



BEADED PARTITIONS WITH k COLORS

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ABSTRACT. In this paper a new category of partitions called beaded partitions with k colors will be introduced. The generating functions for these partitions will be given and several properties and congruences will be presented.

A beaded partition of a positive integer n is a necklace made up of strands of beads in k colors where the total number of beads in the necklace is n . Sliding the beads around the strand does not change the strand and changing the order of the strands in the necklace does not change the partition. However, flipping a strand can result in a different strand. For example if $k = 2$ then there are two distinct strands with one bead, three distinct strands with two beads, and four distinct strands with three beads which are illustrated in Figure 1.

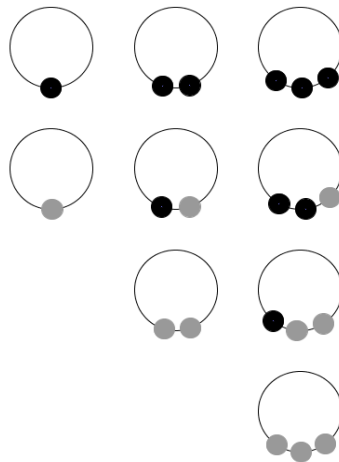


FIGURE 1. Beaded Strands with 2 Colors.

Note that the three strands in Figure 2 are equivalent since the beads can be slid around the strand to form each of these color configurations.

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FIGURE 2. Equivalent strands.

The two strands using $k = 3$ colors in Figure 3, one of which is the reflection of the other, are not equivalent since the beads in the first strand cannot be slid to make the second strand.



FIGURE 3. Non-equivalent strands.

The number of distinct strands with t beads in k colors is given by

$$a_k(t) = \frac{1}{t} \sum_{d|t} \phi\left(\frac{t}{d}\right) k^d;$$

see [4].

For $k = 2$ and $n = 3$, there are fourteen beaded partitions if strands can be repeated and ten beaded partitions if the strands must be distinct. These partitions are illustrated in Figures 4 and 5, respectively.

The generating function for beaded partitions with k colors where the strands can be repeated is given by

$$\sum_{n=0}^{\infty} b_k(n) q^n = \prod_{i=1}^{\infty} (1 - q^i)^{-a_k(i)}.$$

The generating function for beaded partitions with k colors where the strands must be distinct is given by

$$\sum_{n=0}^{\infty} bd_k(n) q^n = \prod_{i=1}^{\infty} (1 + q^i)^{a_k(i)}.$$

These generating functions are similar to those given by Agarwal and Mullen in [1]. Both of these generating functions can be expressed in a different form.

Theorem 1.

$$(a) \quad \prod_{i=1}^{\infty} (1 - q^i)^{-a_k(i)} = \prod_{i=1}^{\infty} (1 - kq^i)^{-1}$$

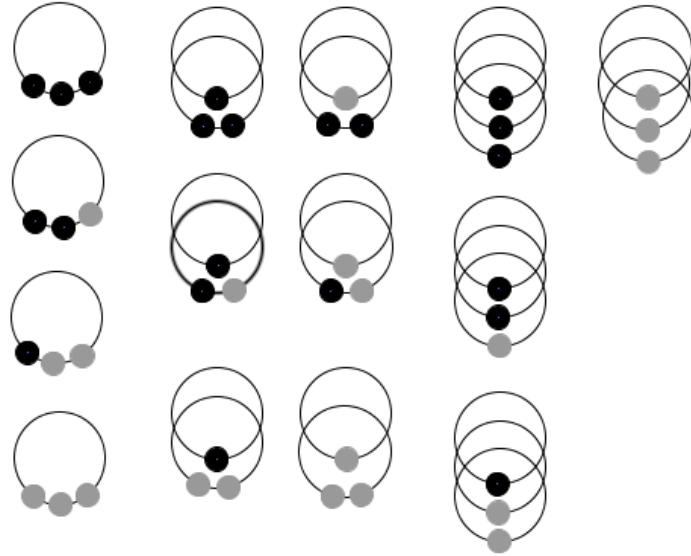


FIGURE 4. Beaded partitions with 2 colors, repetitions allowed.

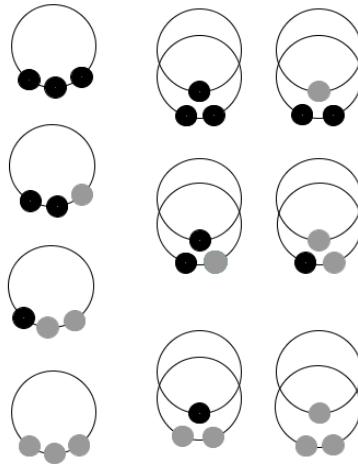


FIGURE 5. Beaded partitions with 2 colors, strands distinct.

(b)

$$\prod_{i=1}^{\infty} (1 + q^i)^{a_k(i)} = \prod_{i=1}^{\infty} (1 + k^{2^{\nu(i)}} q^i)$$

where $\nu(i) = m$ if 2^m divides i but 2^{m+1} does not.

To prove this theorem, we begin by looking at the logarithm of both sides of each equation. For part (a) we have

$$\begin{aligned} \sum_{i=1}^{\infty} -a_k(i) \ln(1 - q^i) &= \sum_{i=1}^{\infty} -a_k(i) \sum_{j=1}^{\infty} \frac{q^{ij}}{j} = - \sum_{n=1}^{\infty} \left(\sum_{t|n} a_k(t) \cdot \frac{t}{n} \right) q^n \\ &= - \sum_{n=1}^{\infty} \left(\sum_{t|n} \frac{1}{t} \left(\sum_{d|t} \phi\left(\frac{t}{d}\right) k^d \right) \cdot \frac{t}{n} \right) q^n \\ &= - \sum_{n=1}^{\infty} \left(\sum_{d|n} \frac{k^d}{n} \sum_{j|\frac{n}{d}} \phi(j) \right) q^n \\ &= - \sum_{n=1}^{\infty} \left(\sum_{d|n} \frac{k^d}{n} \cdot \frac{n}{d} \right) q^n \\ &= - \sum_{i=1}^{\infty} \sum_{d=1}^{\infty} \frac{(kq^i)^d}{d} = - \sum_{i=1}^{\infty} \ln(1 - kq^i). \end{aligned}$$

For part (b) we have

$$\begin{aligned} \sum_{i=1}^{\infty} a_k(i) \ln(1 + q^i) &= \sum_{i=1}^{\infty} a_k(i) \sum_{j=1}^{\infty} \frac{(-1)^j q^{ij}}{j} \\ &= \sum_{n=1}^{\infty} \left(\sum_{t|n} a_k(t) \cdot \frac{t}{n} (-1)^{\frac{n}{t}} \right) q^n \\ &= \sum_{n=1}^{\infty} \left(\sum_{t|n} \frac{1}{t} \left(\sum_{d|t} \phi\left(\frac{t}{d}\right) k^d \right) \cdot (-1)^{\frac{n}{t}} \frac{t}{n} \right) q^n \\ &= \sum_{n=1}^{\infty} \left(\sum_{d|n} \frac{k^d}{n} \sum_{j|\frac{n}{d}} \phi(j) (-1)^{\frac{n}{dj}} \right) q^n \\ &= \sum_{n=1}^{\infty} \left(\sum'_{d|n} \frac{k^d}{n} \cdot \frac{-n}{d} \right) q^n = \sum_{n=1}^{\infty} \left(\sum'_{d|n} \frac{-k^d}{d} \right) q^n \\ &= \sum_{n=1}^{\infty} \left(\sum'_{d|n} \frac{(1 + 2 + 2^2 + \dots + 2^{\nu(n)-1} - 2^{\nu(n)}) k^d}{d} \right) q^n \\ &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (-1)^j \frac{(k^{2^{\nu(i)}} q^i)^j}{j} = \sum_{i=1}^{\infty} \ln(1 + k^{2^{\nu(i)}} q^i). \end{aligned}$$

The ' on the sum indicates that we are summing over the values of d dividing n of the form $2^{\nu(n)}r$ where r is odd since $\sum_{j|\frac{n}{d}} \phi(j)(-1)^{\frac{n}{dj}}$ is 0 if n/d is even and is $-n/d$ if n/d is odd.

Interpreting these new generating functions in terms of partitions we get the following theorem.

Theorem 2.

- (a) *The number of beaded partitions of n with k colors where the strands can be repeated is the same as the number of partitions of n into parts where the parts can occur in k colors and where the ordering of the colors for a part of a given size is taken into account.*
- (b) *The number of beaded partitions of n with k colors where the strands must be distinct is the same as the number of partitions of n where a part i occurs at most once in one of $k^{2^{\nu(i)}}$ colors.*

If we include a parameter z to keep track of the number of strands in each beaded partition with k colors where the strands must be distinct, the product $\prod_{i=1}^{\infty} (1 + q^i)^{a_k(i)}$ becomes $\prod_{i=1}^{\infty} (1 + zq^i)^{a_k(i)}$. If we replace this parameter z by -1 to calculate the difference in the number of beaded partitions of n with an even number of distinct strands and the number with an odd number of distinct strands we obtain the following theorem.

Theorem 3.

$$\prod_{i=1}^{\infty} (1 - q^i)^{a_k(i)} = \prod_{i=1}^{\infty} (1 - kq^i).$$

This theorem follows immediately from Theorem 1.

If we compare the coefficients of q^n on each side of the generating function identities in Theorems 1 and 3, we obtain the following theorem.

Theorem 4.

- (a)

$$b_k(n) = \sum_{\pi \in S(n)} k^{l(\pi)}$$

where $S(n)$ is the set of ordinary partitions of n and $l(\pi)$ is the number of parts in π .

- (b)

$$bd_k(n) = \sum_{\pi \in S(n)} \prod_{n_j \in \pi} \binom{a_k(n_j)}{\#(n_j)} = \sum_{\pi \in S_d(n)} k^{\sum_{n_j \in \pi} 2^{\nu(n_j)}}$$

where $S_d(n)$ is the set of partitions of n into distinct parts, the n_j 's are the parts in the partition π , and $\#(n_j)$ is the number of occurrences of n_j in π .

- (c)

$$\sum_{\pi \in S(n)} (-1)^{l(\pi)} \prod_{n_j \in \pi} \binom{a_k(n_j)}{\#(n_j)} = \sum_{\pi \in S_d(n)} (-k)^{l(\pi)}.$$

Like the ordinary partitions [2] and the generalized Frobenius partitions with k colors [3], the beaded partitions satisfy some interesting congruences. From (a) and (b) in Theorem 4 we have

Theorem 5. *For $n > 0$*

- (a) $b_k(n) \equiv 0 \pmod{k}$,
- (b) $bd_k(n) \equiv 0 \pmod{k}$,
- (c) $bd_k(2n) \equiv 0 \pmod{k^2}$,
- (d) $bd_k(4kn) \equiv 0 \pmod{k^3}$, and
- (e) $bd_k(4kn - 2) \equiv 0 \pmod{k^3}$.

Results (a) and (b) of Theorem 5 follow directly from (a) and (b) of Theorem 4. Result (c) also follows from (b) of Theorem 4 by noting that any partition of $2n$ into distinct parts must include at least one even part or at least two odd parts. Result (d) follows from (b) of Theorem 4 by noting that the number of partitions of $4kn$ into two distinct odd parts is a multiple of k . Result (e) follows by observing that the number of partitions of $4kn - 2$ into two distinct odd parts is $k^2(kn - 1)$ and the number of partitions of $4kn - 2$ as a single part is k^2 which when taken together is a multiple of k^3 .

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