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MONOCHROMATIC EVEN CYCLES

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ABSTRACT. We prove that any r-coloring of the edges of K_m contains a monochromatic even cycle, where $m = 3r + 1$ if r is odd and $m = 3r$ if r is even. We also prove that K_{m-1} has an r-coloring without monochromatic even cycles.

An easy exercise, perhaps folkloristic, says that in any r-coloring of the edges of K_{2^r+1} there is a monochromatic odd cycle (and this is not true for K_{2^r}).

This note explores what happens if we ask the same question for even cycles. Let $f(r)$ denote the smallest integer m for which there is a monochromatic even cycle in every edge coloring of K_m.

Theorem 1. *For odd r, $f(r) = 3r + 1$ and for even r, $f(r) = 3r$.*

Every graph with n vertices and with more than $m = \lfloor 3(n-1)/2 \rfloor$ edges contains a Θ-graph, i.e. three internally vertex disjoint paths connecting the same pair of vertices (see [1], Exercise 10.1). Since a Θ-graph obviously contains an even cycle, any graph with n vertices and more than m edges contains an even cycle. This easily implies that the stated values are upper bounds of $f(r)$ in Theorem 1. Indeed, considering the majority color, one can easily check that

$$\left\lceil \frac{\binom{3r+1}{2}}{r} \right\rceil > \left\lfloor \frac{3(3r)}{2} \right\rfloor \quad \text{if } r \text{ is odd}$$

and

$$\left\lceil \frac{\binom{3r}{2}}{r} \right\rceil > \left\lfloor \frac{3(3r-1)}{2} \right\rfloor \quad \text{if } r \text{ is even.}$$

Therefore to prove Theorem 1 we need a construction, a partition of the edge set of K_{3r} (K_{3r-1}) into r graphs, each without even cycles. Let H_1 be a triangle with vertices v_1, v_2, v_3. For odd $r > 1$ let H_r be the graph formed by three vertex disjoint copies of $(r-1)/2$ triangles sharing one common vertex v_i, $i = 1, 2, 3$ and the triangle v_1, v_2, v_3 which is called the *central triangle* of H_r. Note that each block (maximal biconnected subgraph or cut-edge) of

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H_r is a triangle, so it has no even cycles. Thus for odd r Theorem 1 follows from the next proposition.

Proposition 2. *For odd r, K_{3r} can be partitioned into r copies of H_r.*

Proof. The proof is based on a well-known construction of Steiner triple systems on $6t+3$ vertices (see [2], Theorem 9.1). Set $r = 2t+1$, then $3r = 6t+3$. The vertex set of $K = K_{3r}$ is partitioned into $\{a_i, b_i, c_i\}$, for $i = 1, 2, \ldots, 2t+1$. For $r = 1$, $\{a_i, b_i, c_i\}$ is an H_1, for $r > 1$ consider a near factorization of a complete graph S_{2t+1} with vertex set $\{1, 2, \ldots, 2t+1\}$ into factors F_i, where F_i avoids vertex i. To each factor F_i we define a copy of H_r^i as follows. Place the edges of the following triangles to H_r^i:

$$\{b_i a_k a_l, c_i b_k b_l, a_i c_k c_l : kl \in F_i\}, \{a_i b_i c_i\}. \tag{1}$$

One can easily see that H_r^i is isomorphic to H_r and for $i = 1, \ldots, 2t+1$ they give a partition on the edge set of K (in fact their blocks are triangles forming a Steiner triple system on K). □

For $r = 2$ note that K_5 can be partitioned into two pentagons. However, K_5 can be also partitioned into two "bulls", which is a triangle with two pendant edges (see Figure 1). This latter works well to reduce the even case to the odd one in Proposition 3.

For even r define the graph A_r from H_{r-1} by removing the edges of its central triangle v_1, v_2, v_3 and adding two new vertices u, w together with the five edges v_1v_2, uv_i, wv_2 (see Figure 2). Let B_r be the graph with $r-1$ triangles sharing a common vertex x plus r pendant edges, one from x and one from each triangle (from a vertex different from x). Note that A_r, B_r

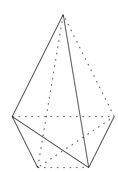

FIGURE 1. A bull with its complementary bull dotted, drawn as later used.

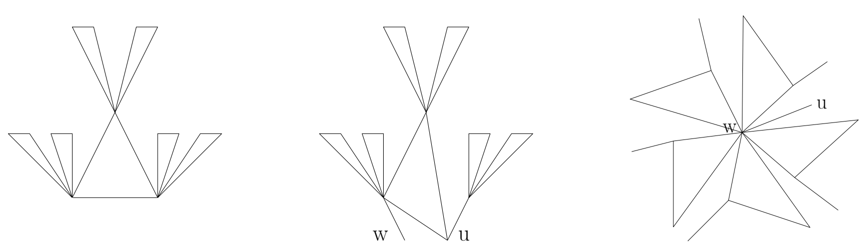

FIGURE 2. The H_{r-1}, A_r and B_r monochromatic subgraphs for $r = 6$.

have $3r-1$ vertices and their blocks are cut-edges and triangles so they do not have even cycles. The graphs A_2, B_2 are both bulls.

Proposition 3. *For even r, K_{3r-1} can be partitioned into $r-1$ copies of A_r and one copy of B_r.*

Proof. Let r be even and consider the construction of Proposition 2 for $r-1$ colors. This gives a partition of K_{3r-3} into $r-1$ copies of H_{r-1}. Notice that the central triangles $T_i = \{a_i, b_i, c_i\}$ of the i-th copies are vertex disjoint $(i = 1, 2, \dots, r-1)$. Adding two new vertices d, e to $V(K_{3r-3})$ transform the i-th copy of H_{r-1} as follows: remove the edges a_ic_i, b_ic_i from T_i and add da_i, db_i, dc_i, eb_i. This gives $r-1$ copies of A_r for $(i = 1, 2, \dots, r-1)$. The "missing edges", $de, ea_i, ec_i, a_ic_i, b_ic_i$ for $i = 1, 2, \dots, r-1$ define one copy of B_r. □

Proposition 3 shows that for even r, $f(r) \geq 3r$, thus completing the proof of Theorem 1.

References

1. L. Lovász, *Combinatorial problems and exercises*, 2nd ed., North-Holland, 1993.
2. W. D. Wallis, *One-Factorizations*, Kluwer Academic Publishers, 1997.

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