these entries must be at least 1. Let $x \in \{4, 5, 6\}$. So x permutations with four complete columns would have to correspond to a total sum of at least 4. However, if x such permutation sums were all less than $2/3$, then the total of all x sums would be less than $x(2/3) \leq 4$, a contradiction.

(ii) Similarly, if nine such permutation sums were all less than $2/3$, then the total of all nine sums would be less than $9(2/3) = 6$, while the sum of the three doubly complete columns is at least $2 \cdot 3 = 6$, a contradiction. \square

A special case of (i) would be any cyclic 4-tuple such as (1234), (2341), (3412), (4123). A special case of (ii) would be three cyclic triples with different leading fixed points, such as

(1234), (1342), (1423); (2134), (2341), (2413); or (3124), (3241), (3412).

Proposition 4.3. *Given any four cards, at most 15 of the 24 permutation sums can be less than $2/3$.*

Proof. We first make the following general observation. If we are given a set of n cards as in the n -card problem, then there are $n!$ sums to be considered. Rearranging the cards we are given obviously does not change these sums. Also, rearranging the numbers on each card in the same way (that is, applying the same permutation to the numbers on each card) will not change the $n!$ sums either. For example, we could switch the first and second numbers on each card without affecting the $n!$ sums.

Now let $n = 4$. For a given set of four cards, we will consider the set \mathcal{S} of all permutations giving sums less than $2/3$. We classify the permutations in \mathcal{S} according to the number of permutations starting with each of 1, 2, 3 and 4. Let the *type* of \mathcal{S} be the quadruple $\mathbf{n} = [n_1, n_2, n_3, n_4]$, where n_i is the number of permutations π in \mathcal{S} whose first element $\pi(1)$ is i . That is, n_i is the number of times the i th number on the first card is used in a sum less than $2/3$. Then $\sum_{i=1}^4 n_i$ is the number of permutations in \mathcal{S} . We want to eliminate all possible types \mathbf{n} , where $\sum_{i=1}^4 n_i = 16$, as the type of \mathcal{S} .

Note that, from the above observation, rearranging the numbers on each card in the same way would give us a permutation of the n_i 's without affecting the sums. Thus, without loss of generality, we can assume that the type \mathbf{n} of \mathcal{S} satisfies $n_1 \geq n_2 \geq n_3 \geq n_4$.

Also, by rearranging the cards, the sums are unchanged but a different card may become the first card and thus would determine the n_i 's, so we could get a different type \mathbf{n} .

Altogether, for a given four cards, we have up to four possible types \mathbf{n} such that $n_1 \geq n_2 \geq n_3 \geq n_4$. Of these, we choose the type \mathbf{n} for which n_4 is as small as possible.

We now proceed to eliminate all possible \mathbf{n} 's with $\sum_{i=1}^4 n_i = 16$ that might occur as the type of \mathcal{S} . We do so in the following order, where \geq is componentwise:

- $[n_1, n_2, n_3, n_4] \geq [5, 5, 5, 0]$ (so $\mathbf{n} = [5, 5, 5, 1]$ and $[6, 5, 5, 0]$).
- $[n_1, n_2, n_3, n_4] \geq [6, 6, 3, 0]$ (so $\mathbf{n} = [6, 6, 3, 1]$ and $[6, 6, 4, 0]$).
- $[n_1, n_2, n_3, n_4] \geq [6, 5, 1, 1]$ (so $\mathbf{n} = [6, 6, 2, 2], [6, 5, 4, 1], [6, 5, 3, 2]$).
- $[n_1, n_2, n_3, n_4] \geq [5, 5, 3, 1]$ (so $\mathbf{n} = [5, 5, 4, 2]$ and $[5, 5, 3, 3]$).
- $[n_1, n_2, n_3, n_4] \geq [5, 4, 4, 1]$ (so $\mathbf{n} = [5, 4, 4, 3]$ and $[6, 4, 4, 2]$).
- $[n_1, n_2, n_3, n_4] \geq [6, 4, 3, 1]$ (so $\mathbf{n} = [6, 4, 3, 3]$).
- $[n_1, n_2, n_3, n_4] = [4, 4, 4, 4]$.

Case 1: $[n_1, n_2, n_3, n_4] \geq [5, 5, 5, 0]$: If $n_1 = 5$ it means that only one of the six permutations starting with 1 is missing from \mathcal{S} , so there must be a cyclic triple (either (1234), (1342), (1423) or (1243), (1432), (1324)) included in \mathcal{S} . Similarly there are cyclic triples in \mathcal{S} starting with 2 and 3. This is a contradiction to (ii) of Lemma 4.2.

Case 2: $[n_1, n_2, n_3, n_4] \geq [6, 6, 3, 0]$: The three permutations in \mathcal{S} starting with 3 must contain two members of a cyclic triple. Say they are (3124) and (3241). Then the following (multi-)set of nine permutations in \mathcal{S} satisfies (ii) of Lemma 4.2:

$$(3124), (3241), (3241), (1324), (1432), (1432), (2314), (2413), (2143).$$

We may permute columns 2, 3 and 4 to account for all other possibilities.

At this point we have eliminated all types where $n_4 = 0$, so from now on we can assume that $n_4 \geq 1$, and moreover (by the minimality of n_4) every column of \mathcal{S} contains each of 1, 2, 3, 4 at least once.

Case 3: $[n_1, n_2, n_3, n_4] \geq [6, 5, 1, 1]$: Without loss of generality, assume that (2431) is the only permutation starting with 2 which is missing from \mathcal{S} . \mathcal{S} contains all permutations starting with 1, and one permutation starting with 3 and with 4. It won't matter which permutation starting with 4 is in \mathcal{S} (we denote this permutation by (4***)), so for each permutation in \mathcal{S} starting with 3 we give a set of 6 permutations satisfying (i) of Lemma 4.2.

$$\begin{aligned} &(3124), (1243), (1432), (2314), (2341), (4***); \\ &(3142), (1423), (1234), (2314), (2341), (4***); \\ &(3214), (1423), (1432), (2134), (2341), (4***); \\ &(3241), (1423), (1432), (2143), (2314), (4***); \end{aligned}$$

$$(3412), (1234), (1423), (2134), (2341), (4***);$$

$$(3421), (1243), (1432), (2143), (2314), (4***).$$

Thus Case 3 is impossible.

Case 4: $[n_1, n_2, n_3, n_4] \geq [5, 5, 3, 1]$: Of the three permutations starting with 3 which lie in \mathcal{S} , two of them must belong to the same cyclic triple, so (by permuting the last three columns) we may assume that they are (3142) and (3214). Then:

- If we have $(1423) \in \mathcal{S}$, then the subset $(3142), (3214), (1423), (2xy1)$ of \mathcal{S} (where $\{x, y\} = \{3, 4\}$) has two complete columns; column 4 and either column 2 or 3, depending on whether $x = 3$ or $y = 3$. (Here $(2xy1)$ denotes any permutation of this form which belongs to \mathcal{S} , which must exist as there is only one permutation $(2***)$ missing from \mathcal{S} .) We now add a permutation in \mathcal{S} of the form $(13**)$ or $(1*3*)$ to make all three columns 2, 3 and 4 complete, and this is always possible because there is only one permutation $(1***)$ missing from \mathcal{S} . Finally, adding in the permutation $(4***)$ in \mathcal{S} will give us six permutations in \mathcal{S} with all columns complete, and we are done by (i) of Lemma 4.2.
- Otherwise, \mathcal{S} is missing (1423) but contains the other five permutations starting with 1. Now if \mathcal{S} contains (2431) we use

$$(1243), (1324), (2431), (3142), (3214), (4***),$$

and otherwise we use

$$(1324), (1432), (2143), (2341), (3214), (4***).$$

In either case these six permutations have all columns complete, and we are again done by (i) of Lemma 4.2.

Case 5: $[n_1, n_2, n_3, n_4] \geq [5, 4, 4, 1]$: If both 4's contain cyclic triples, we are done by (ii) of Lemma 4.2. So suppose that the permutations in \mathcal{S} starting with 3 do not contain a cyclic triple. This means that the two permutations starting with 3 which are missing from \mathcal{S} must agree in two columns.

- Suppose \mathcal{S} does not contain the two permutations $(34**)$ (that is, (3412) and (3421)). Then if \mathcal{S} contains (2413) , we use $(3124), (3241), (2413)$ together with any permutation in \mathcal{S} of the form $(1**2)$. This set will contain two complete columns, namely column 4 and either column 2 or column 3, and we can now add in a permutation from \mathcal{S} of the form $(1***)$ so as to obtain a third complete column. The permutation $(4***)$ in \mathcal{S} will now complete column 1 and allow us to use (i) of Lemma 4.2. If we have $(2431) \in \mathcal{S}$, we similarly use

$$(3142), (3214), (2431), (1*2*), (1***), (4***),$$

obtained from the previous construction by switching columns 3 and 4. If \mathcal{S} misses both (2431) and (2413), then we use either

$$(3142), (3214), (2134), (2341), (1423), (4***)$$

or

$$(3124), (3241), (2143), (2314), (1432), (4***),$$

depending on which of (1423) or (1432) is present in \mathcal{S} , and Lemma 4.2(i) applies again. We can permute columns to handle the cases where \mathcal{S} misses (3*4*) or (3**4).

- Suppose \mathcal{S} misses (32**); the argument here is very similar to the previous case. If (2341) $\in \mathcal{S}$ then we use

$$(3124), (3412), (2341), (12**), (1***), (4***),$$

and if (2314) $\in \mathcal{S}$ then we use

$$(3142), (3421), (2314), (12**), (1***), (4***).$$

If \mathcal{S} misses both (2341) and (2314), then we use either

$$(3142), (2413), (2431), (1324), (12**), (4***)$$

or

$$(3124), (2431), (2413), (1342), (12**), (4***),$$

depending on which of (1324) or (1342) is present in \mathcal{S} . We permute columns to handle the cases (3*2*) and (3**2).

So now we assume that \mathcal{S} misses (31**). If (2143) $\in \mathcal{S}$ then we use

$$(3214), (3421), (2143), (1**2), (1***), (4***).$$

If (2134) $\in \mathcal{S}$ then we use

$$(3241), (3412), (2134), (1*2*), (1***), (4***).$$

So we assume that \mathcal{S} misses both (31**) and (21**), which means \mathcal{S} contains all permutations of the forms (23**), (24**), (32**) and (34**).

Now, since \mathcal{S} must have at least one 1 in column 2, \mathcal{S} must contain some permutation of the form (41**). If \mathcal{S} contains (1234) and (1423), then we use

$$(1234), (1423), (2341), (3412), (41**).$$

If \mathcal{S} contains (1234) and (1342), then we use

$$(1234), (1342), (2413), (3421), (41**).$$

If \mathcal{S} misses (1234), then we use

$$(1243), (1324), (2431), (3412), (41**).$$

All of these cases result in instances of (i) of Lemma 4.2.

Case 6: $[n_1, n_2, n_3, n_4] \geq [6, 4, 3, 1]$: Consider the four permutations in \mathcal{S} that start with 2. Their associated cyclic shifts give us four permutations starting with 3, so one of these must belong to \mathcal{S} , since \mathcal{S} is only

missing three permutations starting with 3. Thus \mathcal{S} must contain some permutation $(2xyz)$ and two of its cyclic shifts $(xyz2)$, $(yz2x)$, $(z2xy)$, namely the ones starting with 3 and 1. These three permutations give us three different numbers in each column, with the only missing number in each column being that number in the fourth cyclic permutation. Now take the fourth cyclic permutation (starting with 4) and switch 4 and 1. The resulting permutation will start with 1 (and thus it will be in \mathcal{S}), and adding it to the three previously chosen permutations will create two complete columns. The other two columns miss only 4 (column 1) and 1. The permutation in \mathcal{S} starting with 4 supplies the 4 in column 1, and \mathcal{S} must contain a permutation with 1 in the column that needs it, because at this point we are assuming that all four numbers 1, 2, 3, 4 appear in every column at least once. So we can find six permutations in \mathcal{S} with all columns complete, as in (i) of Lemma 4.2.

This completes all types where $n_4 \leq 3$. So the only remaining type is the following.

Case 7: $[n_1, n_2, n_3, n_4] = [4, 4, 4, 4]$: Now we can assume that every column contains exactly four of each of 1, 2, 3, 4. Thus \mathcal{S} must miss two of the six permutations starting with 1.

Suppose that \mathcal{S} misses both permutations (12^{**}) , that is, (1234) and (1243) . Then \mathcal{S} must contain (1324) , (1342) , (1423) , (1432) , and to get enough 2's in column 2 \mathcal{S} must also contain (3214) , (3241) , (4213) , (4231) . Now if \mathcal{S} contains a permutation (21^{**}) , then the five permutations

$$(1324), (3241), (4213), (1432), (21^{**})$$

in \mathcal{S} have four complete columns, violating (i) of Lemma 4.2. So \mathcal{S} must miss both (2134) and (2143) . Then \mathcal{S} must contain (2314) , (2341) , (2413) , (2431) , and (to get enough 1's in column 2) \mathcal{S} must also contain (3124) , (3142) , (4123) , and (4132) . But now \mathcal{S} contains a complete 4-cycle (1324) , (3241) , (2413) , (4132) , which is impossible by (i) of Lemma 4.2.

Thus \mathcal{S} cannot miss both (1234) and (1243) . By symmetry we may assume \mathcal{S} misses (1234) and (1342) , so \mathcal{S} contains (1243) , (1324) , (1423) , (1432) . Now look at the four permutations in \mathcal{S} starting with 2. As above, \mathcal{S} cannot miss any pair of permutations that agree in exactly two coordinates, so we have six possibilities for the two missed permutations starting with 2:

- (i) $(2134), (2341)$;
- (ii) $(2134), (2413)$;
- (iii) $(2341), (2413)$;
- (iv) $(2143), (2431)$;
- (v) $(2143), (2314)$;
- (vi) $(2431), (2314)$.

If \mathcal{S} contains both (2413) and (2431), then (to avoid too many 4's in column 2) \mathcal{S} must miss both (34**)’s, which is impossible by the above argument for (12**). So \mathcal{S} must miss one of (2413) and (2431), which shows that (i) and (v) above are impossible. Similarly, if \mathcal{S} contains both (2413) and (2143), then (to avoid too many 3's in column 4) \mathcal{S} must miss both (4**3)’s, which is impossible by the above argument for (12**). So \mathcal{S} must miss one of (2413) and (2143), which shows that (vi) above is impossible.

Suppose (ii). Then \mathcal{S} contains (2143), (2314), (2341), (2431). To get enough 4's in column 4 we need (3124) and (3214) in \mathcal{S} . To get enough 3's in column 3 we need (4132) and (4231) in \mathcal{S} . But now \mathcal{S} contains

$$(1243), (2431), (3124), (4132), (2314),$$

as in (i) of Lemma 4.2.

Suppose (iii). Then \mathcal{S} contains (2134), (2143), (2314), (2431). To get enough 4's in column 3 we need both (3*4*)’s in \mathcal{S} , and to get enough 3's in column 2 we need both (43**)’s in \mathcal{S} . But now \mathcal{S} contains

$$(1243), (2431), (3142), (4312), (1324),$$

as in (i) of Lemma 4.2.

Finally suppose (iv). Then \mathcal{S} contains (2134), (2314), (2341), (2413). To get enough 4's in column 3 we need both (3*4*)’s in \mathcal{S} , and to get enough 3's in column 3 we need both (4*3*)’s in \mathcal{S} . But now \mathcal{S} contains a 4-cycle

$$(1324), (3241), (2413), (4132),$$

as in (i) of Lemma 4.2.

This completes the proof of Proposition 4.3. \square

Propositions 4.1 and 4.3 together give us the proof of Theorem 1.2. Note that the set

$$(1, 0, 0, 0), (0, 1, 0, 0), (1/4, 1/4, 1/4, 1/4), (1/4, 1/4, 1/4, 1/4)$$

has 14 sums less than $2/3$. We do not have an example of a set of four cards that has 15 sums less than $2/3$.

Problem 1. *Can the number 15 in Proposition 4.3 be improved to 14?*

More generally, we have the following problem.

Problem 2.

- (i) *For each real number $k \in (0, 1]$, what is the maximum number of sums less than k for any set of four cards?*
- (ii) *For each real number $k \in [1, 4]$, what is the maximum number of sums greater than k for any set of four cards?*

Of course, we could ask similar questions for n cards.

5. FURTHER RESULTS AND OPEN PROBLEMS

The cases $n = 3$ and $n = 4$ (and also $n = 2$, as the reader can easily check) certainly suggest the following:

Problem 3. *For an arbitrary integer $n \geq 2$, are the n intervals*

$$\left\{ I_k = \left[\frac{k}{n-1}, 1 + \frac{k}{n-1} \right] : k = 0, 1, \dots, n-1 \right\}$$

the complete solution to the n -card problem in general?

If these intervals were solutions, then they would be best possible, and moreover no other solutions would be possible, because of the following sets of n cards, generalized from examples given in [2]. For each $i \in \{0, 1, \dots, n\}$, let the collection \mathcal{S}_i of n cards be made up of

$$\left\{ \begin{array}{l} \left(1, 0, 0, \dots, 0 \right) \quad (n-i \text{ cards}) \text{ and} \\ \left(0, \frac{1}{n-1}, \frac{1}{n-1}, \dots, \frac{1}{n-1} \right) \quad (i \text{ cards}). \end{array} \right.$$

Then for each $i \in \{1, 2, \dots, n-1\}$, the sums for \mathcal{S}_i will all be either

$$\frac{i-1}{n-1} \quad \text{or} \quad 1 + \frac{i}{n-1} .$$

(The sums for \mathcal{S}_0 and \mathcal{S}_n are all 1.) So \mathcal{S}_k and \mathcal{S}_{k+1} show that no proper subinterval of I_k can have the n -card property. Moreover, suppose that some interval $[a, b]$ has the n -card property, where $1 \leq k \leq n-1$ is an integer such that

$$\frac{k-1}{n-1} < a \leq \frac{k}{n-1} .$$

Then the smaller sum $(k-1)/(n-1)$ for \mathcal{S}_k does not lie in $[a, b]$, so the larger sum must, that is,

$$1 + \frac{k}{n-1} \leq b .$$

Thus $[a, b]$ must contain our interval I_k . So if the intervals I_k all have the n -card property, then they are the only minimal intervals that do. Thus we know that the only minimal intervals with the 3-card and 4-card property respectively are

$$\{[0, 1], [1/2, 3/2], [1, 2]\}$$

and

$$\{[0, 1], [1/3, 4/3], [2/3, 5/3], [1, 2]\} ,$$

as mentioned in Sections 1 and 2.

Two special cases of Problem 3 seem worth pointing out.

Problem 4. *Is $[1/2, 3/2]$ a solution to the n -card problem for all odd n ?*

Problem 5. *Is $[1, 2]$ a solution to the n -card problem for all n ?*

Regarding Problem 5, as mentioned at the end of Section 3, the method we use to show that $[1, 2]$ has the n -card property for $n = 3$ and $n = 4$ will not work for larger n . We can only prove the following weaker result.

Lemma 5.1. $[a, b] = [1, 3]$ is a (perhaps non-minimal) solution to the n -card problem for every $n \geq 2$.

Proof. Since the sum on every card is 1, the average of all $n!$ permutation sums must be 1. Thus at least one permutation sum must be at least 1. Any set of n identical cards shows that the lower bound 1 cannot be increased, since every sum for such a set would be exactly 1. Now notice that if we switch two terms of any permutation, the effect on the permutation sum is at most 2 (either higher or lower), since all numbers on each card are at most 1. Given any permutation, we can change it to any other permutation by a sequence of transpositions. Thus, start with some permutation sum $S \geq 1$, and change it to some permutation sum at most 1 by a sequence of transpositions. Then there must be some permutation sum along the way lying in $[1, 3]$. \square

Of course, one cannot help but notice the following further “special case” of Problem 3.

Problem 6. *Do all minimal intervals with the n -card property have length exactly 1?*

If the answer to this problem were yes, then the discussion above would instantly show that the answer to Problem 3 is yes as well. It would then have been unnecessary to go through the proofs of Theorems 1.1 and 1.2, and we could have all gone home early!

Finally, we note that the n -card problem has a matrix formulation. Recall that a *stochastic matrix* is a square matrix such that each row consists of nonnegative real numbers whose sum is 1. Thus if we arrange a set of n cards to form the rows of a matrix, we obtain a stochastic matrix. Then all of our problems and results could be phrased in these terms. For example, it is easy to see that Theorem 1.2 is equivalent to:

For every 4 by 4 stochastic matrix M , there is a 4 by 4 permutation matrix P so that $\text{trace}(PM) \in [2/3, 5/3]$.

But the author does not know whether such a reformulation would simplify the proof. (Many papers give results on trace inequalities, for example [3].)

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