Contributions to Discrete Mathematics

Volume 8, Number 2, Pages 88–119
ISSN 1715-0868

ELEMENTS OF FINITE ORDER IN AUTOMORPHISM GROUPS OF HOMOGENEOUS STRUCTURES

DOĞAN BILGE AND JULIEN MELLERAY

The authors dedicate this article to the memory of Éric Jaligot (1972–2013).

Abstract. We study properties of the automorphism groups of Fraïssé limits of classes with certain strong amalgamation properties, including classes with the free amalgamation property and classes of metric spaces. We discuss conditions on a Fraïssé class $\mathcal{K}$ that imply that the automorphism group $G_{\mathcal{K}}$ of its limit admits generic elements of order n for all n and show that, for many such $\mathcal{K}$, any element of $G_{\mathcal{K}}$ is a product of 4 conjugates of the generic element of order n for all $n \geq 2$. Our constructions enable us to compute the Borel complexity of the relation of conjugacy between automorphisms of the Henson graphs, and to obtain some new results about the structure of the isometry group of the Urysohn space and the Urysohn sphere.

1. Introduction

In recent years, countable homogeneous structures and their automorphism groups have attracted a considerable amount of attention; see for instance, the recent survey [22]. This domain of research features an attractive interplay between model theory and descriptive set theory. For instance, Kechris and Rosendal [19] characterized the Fraïssé classes $\mathcal{K}$ such that the automorphism group G of the Fraïssé limit of $\mathcal{K}$ has *ample generics*, i.e. such that there are comeager diagonal conjugacy classes in G^n for all n (earlier, Ivanov [14] had obtained similar results in a slightly different setting). This notion is a good example of the aforementioned interplay: it first appeared in a paper by Hodges, Hodkinskon, Lascar and Shelah about automorphism groups of certain first-order countable structures [13], then was generalized and applied to the context of general Polish groups by Kechris and Rosendal [19]. Another situation where Baire category and

Received by the editors June 22, 2012, and in revised form August 12, 2013.
2010 *Mathematics Subject Classification.* 20B27,51F99,54E52.
Key words and phrases. Homogeneous structure, Fraïssé limit, Automorphism group, Polish group, Baire category, Urysohn space.
The second author's work on this project was partially funded by the ECOS Nord program (ECOS Nord C10M01) and by the ANR project ANR-11-JS01-0008 GRUPOLOCO.

model-theoretic techniques were used conjointly to great effect is the recent work of MacPherson and Tent, who proved the following result.

Theorem (MacPherson–Tent [21]**).** *Let $\mathcal{K}$ be a Fraïssé class in a countable relational language, and assume that $\mathcal{K}$ has the free amalgamation property. Let $\mathbf{K}$ denote the Fraïssé limit of $\mathcal{K}$, and assume that $\mathrm{Aut}(\mathbf{K})$ acts transitively on $\mathbf{K}$ but is not the full permutation group of $\mathbf{K}$. Then, for any $g \in \mathrm{Aut}(\mathbf{K}) \setminus \{1\}$, every other element of $\mathrm{Aut}(\mathbf{K})$ is a product of at most 32 conjugates of g.*

Informally, saying that $\mathcal{K}$ has the free amalgamation property means that, whenever A, B, C belong to $\mathcal{K}$ and A is a substructure of both B and C, then the structure X with universe $B \cup C$, $B \cap C = A$, and relations exactly those that come from B and C and no others (i.e. a tuple that meets both $B \setminus A$ and $C \setminus A$ does not satisfy any relation in the language of $\mathcal{K}$) belongs to $\mathcal{K}$.

In particular, whenever the assumptions of the above theorem are satisfied, the automorphism group of $\mathbf{K}$ is simple. Very recently, Tent and Ziegler [28] improved the bound above from 32 to 16 and, more interestingly, proved similar results for the group of isometries of the unbounded Urysohn space; for instance, they proved that the quotient of this group by the normal subgroup of bounded isometries is a simple group. Even more recently, after the current article was submitted for publication, they announced a proof that the isometry group of the bounded Urysohn space is simple [27].

In this paper, we are particularly interested in the properties of elements of *finite order* in automorphism groups of Fraïssé limits of certain Fraïssé classes, including those with the free amalgamation property. In this setting, we adapt a construction originally due to Katětov [16] and later used by Uspenskij [30], to obtain the following basic result (below $\mathcal{K}_\omega$ denotes the class of countable structures whose age belongs to $\mathcal{K}$).

Theorem. *Let $\mathcal{K}$ be a Fraïssé class in a countable relational language, assume that $\mathcal{K}$ has the free amalgamation property, and denote its limit by $\mathbf{K}$. Then, for any $\mathbf{X} \in \mathcal{K}_\omega$, there exists a substructure $\widetilde{\mathbf{X}}$ of $\mathbf{K}$ isomorphic to $\mathbf{X}$ such that every automorphism of $\widetilde{\mathbf{X}}$ extends to an automorphism of $\mathbf{K}$; the extension map may be taken to be a continuous group homomorphism from $\mathrm{Aut}(\widetilde{\mathbf{X}})$ to $\mathrm{Aut}(\mathbf{K})$.*

Thus, $\mathrm{Aut}(\mathbf{K})$ is universal for groups of the form $\mathrm{Aut}(\mathbf{X})$, $\mathbf{X} \in \mathcal{K}_\omega$; this is a positive partial answer to a general problem posed by E. Jaligot in [15], which we quote now:

> "Assume that $\mathcal{C}$ is a class of structures in a fixed language, with universal model $\mathcal{M}$, and let $\mathcal{G}$ denote the class of groups of automorphisms of structures in $\mathcal{C}$. Then is it true that the group of automorphisms $\mathrm{Aut}(\mathcal{M})$ is universal for the class of

> groups $\mathcal{G}$? And, if the answer is positive, in what sense is it universal?"

The approach used in our construction leads to the following result.

Theorem. *Let $\mathcal{K}$ be a Fraïssé class in a countable relational language, assume that $\mathcal{K}$ has the free amalgamation property, and denote its limit by $\mathbf{K}$. Then for every integer $n \geq 2$ and any $\mathbf{X} \in \mathcal{K}_\omega$, there exists an automorphism ϕ of $\mathbf{K}$ such that $\phi^n = 1$ and the set of fixed points of ϕ is isomorphic (as a substructure of $\mathbf{K}$) to $\mathbf{X}$.*

This theorem has consequences on the Borel complexity of the conjugacy relation in the automorphism groups of the random graph and the Henson graphs.

Theorem. *Let Γ be either the random graph or the universal homogeneous K_m-free graph for some $m \geq 3$, let G be the automorphism group of Γ and let $n \geq 2$ be an integer. Then the equivalence relation induced by the conjugacy action of G on $\{g \in G \colon g^n = 1\}$ is universal (in the sense of Borel reducibility) for relations induced by a Borel action of S_∞.*

A consequence of the above result is that, if Γ is one of the graphs above, then conjugacy in $\mathrm{Aut}(\Gamma)$ is universal for relations induced by a Borel action of S_∞; in the case of the random graph, this is a recent result due to Coskey, Ellis and Schneider [3].

The second type of Fraïssé class that we are particularly interested in is Q-metric spaces, whose definition we give now.

Definition. Recall that a subsemigroup of $(\mathbf{R}, +)$ is a subset containing 0 and stable under addition. A *metric value set* is either a countable subsemigroup of $(\mathbf{R}, +)$ or the intersection of a countable subsemigroup with an interval $[0, M]$. When Q is a metric value set, and (X, d) is a metric space whose distance function takes its values in Q, we say that (X, d) is a Q-metric space.

It is a folklore fact that one may turn finite Q-metric spaces into a Fraïssé class in a countable relational language (by adding a binary predicate for each possible value of the distance function; see section 4.2 for details) and the limit of this Fraïssé class is a countable Q-metric space, which we denote by $\mathbf{U}_Q$ and is characterized among countable Q-metric spaces by the fact that it contains a copy of every finite Q-metric space and every isometry between finite subsets extends to an isometry of $\mathbf{U}_Q$. Actually, in the case $Q = \mathbf{Q}$ is the set of rational numbers, this space was constructed some thirty years before Fraïssé developed his famous theory [4], in a remarkable paper by Urysohn [29].

A theorem due to Herwig and Lascar [10], which was recently refined by Solecki [26], led us to introduce the following definition.

Definition. Let $\mathcal{K}$ be a class of finite structures in a relational language $\mathcal{L}$. We say that $\mathcal{K}$ has the *isomorphic extension property* (IEP) if for any

$A \in \mathcal{K}$ there exists $B \in \mathcal{K}$ such that $A \leq B$, and a map E from the set of *partial* isomorphisms of A to the set of *global* isomorphisms of B such that:

(i) For any partial isomorphism g of A, $E(g)$ extends g.
(ii) For any $A' \leq A$, E induces a homomorphism from $\mathrm{Aut}(A')$ to $\mathrm{Aut}(B)$.

Solecki's refinement of the Herwig–Lascar theorem presented in [26] shows that, whenever $\mathcal{L}$ is a finite relational language and T is a finite set of $\mathcal{L}$-structures such that the class $\mathcal{K}_T$ of T-free structures is a Fraïssé class, $\mathcal{K}_T$ has the isomorphic extension property. Using his technique from [25], Solecki also showed in [26] that, whenever Q is a metric value set, the class of Q-metric spaces has the isomorphic extension property.

We also introduce the *isomorphic amalgamation property* (IAP), which is satisfied by classes with the free amalgamation property as well as by classes of Q-metric spaces. Applying standard descriptive set theoretic techniques, we show that, if $\mathcal{K}$ is a Fraïssé class with both the (IEP) and (IAP) and limit $\mathbf{K}$, then for any k-tuple $\bar{n}$ of integers there exists a generic element in

$$\Omega_{\bar{n}}(\mathbf{K}) = \{\bar{g} \in \mathrm{Aut}(\mathbf{K})^k \colon \forall i \in \{1, \dots, k\}\ g_i^{n_i} = 1\} \ ,$$

that is, some $\bar{g}$ has a comeager orbit under the diagonal conjugacy action of $\mathrm{Aut}(\mathbf{K})$ on $\Omega_{\bar{n}}(\mathbf{K})$. We apply this to obtain the following.

Theorem. *Assume that T is a finite set of $\mathcal{L}$-structures in a finite relational language $\mathcal{L}$, that $\mathcal{K}$ is the class of finite T-free structures, and that $\mathcal{K}$ is a Fraïssé class with the free amalgamation property; or that $\mathcal{K}$ is the class of finite Q-metric spaces for some metric value set Q. Denote by G the automorphism group of the Fraïssé limit of $\mathcal{K}$, endowed with the permutation group topology.*

Then, for any quadruple of integers $\bar{n}$ bigger than 2, any $g \in G$ may be written in the form $g = h_1 h_2 h_3 h_4$, where h_i is a generic element of order n_i.

Note that the results of MacPherson–Tent–Ziegler show that, in most classes with the free amalgamation property, for *any* element $g \neq 1 \in \mathrm{Aut}(\mathbf{K})$ any other element may be written as a product of 16 conjugates of g and g^{-1}; Tent–Ziegler proved that the same thing holds in the isometry group of the rational Urysohn space whenever g is an unbounded isometry, i.e. $\sup d(x, g(x)) = +\infty$. In that respect, the results of MacPherson–Tent–Ziegler are much stronger than the above theorem; however, our proof is different and (we think) interesting in its own right. It can also be of interest to know that there exist generic elements of order n, a fact that does not appear in the MacPherson–Tent–Ziegler papers.

In the last section of the article, we turn to applications to the isometry group of the continuous Urysohn space $\mathbf{U}$ (and its bounded counterpart, the Urysohn sphere $\mathbf{U}_1$).

Theorem. *Every element of* $\mathrm{Iso}(\mathbf{U})$ *is a commutator. For every integer* n *there exists a generic element* g_n *in* $\{g \in \mathrm{Iso}(\mathbf{U}) \colon g^n = 1\}$*, and every element of* $\mathrm{Iso}(\mathbf{U})$ *is a product of four conjugates of* g_n*.*

This result also holds for $\mathbf{U}_1$ instead of $\mathbf{U}$. Our last result is the product of a failed attempt to prove that a generic element of $\mathrm{Iso}(\mathbf{U})$ embeds in a flow, i.e. for a generic element $g \in \mathrm{Iso}(\mathbf{U})$ there exists a continuous homomorphism $F \colon (\mathbf{R}, +) \to \mathrm{Iso}(\mathbf{U})$ such that $g = F(1)$. We do not answer that question, nor do we even know whether a generic element of $\mathrm{Iso}(\mathbf{U})$ admits a square root; but we do know what the situation is for generic elements of order n in $\mathrm{Iso}(\mathbf{U})$, obtaining a surprisingly strong answer in the bounded case (by d_u below we mean the uniform metric on $\mathrm{Iso}(\mathbf{U}_1)$).

Theorem. *Let* n *be an integer. Then a generic element of order* n *in* $\mathrm{Iso}(\mathbf{U})$ *embeds in a flow. In the case of* $\mathrm{Iso}(\mathbf{U}_1)$*, a generic element of order* n *embeds in a flow which is* n*-Lipschitz from* $(\mathbf{R}, |\cdot|)$ *to* $(\mathrm{Iso}(\mathbf{U}_1), d_u)$*.*

A corollary of this, and our result that every element of $\mathrm{Iso}(\mathbf{U}_1)$ is a product of four conjugates of the generic element of order 2, is the following result.

Corollary. *The topological group* $(\mathrm{Iso}(\mathbf{U}_1), d_u)$ *is path-connected. Actually, for any* $\phi \in \mathrm{Iso}(\mathbf{U}_1)$ *there exists a path* $(\phi_t)_{t \in [0,1]}$ *such that* $\phi_0 = id$*,* $\phi_1 = \phi$*, and* $d_u(\phi_t, \phi_s) \leq 8|t-s|$ *for all* $t, s \in [0,1]$*.*

Note that path-connectedness of $(\mathrm{Iso}(\mathbf{U}_1), d_u)$ is also an immediate corollary of Tent–Ziegler's theorem stating that $\mathrm{Iso}(\mathbf{U}_1)$ is a simple group, a result which was announced after we submitted this paper for publication.

2. Background

2.1. **Fraïssé classes and limits.** We refer to [12] for background on first-order logic; in this article, all languages are relational, (at most) countable and contain the equality symbol. We quickly recall the terminology of Fraïssé classes.

Definition 2.1. Let $\mathcal{L}$ be a language, and $\mathbf{M}$ be a $\mathcal{L}$-structure. The *age* of $\mathbf{M}$ is the collection of all finite $\mathcal{L}$-structures which are isomorphic to a substructure of $\mathbf{M}$.

Definition 2.2. Let $\mathcal{L}$ be a language and $\mathcal{K}$ be a class of finite $\mathcal{L}$-structures. One says that:

(i) $\mathcal{K}$ is *countable* if $\mathcal{K}$ has countably many members up to isomorphism.
(ii) $\mathcal{K}$ is *hereditary* if $\mathcal{K}$ is closed under embeddings, i.e. if $A \in \mathcal{K}$ and B is a $\mathcal{L}$-structure embedding in A then $B \in \mathcal{K}$.
(iii) $\mathcal{K}$ has the *joint embedding property* if for any $A, B \in \mathcal{K}$, there exists $C \in \mathcal{K}$ such that both A, B embed in C.
(iv) $\mathcal{K}$ has the *amalgamation property* if for any $A, B, C \in \mathcal{K}$, and any embeddings $i \colon A \to B$, $j \colon A \to C$, there exists $D \in \mathcal{K}$ and embeddings $\beta \colon B \to D$, $\gamma \colon C \to D$ such that $\beta \circ i = \gamma \circ j$.

A class satisying the four properties above is called a *Fraïssé class.*

It is clear that, whenever $\mathbf{M}$ is a countable $\mathcal{L}$-structure, its age satisfies the first three properties above. The last one, however, is not satisfied in general.

Definition 2.3. Let $\mathcal{L}$ be a language, and $\mathbf{M}$ be a countable $\mathcal{L}$-structure. We say that $\mathbf{M}$ is *homogeneous* if any isomorphism between finite substructures of $\mathbf{M}$ exends to an automorphism of $\mathbf{M}$.

Whenever $\mathbf{M}$ is homogeneous, its age satisfies the amalgamation property, as was first shown by Fraïssé. The converse is true, in the following sense.

Theorem 2.4 (Fraïssé [4]). *Let $\mathcal{L}$ be a language and $\mathcal{K}$ a Fraïssé class of $\mathcal{L}$-structures. Then there exists a unique (up to isomorphism) $\mathcal{L}$-structure $\mathbf{K}$ which is homogeneous and such that age$(\mathbf{K}) = \mathcal{K}$. This structure is called the* Fraïssé limit *of $\mathcal{K}$.*

Note that the Fraïssé limit $\mathbf{M}$ of a class $\mathcal{K}$ of $\mathcal{L}$-structures may be characterized, up to isomorphism, by the following universal property (sometimes called Alice's Restaurant axiom):

For any finite $A \subseteq M$, any $B \in \mathcal{K}$ and any embedding $j\colon A \to B$, there exists $\widetilde{B} \subseteq M$ such that $A \subseteq \widetilde{B}$ and an isomorphism $\phi\colon B \to \widetilde{B}$ such that $\phi \circ j = i$, where i denotes the inclusion map from A to $\widetilde{B}$.

2.2. **Automorphism groups of countable structures.**

Definition 2.5. Let $\mathcal{L}$ be a language and $\mathbf{M}$ a countable $\mathcal{L}$-structure. We denote its automorphism group by $\mathrm{Aut}(\mathbf{M})$, and endow it with the *permutation group topology*, whose basic open sets are of the form

$$\{g \in \mathrm{Aut}(\mathbf{M})\colon g(m_1) = n_1, \dots, g(m_k) = n_k\}$$

where k is an integer and m_i, n_i are elements of M for all $i \in \{1, \dots, k\}$.

The topology defined above is a group topology (i.e. the group operations are continuous), and it is Polish, which means that there exists a complete separable metric inducing the permutation group topology on $\mathrm{Aut}(\mathbf{M})$.

A particularly important example is the case when $\mathcal{L}$ is reduced to the equality symbol, and M is a countable infinite set. Then its automorphism group is the group of all permutations of a countable infinite set; we denote this group by S_∞. Closed subgroups of S_∞ are interesting from the decriptive set theoretic point of view for many reasons and are intimately linked with automorphism groups of countable structures because of the following folklore result.

Theorem 2.6 (see e.g. [1]). *Let $\mathcal{L}$ be a language and $\mathbf{M}$ be a countable $\mathcal{L}$-structure. Then $\mathrm{Aut}(\mathbf{M})$, endowed with its permutation group topology, is isomorphic (as a topological group) to a closed subgroup of S_∞.*

Conversely, for any closed subgroup G of S_∞, there exists a language $\mathcal{L}$ and a homogeneous *$\mathcal{L}$-structure $\mathbf{M}$ such that G is isomorphic (as a topological group) to $\mathrm{Aut}(\mathbf{M})$ endowed with its permutation group topology.*

In the remainder of this article, whenever we mention a topological property of $\mathrm{Aut}(\mathbf{M})$, it is implicitly assumed that we are talking about the permutation group topology. The fact that $\mathrm{Aut}(\mathbf{M})$ is a Polish group will be essential to us, because it enables us to use Baire-category techniques. We refer the reader to [17], [6] and references therein for information on Polish groups and spaces as well as the Baire-category vocabulary and techniques.

3. Free amalgamation and consequences on the structure of the automorphism group

3.1. **Free amalgams.** We quickly recall the definition of a free amalgam.

Let $\mathcal{L}$ be a language, $\mathcal{A}$ a $\mathcal{L}$-structure, $(\mathcal{X}_j)_{j\in J}$ a family of $\mathcal{L}$-structures and $f_j\colon \mathcal{A} \to \mathcal{X}_j$ an embedding from $\mathcal{A}$ to $\mathcal{X}_j$ for all j. We define a $\mathcal{L}$-structure $\mathcal{Y}$, which we will call the free amalgam of the family $(\mathcal{X}_j)$ over the embeddings f_j, or less formally *the free amalgam of the X_j's over A*, as follows:

- First, we consider the disjoint union $Z = \sqcup_j X_j$, and define an equivalence relation $\sim$ on Z by saying that $f_j(a) \sim f_k(a)$ for all j, k and there are no other nontrivial $\sim$-classes. Then we set $Y = \sqcup_j X_j / \sim$ and let Y be the universe of $\mathcal{Y}$.
- Next, we need to turn Y into a $\mathcal{L}$-structure; modulo the obvious identifications, we view X_j as a subset of Y, so that $Y = \cup X_j$, $X_j \cap X_k = A$ for all $j \neq k$. Then, if n is an integer and R is a n-ary relation symbol of $\mathcal{L}$, for any $\bar{y} \in Y^n$ we set

$$(Y \models R(\bar{y})) \Leftrightarrow (\exists j \in J\ \forall k \in \{1, \dots, n\}\ y_k \in X_j \text{ and } X_j \models R(\bar{y}))\ .$$

Informally, the free amalgam of the X_j's over A is an $\mathcal{L}$-structure Y with universe $\cup X_j$ such that each X_j is a substructure of Y, $X_j \cap X_k = A$ for all $j \neq k$, and no tuple which meets $X_j \setminus A$ and $X_k \setminus A$ for some $j \neq k$ satisfies any relation in $\mathcal{L}$.

Definition 3.1. Let $\mathcal{L}$ be a language. We say that a class $\mathcal{K}$ of $\mathcal{L}$-structures has the *free amalgamation property* if whenever $A, B, C \in \mathcal{K}$ and $i\colon A \to B, j\colon A \to C$ are embeddings, the free amalgam of B, C over i, j belongs to $\mathcal{K}$.

The class of all finite graphs (seen as a class of structures in the language $\mathcal{L}$ whose only relation symbol besides the equality is binary) has the free amalgamation property, while the class of tournaments does not (it does have the amalgamation property). If K_m denotes the complete graph on m vertices, a graph which does not contain a substructure isomorphic to K_m is called a *K_m-free graph.* The class of all K_m-free graphs also has the free amalgamation property.

The other conditions defining a Fraïssé class are easy to check in all the examples above; the Fraïssé limit of the class of all finite graphs is often called the *Radó graph*, or the *random graph*, while the various Fraïssé limits of K_m-free graphs (as m varies) are sometimes called the Henson graphs, as they were first introduced and studied by Henson in [9].

3.2. **Structures whose age is contained in $\mathcal{K}$.** In this subsection, we fix a language $\mathcal{L}$ and a class $\mathcal{K}$ of finite $\mathcal{L}$-structures, and we assume that $\mathcal{K}$ is a Fraïssé class with the free amalgamation property.

Definition 3.2. We let $\mathcal{K}_\omega$ denote the class of all countable $\mathcal{L}$-structures whose age is contained in $\mathcal{K}$.

Observe that, if $\mathbf{M}$ is a countable $\mathcal{L}$-structure admitting an increasing chain of substructures $\mathbf{M}_i$ such that each $\mathbf{M}_i$ belongs to $\mathcal{K}_\omega$ and $\mathbf{M} = \cup \mathbf{M}_i$, then $\mathbf{M}$ also belongs to $\mathcal{K}_\omega$.

The following fact is clear.

Proposition 3.3. *Assume that $\mathbf{A} \in \mathcal{K}_\omega$,$(\mathbf{X}_i)_{i\in I}$ is a countable family of element of $\mathcal{K}_\omega$ and that $f_i \colon \mathbf{A} \to \mathbf{X}_i$ is an embedding for all i. Then the free amalgam of the $\mathbf{X}_i$'s over $\mathbf{A}$ belongs to $\mathcal{K}_\omega$.*

Definition 3.4. For $X \in \mathcal{K}_\omega$, denote by $\mathcal{L}_X$ the language obtained by adding to $\mathcal{L}$ a constant symbol c_x for all $x \in X$ (Here we do not respect our convention that languages are relational).

If $Y = X \cup \{y\}$ is an element of $\mathcal{K}_\omega$, $y \notin X$, seen as an $\mathcal{L}_X$-structure by interpreting each c_x by x, the *quantifier-free type of y over X* is the family of all quantifier-free $\mathcal{L}_X$-formulas ϕ with one free variable z such that $Y \models \phi(y)$.

We say that a set p of quantifier-free $\mathcal{L}_X$-formulas with one free variable is a *quantifier-free type over X* if there exists $Y = X \cup \{y\} \in \mathcal{K}_\omega$ such that p is the quantifier-free type of y. We call Y the *structure associated to p*; up to obvious identifications it is uniquely determined by p.

In less formal terms: if Y is a structure of the form $X \cup \{y\}$, $y \notin X$, then the quantifier-free type of y over X is the complete description of the relations between elements of X, and relations between elements of X and y. To shorten notation a bit, we write q.f. type instead of quantifier-free type.

Definition 3.5. Let $X \in \mathcal{K}_\omega$ and p be a q.f. type over X. Then, for any automorphism f of X, we define $f(p)$ as the q.f. type with one free variable z defined by:

$$\phi(z, c_{x_1}, \ldots, c_{x_n}) \in f(p) \Leftrightarrow \phi(z, c_{f^{-1}(x_1)}, \ldots, c_{f^{-1}(x_n)}) \in p \ .$$

Intuitively, $f(p)$ describes an element such that, for any $x_1, \ldots, x_n$ in X, $f(p)$ satisfies the same relations with $f(x_1), \ldots, f(x_n)$ as p does with $x_1, \ldots, x_n$.

Definition 3.6. Let $X \in \mathcal{K}_\omega$. We say that a q.f. type p over X is *finitely induced* if there exists a finite substructure A of X, and an element $B = A \cup \{b\}$ of $\mathcal{K}$, such that the structure associated to p is isomorphic to the free amalgam of B and X over A.

Note that, if $\mathcal{K}$ is not reduced to a singleton, then the free amalgamation property implies that $\mathcal{K}$ is infinite, and for any $X \in \mathcal{K}_\omega$ there always exists at least one finitely-induced q.f. type over X (there may exist only one, a situation that presents itself for instance when $\mathcal{K}$ is the class of pure sets). In the remainder of the article, we always assume that the classes we consider are infinite.

Definition 3.7. Let $\mathbf{X}$ be an element of $\mathcal{K}_\omega$, $\{p_i\}_{i\in I}$ be an enumeration of all the finitely induced q.f. types over $\mathbf{X}$, and for each i let $\mathbf{Y}_i$ denote the structure associated to p_i. Let $E(\mathbf{X})$ denote the free amalgam of the $\mathbf{Y}_i$'s over $\mathbf{X}$.

Since there are at most countably many finitely induced q.f. types over $\mathbf{X}$, and $\mathcal{K}_\omega$ is stable under free amalgamation, $E(\mathbf{X})$ belongs to $\mathcal{K}_\omega$. Also, $\mathbf{X}$ naturally embeds in $E(\mathbf{X})$, and $E(\mathbf{X}) \setminus \mathbf{X}$ is always nonempty.

The definition of $E(\mathbf{X})$ is motivated by Katětov's construction of the Urysohn space [16]; in that context the next proposition was first pointed out by Uspenskij [30].

Proposition 3.8. *Assume $\mathbf{X} \in \mathcal{K}_\omega$. Then each automorphism of $\mathbf{X}$ extends uniquely to an automorphism of $E(\mathbf{X})$, and the extension morphism is an embedding of topological groups from $\mathrm{Aut}(\mathbf{X})$ to $\mathrm{Aut}(E(\mathbf{X}))$.*

Proof. Let ϕ be an automorphism of X. For each $y \in E(X) \setminus X$, with q.f. type denoted by p, there exists a unique $z \in E(X) \setminus X$ such that the q.f. type of z is equal to $\phi(p)$. To extend ϕ to an automorphism $E(\phi)$, one has no choice but to set $E(\phi)(y) = z$; this proves the uniqueness. The fact that this extension is indeed an automorphism of $E(X)$ is also obvious, since by definition, two different elements of $E(X) \setminus X$ do not belong to any tuple satisfying a relation of $\mathcal{L}$.

The uniqueness of the extension ensures that $\phi \mapsto E(\phi)$ is an injective homomorphism. To show that it is continuous, pick $x_1, \ldots, x_n \in X$ and $y_1, \ldots, y_m \in E(X) \setminus X$. For $i \in \{1, \ldots, m\}$, let A_i be a finite substructure of X witnessing the fact that the q.f. type of y_i is finitely induced, and let $A = \{x_1, \ldots, x_n\} \cup \cup_i A_i$. Then A is a finite substructure of X, and for any ϕ which coincides with the identity on A one must have $\phi(x_i) = x_i$ for all $i \in \{1, \ldots, n\}$ and $\phi(y_i) = y_i$ for all $i \in \{1, \ldots, m\}$. We have just proved that, for any neighborhood V of the identity in $\mathrm{Aut}(E(X))$, there exists a neighborhood U of the identity in $\mathrm{Aut}(X)$ such that $E(U) \subseteq V$, hence E is continuous. $\square$

Theorem 3.9. *Assume that $\mathcal{K}$ is a Fraïssé class with the free amalgamation property, and denote by $\mathbf{K}$ the Fraïssé limit of $\mathcal{K}$. Then, for any $\mathbf{X} \in \mathcal{K}_\omega$ there exists an embedding $i \colon \mathbf{X} \to \mathbf{K}$ such that any automorphism ϕ of $i(\mathbf{X})$ extends to an automorphism $E(\phi)$ of $\mathbf{K}$.*

The extension map $\phi \mapsto E(\phi)$ may be taken to be a continuous group embedding from $\mathrm{Aut}(\mathbf{X})$ to $\mathrm{Aut}(\mathbf{K})$.

In particular, this result shows that $\mathrm{Aut}(\mathbf{K})$ contains an isomorphic copy of any permutation group of the form $\mathrm{Aut}(\mathbf{X})$, $\mathbf{X} \in \mathcal{K}_\omega$. In the Urysohn space context, the above result was proved by Uspenskij [30]; the proof below is essentially Uspenskij's proof, translated to our setting.

Proof. Starting from $X_0 = X$, we build an increasing chain of structures in $\mathcal{K}_\omega$ by setting $X_{i+1} = E(X_i)$ for all $i < \omega$, viewing X_i as a substructure of X_{i+1} via the natural embedding from X_i to $E(X_i)$. Then $X_\infty = \cup X_i$ belongs to $\mathcal{K}_\omega$. Using Proposition 3.8, we see that any automorphism φ of X_i uniquely extends to an automorphism $E(\phi)$ such that $E(\phi)(X_i) = X_i$ for all i, and that $\phi \mapsto E(\phi)$ is a topological group embedding from $\mathrm{Aut}(X)$ to $\mathrm{Aut}(X_\infty)$.

Fix a finite substructure A of X_∞, and an embedding $j \colon A \to B$ for some $B \in \mathcal{K}$. Let $B = j(A) \cup \{b_1, \dots, b_n\}$. There must exist some $i < \omega$ such that $A \subset X_i$, and an easy induction argument shows that there exists $\tilde{b}_1, \dots, \tilde{b}_n \in X_{i+n}$ such that $\widetilde{B} = A \cup \{\tilde{b}_1, \dots, \tilde{b}_n\}$ is isomorphic to B via an isomorphism $\phi \colon B \to \widetilde{B}$ such that $\phi \circ j = i$ (where i stands for the inclusion map from A to $\widetilde{B}$). This shows that X_∞ is isomorphic to the Fraïssé limit of $\mathcal{K}$, and we are done. $\square$

Theorem 3.10. *Let $n \geq 2$ be an integer, assume that $\mathcal{K}$ is a Fraïssé class with the free amalgamation property, and let $\mathbf{K}$ denote the Fraïssé limit of $\mathcal{K}$. Then, for any $\mathbf{X} \in \mathcal{K}_\omega$, there exists an automorphism ϕ of $\mathcal{K}_\omega$ such that $\phi^n = 1$ and the set of fixed points of ϕ, seen as a substructure of $\mathbf{K}$, is isomorphic to $\mathbf{X}$.*

Proof. We build inductively an increasing sequence (X_i, ϕ_i) such that each X_i belongs to $\mathcal{K}_\omega$, ϕ_i is an automorphism of X_i, and:

(i) $X_0 = X$, $\phi_0 = id_X$.
(ii) For all $i < \omega$, $X_i \subseteq E(X_i) \subseteq X_{i+1}$ and ϕ_{i+1} exends ϕ_i.
(iii) For all $i < \omega$, one has $\phi_i^n = 1$.
(iv) For all $i < \omega$, the set of fixed points of ϕ_i is equal to X_0.

In point (ii), the inclusion $X_i \subseteq E(X_i)$ is to be understood as the natural embedding of X_i in $E(X_i)$.

Then, set $X_\infty = \cup X_i$, $\phi_\infty = \cup \phi_i$. As in the proof of Theorem 3.9, (ii) ensures that X_∞ is the Fraïssé limit of $\mathcal{K}$; points (iii) and (iv) ensure that $\phi_\infty^n = 1$ and that the set of fixed points of ϕ_∞ is equal to X_0, hence, as a substructure of X_∞, is isomorphic to X.

Thus, we only need to explain how to carry out the construction. The first step is imposed, so we set $X_0 = X$, $\phi_0 = id_X$. Assume that (X_i, ϕ_i) has been constructed. We let X_{i+1} be the free amalgam of n copies of $E(X_i)$ over X_i. To define ϕ_{i+1}, let $Y_0, \dots, Y_{n-1}$ be n copies of $E(X_i)$ such that $X_{i+1} = \cup Y_k$, $Y_j \cap Y_k = X_i$ for all $j \neq k \in \{0, \dots, n-1\}$, and pick $y \in X_{i+1} \setminus X_i$. There exists a unique j such that $y \in Y_j$; let p denote the q.f. type of y over X_i. There exists a unique element z of Y_{j+1} (addition here being modulo n) such that the q.f. type of z is equal to $\phi(p)$; we set $\phi_{i+1}(y) = z$.

It is clear that ϕ_{i+1} is then an automorphism of X_{i+1}, whose set of fixed points is the same as the set of fixed points of ϕ_i, hence is equal to X_0. To check that $\phi_{i+1}^n = 1$, it suffices to notice that, for any $y \in Y_j \setminus X_i$, with q.f. type denoted by p, $\phi_{i+1}^n(y)$ is by definition the unique element of $Y_{j+n} = Y_j$ whose type is equal to $\phi_i^n(p) = p$. Hence $\phi_{i+1}^n(y) = y$, and we are done. $\square$

We note in passing that the result above also holds for the rational Urysohn space, with a similar proof. The second author proved in [23] a slightly weaker result, namely, that any countable rational metric space is isometric to the set of fixed points of some isometry of the rational Urysohn space. The construction provided there enabled one to obtain a similar result for the Urysohn space (which is the completion of the rational one), while the construction presented here is not enough for that purpose; curiously, it is not true that any Polish metric space is isometric to the set of fixed points of an isometry of the Urysohn space of finite order. In other words, the analogue of Theorem 3.10 is false in the Urysohn space.

3.3. **An application to the complexity of some conjugacy problems.** In this subsection, we apply Theorem 3.10 to show that, if $\mathcal{K}$ is a Fraïssé class with the free amalgamation property, then the isomorphism relation on $\mathcal{K}_\omega$ is always reducible to the conjugacy relation in $\mathrm{Aut}(\mathbf{K})$. This enables one to compute the exact complexity of the conjugacy relation in the automorphism group of the Henson graphs and the random graph.

Since this section is only tangentially related to our interests in this paper, we will not give any background on the theory of Borel reducibility of definable equivalence relations. For information on this rich theory we refer to [1], [6], [11] and [18]. Let us simply set our notation: we fix a language $\mathcal{L} = (R_i, n_i)_{i \in I}$. For any $i \in I$, we let $\mathcal{X}_i = 2^{\mathbf{N}^{n_i}}$, endowed with the product topology, and set

$$\mathcal{X}_{\mathcal{L}} = \prod_{i \in I} \mathcal{X}_i \ .$$

Endowed with the product topology, this is a compact topological space (homeomorphic to the Cantor space); for each i we denote by $\pi_i \colon \mathcal{X}_{\mathcal{L}} \to \mathcal{X}_i$ the coordinate projection.

To each element X of $\mathcal{X}_{\mathcal{L}}$ we may associate an infinite, countable $\mathcal{L}$-structure $\widetilde{X}$ by setting, for all $\bar{n} \in \mathbf{N}^{n_i}$:

$$\widetilde{X} \models R_i(\bar{n}) \Leftrightarrow \pi_i(X)(\bar{n}) = 1 \ .$$

Conversely, any $\mathcal{L}$-structure with universe $\mathbf{N}$ defines an element of $\mathcal{X}_{\mathcal{L}}$; thus we may see $\mathcal{X}_{\mathcal{L}}$ as the *space of all infinite countable $\mathcal{L}$-structures.* We say that $X, Y \in \mathcal{X}_{\mathcal{L}}$ are *isomorphic* if $\widetilde{X}, \widetilde{Y}$ are isomorphic as $\mathcal{L}$-structures. The permutation action of S_∞ on $\mathbf{N}$ naturally extends to a continuous action of S_∞ on $\mathcal{X}_{\mathcal{L}}$, defined by setting, for all $X \in \mathcal{X}_{\mathcal{L}}$, $\sigma \in S_\infty$, $i \in I$ and $\bar{n} \in \mathbf{N}^{n_i}$,

$$\pi_i(\sigma \cdot X)(\bar{n}) = \pi_i(X)(\sigma^{-1}(\bar{n})) \ .$$

Then, X and Y are isomorphic if, and only if, there exists $\sigma \in S_\infty$ such that $\sigma \cdot X = Y$; thus, thinking of $\mathcal{X}_\mathcal{L}$ as the space of all infinite countable $\mathcal{L}$-structures, the relation of isomorphism of infinite countable $\mathcal{L}$-structures is given by a continuous action of S_∞ on the compact space $\mathcal{X}_\mathcal{L}$, called the *logic action* of S_∞ on $\mathcal{X}_\mathcal{L}$.

Now, if $\mathcal{K}$ is a Fraïssé class of $\mathcal{L}$-structures, define

$$\mathcal{X}_\mathcal{K} = \{X \in \mathcal{X}_\mathcal{L} \colon \widetilde{X} \in \mathcal{K}_\omega\} \ .$$

Then $\mathcal{X}_\mathcal{K}$ is a Borel subset of $\mathcal{X}_\mathcal{L}$ (it is even G_δ if $\mathcal{L}$ is finite), and $\mathcal{X}_\mathcal{K}$ is invariant under the logic action of S_∞. We call the relation induced by the logic action the isomorphism relation on infinite elements of $\mathcal{K}_\omega$.

Theorem 3.11. *Let $n \geq 2$ be an integer, $\mathcal{L}$ be a countable relational language, and $\mathcal{K}$ be a Fraïssé class of $\mathcal{L}$-structures with the free amalgamation property. Denote by $\mathbf{K}$ the Fraïssé limit of $\mathcal{K}$. Then, for any integer n, the isomorphism relation on infinite elements of $\mathcal{K}_\omega$ (as encoded above) is Borel reducible to the conjugacy relation in $\{g \in \mathrm{Aut}(\mathbf{K}) \colon g^n = 1\}$.*

Proof. It is possible to define a Borel map $X \mapsto \phi(X)$ from $\mathcal{X}_\mathcal{K}$ to $\mathrm{Aut}(\mathbf{K})$ such that $\phi(X)$ is conjugate to the automorphism of $\mathbf{K}$ built in the proof of Theorem 3.10. We do not give the tedious technical details of this construction; they are presented in full in the first author's doctoral dissertation [2]). Let us now see why this proves our result.

First, letting $(\widetilde{X}_i, \phi_i(X))$ and $(\widetilde{Y}_i, \phi_i(Y))$ denote the sequences constructed by applying the construction of 3.10 to X, Y, we see that if X, Y are isomorphic we may inductively build a sequence of isomorphisms $g_n \colon \widetilde{X}_i \to \widetilde{Y}_i$ such that $g_i\phi_i(X)g_i^{-1} = \phi_i(Y)$. This yields an isomorphism $g \colon \widetilde{X}_\infty \to \widetilde{Y}_\infty$ such that $g\phi(X)g^{-1} = \phi(Y)$; identifying $\widetilde{X}_\infty, \widetilde{Y}_\infty$ with $\mathbf{K}$ we see that $\phi(X)$, $\phi(Y)$ are conjugate. Conversely, if $\phi(Y) = g\phi(X)g^{-1}$ for some $g \in \mathrm{Aut}(\mathbf{K})$, then g must map the set of fixed points of $\phi(X)$ onto the set of fixed points of $\phi(Y)$. Hence the sets of fixed points of $\phi(X), \phi(Y)$ are isomorphic substructures of M, that is, X and Y are isomorphic. □

The fact that the conjugacy relation in the automorphism group of the random graph is S_∞-universal, which follows from point (i) below, was first proved by Coskey, Ellis and Schneider [3].

Corollary. *Let $n \geq 2$ be an integer. Then:*

(i) *The conjugacy relation on $\{g \in \mathrm{Aut}(\mathcal{R}) \colon g^n = 1\}$, where $\mathcal{R}$ denotes the random graph, is S_∞-universal.*

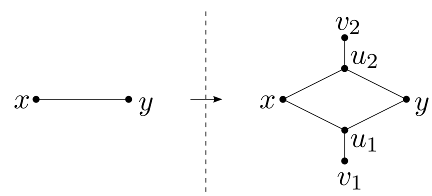
(ii) *Let $m \geq 3$ be an integer and G_m denote the Fraïssé limit of the class of K_m-free graphs. Then the conjugacy relation on*

$$\{g \in \mathrm{Aut}(G_m) \colon g^n = 1\}$$

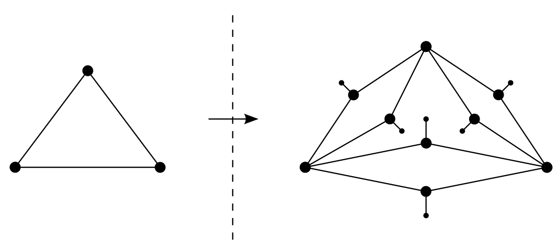
is S_∞-universal.

Proof. Theorem 3.11 shows that the relation of isomorphism of graphs Borel reduces to the conjugacy relation on $\{g \in \mathrm{Aut}(\mathcal{R}) \colon g^n = 1\}$ for all $n \geq 2$, so the first result is an immediate corollary of Theorem 3.11 and the fact that isomorphism of countable graphs is S_∞-universal (this is one of the first results in the theory of Borel reducibility, see [5]).

For the second one, we claim that, for any m, the isomorphism relation for graphs Borel reduces to the isomorphism relation for K_m-free graphs, from which the result follows. One way to build a reduction is as follows: let G denote a countable graph; for each edge of G, add four new vertices, remove the edge, and add new edges as in the picture below.



Let $\widetilde{G}$ denote the graph produced by applying this transformation to G; for example, the following picture shows what $\widetilde{G}$ is if G is a triangle.



Then $\widetilde{G}$ is K_m-free for all $m \geq 3$, and the map $G \mapsto \widetilde{G}$ can be coded by a Borel map (in the space of codes of graphs), thus we only need to show that, for two countable graphs G, H, one has

$$(G \cong H) \Leftrightarrow (\widetilde{G} \cong \tilde{H}) \ .$$

The implication from left to right is obvious. To prove the converse, assume that $\phi \colon \widetilde{G} \to \widetilde{H}$ is an isomorphism. The v_i's constructed above are the only elements of $\widetilde{G}$ with exactly one neighbour, and the u_i's are the only elements of $\widetilde{G}$ with a neighbour v_i. The same thing happens in $\widetilde{H}$, so ϕ must map the u_i's, v_i's that were added to G to the u_i's, v_i's that were added to H, hence ϕ maps the set $V(G)$ of vertices of G to the set $V(H)$ of vertices of H. Finally, two elements of $V(G)$ have an edge between them in G if and only if they have a common u_i as a neighbour in $\widetilde{G}$, and this is preserved by ϕ, hence ϕ induces an isomorphism from G to H, and we are done. $\square$

Before moving on, let us note that we do not know any example of a Fraïssé class $\mathcal{K}$ with the free amalgamation property and such that the isomorphism relation for infinite elements of $\mathcal{K}_\omega$ is neither smooth nor S_∞-universal.

4. Generic elements of finite order

4.1. **Coherent extensions.** In this subsection, we fix a *finite* language $\mathcal{L}$. We discuss an improvement by Solecki of a theorem of Herwig–Lascar.

Definition 4.1 (see [10], p. 1994)**.** Let A, B be two $\mathcal{L}$-structures. A map $f \colon A \to B$ is a *weak homomorphism* if for any integer n, any n-ary relational symbol R of $\mathcal{L}$ and any $a_1, \ldots, a_n \in A$ one has

$$A \models R(a_1, \ldots, a_n) \Rightarrow B \models R(f(a_1), \ldots, f(a_n)) .$$

If A is a $\mathcal{L}$-structure and T is a set of $\mathcal{L}$-structures, say that A is *T-free* if there is no weak homomorphism from a structure in T to A (this coincides with the usual terminology in the case of graphs).

Definition 4.2 (Solecki [26])**.** Let A be a set, and p_1, p_2, p_3 be three partial bijections of A. Say that (p_1, p_2, p_3) is *coherent* if

$$\mathrm{dom}(p_1) = \mathrm{rng}(p_2),\ \mathrm{dom}(p_2) = \mathrm{dom}(p_3),\ \mathrm{rng}(p_1) = \mathrm{rng}(p_3) \text{ and } p_3 = p_1 \circ p_2 .$$

If A, B are sets and ϕ is a function from the sets of partial bijections of A to the set of partial bijections of B, say that ϕ is *coherent* if $(\phi(p_1), \phi(p_2), \phi(p_3))$ is coherent whenever (p_1, p_2, p_3) is coherent.

The following is Solecki's aforementioned refinement of the Herwig–Lascar theorem.

Theorem 4.3 (Solecki [26])**.** *Let T be a finite family of structures, A be a finite T-free structure, and P be a set of partial isomorphisms of A. If there exists a T-free structure M containing A such that each element of P extends to an automorphism of M, then there exists a finite T-free structure B containing A and $E \colon P \to \mathrm{Aut}(B)$ such that:*

(i) *$E(p)$ is an extension of p for all $p \in P$.*
(ii) *E is coherent.*

Using the notations of the above theorem, note that if A' is a substructure of A, and P contains $\mathrm{Aut}(A')$, then the coherence of the map E implies that $E \colon \mathrm{Aut}(A') \to \mathrm{Aut}(B)$ is a homomorphism (which is necessarily injective).

The assumption of Theorem 4.3 is satisfied if T is a finite set of $\mathcal{L}$-structures such that the set of all T-free structures is a Fraïssé class. Solecki also used this result to show a similar theorem for classes of metric spaces, proving the following:

Theorem 4.4 (Solecki [26])**.** *Let A be a finite metric space. There exists a finite metric space B such that $A \subseteq B$ as metric spaces, each partial isometry p of A extends to an isometry $E(p)$ of B and the function E is coherent. Moreover, the distances between points in B belong to the additive semi-group generated by the distances between points in A.*

4.2. **Metric spaces as relational structures.** To prove Theorem 4.4, one of the steps is to view finite metric spaces as relational structures. We quickly explain how one may do this; first, to stay within the realm of countable structures, one has to impose a condition on the set of possible values for the metric, and we introduce some ad hoc terminology.

Definition 4.5. Let Q be a countable subset of $[0, +\infty)$ containing 0. We say that Q is a *metric value set* if one of the following conditions is satisfied:

(i) Q is a subsemigroup of $(\mathbf{R}; +)$.
(ii) Q is the intersection of a subsemigroup of $(\mathbf{R}, +)$ and a bounded interval, and $M_Q = \sup(Q) \in Q$.

Definition 4.6. If Q is a metric value set, a Q-metric space is a metric space (X, d) such that $d(x, x') \in Q$ for all $(x, x') \in X^2$.

Q-metric spaces may be turned into relational structures in a countable language $\mathcal{L}_Q$, containing a binary predicate R_q for all $q \in Q$: simply put $R_q(x, y)$ if and only if $d(x, y) = q$. In this way, one may think of Q-metric spaces as relational structures in a countable language. It is a folklore fact that, if Q is a metric value set, the class of finite Q-metric spaces is a Fraïssé class. The limit denoted by $\mathbf{U}_Q$ is a countable metric space characterized up to isometry within the class of countable Q-metric spaces, by the fact that it contains a copy of any finite Q-metric space and any isometry between finite subsets extends to an isometry of the whole space.

We now discuss the amalgamation procedure we use for Q-metric spaces (which justifies that it is a Fraïssé class): let (A, d_A), (B, d_B), (C, d_C) be three finite Q-metric spaces, $i\colon A \to B$, $j\colon A \to C$ two isometric embeddings and assume that A is nonempty. We let X denote the disjoint union of B and C and define a pseudometric ρ on X as follows:

(i) If Q is unbounded, $\rho(b, c) = \min\{d_B(b, i(a)) + d_C(j(a), c)\colon a \in A\}$.
(ii) If Q is bounded and $M_Q = \sup(Q)$,
$$\rho(b, c) = \min\{M_Q, \min\{d_B(b, i(a)) + d_C(j(a), c)\colon a \in A\}\} .$$

In less formal terms, we identified the two copies of A in B, C and for $b \in B \setminus A, c \in C \setminus C$, set $d_D(b, c)$ to be equal to the length of the shortest path between b and c going through A (cut off at M_Q if Q is bounded). Note that we do not really need $\sup(Q)$ to belong to Q to make amalgamation work, but this assumption slightly simplifies the exposition and is more than enough for the applications to the Urysohn space that we have in mind.

Since we allow metric spaces to be empty, we should also explain how we amalgamate over the empty set (which proves that the class of Q-metric spaces satisfies the joint embedding property). Assume that Q is a metric value set and B, C are two finite Q-metric spaces. We extend the metric on B, C to a metric on $B \sqcup C$ by setting $d(b, c) = \max(\mathrm{diam}(B), \mathrm{diam}(C))$ for all $b \in B, c \in C$.

Here, it may be worth pointing out a somewhat counterintuitive phenomenon: the Fraïssé limit of the class of Q-metric spaces, when seen as a structure in a relational language as above, is *not* $\aleph_0$-categorical as soon as Q is infinite: there are models of its theory which are not metric spaces, as there is no first-order way to ensure that for any x, y there exists q such that $M \models R_q(x, y)$.

4.3. The isomorphic extension property.

Definition 4.7. Let $\mathcal{K}$ be a class of finite structures in a relational language $\mathcal{L}$. We say that $\mathcal{K}$ has the *isomorphic extension property* (IEP) if for any $A \in \mathcal{K}$ there exists $B \in \mathcal{K}$ such that $A \leq B$, and a map E from the set of *partial* isomorphisms of A to the set of *global* automorphisms of B such that:

- (i) For any partial isomorphism g of A, $E(g)$ extends g.
- (ii) For any $A' \leq A$, E induces a homomorphism from $\mathrm{Aut}(A')$ to $\mathrm{Aut}(B)$.

If there exists E as above such that only condition (i) is satisfied, then we say that $\mathcal{K}$ has the *extension property.*

As mentioned above, whenever $\mathcal{L}$ is a finite relational language, and T is a finite set of $\mathcal{L}$-structures such that the class $\mathcal{K}$ of all finite T-free structures is a Fraïssé class, it follows from the Herwig–Lascar–Solecki theorem that $\mathcal{K}$ has the isomorphic extension property. Similarly, it follows from Theorem 4.4 that, whenever Q is a metric value set, the class $\mathcal{M}_Q$ of all finite Q-metric spaces has the isomorphic extension property.

Definition 4.8. Let $\mathcal{L}$ be a relational language and $\mathcal{K}$ a class of $\mathcal{L}$-structures. We say that $\mathcal{K}$ has the *isomorphic amalgamation property* (IAP) if, for any $A, B, C \in \mathcal{K}$ and any embeddings $i\colon A \to B$, $j\colon A \to C$, there exists $D \in \mathcal{K}$ and embeddings $\alpha\colon B \to D$, $\beta\colon C \to D$ such that:

- (i) $\alpha \circ i = \beta \circ j$.
- (ii) Whenever $\phi \in \mathrm{Aut}(B)$ fixes $i(A)$ setwise, $\psi \in \mathrm{Aut}(C)$ fixes $j(A)$ setwise, and $i^{-1}\phi i = j^{-1}\psi j$, there exist a partial isomorphism σ of D fixing $\alpha(B)$, $\beta(C)$ respectively such that $\alpha^{-1}\sigma\alpha = \phi$, $\beta^{-1}\sigma\beta = \psi$.

Note that any Fraïssé class with the free amalgamation property satisfies the isomorphic amalgamation property, as does the class of Q-metric spaces for any metric value set Q. Also, if $\mathcal{K}$ has both the (IAP) and the extension property (which will be the case in all our examples), then σ can be taken to be an automorphism of D.

Notation. Let $\mathcal{K}$ be a Fraïssé class with limit $\mathbf{K}$, and denote by G the automorphism group of $\mathbf{K}$. For any integer k and any $\bar{n} = (n_1, \ldots, n_k) \in \mathbf{N}^k$, we set

$$\Omega_{\bar{n}}(G) = \{\bar{g} \in G^k \colon \forall i \in \{1, \ldots, k\}\ g_i^{n_i} = 1\}\ .$$

We endow it with the topology induced by the topology of G^k, which turns it into a Polish space on which G acts by diagonal conjugacy:

$$g \cdot (g_1, \ldots, g_k) = (gg_1g^{-1}, \ldots, gg_ng^{-1})\ .$$

We say that $\bar{g} \in \Omega_{\bar{n}}(G)$ is *generic* if its diagonal conjugacy class is comeager (equivalently, dense G_δ).

Similarly, if A is a structure, we write $\Omega_{\bar{n}}(A)$ to denote the set of n-tuples $(g_1, \dots, g_n)$ of automorphisms of A such that $g_i^{n_i} = 1$ for all i.

Proposition 4.9. *Let $\mathcal{K}$ be a Fraïssé class satisfying the isomorphic extension and isomorphic amalgamation properties, and let G denote the automorphism group of its limit $\mathbf{K}$. Then for any integer k, and any k-tuple $\bar{n}$ of non-negative integers, there exists a generic element in $\Omega_{\bar{n}}(G)$. The orbit of any $x \in \mathbf{K}$ under $\langle \bar{g} \rangle$ is finite.*

Proof. First, let us show that for a comeager set of $\bar{g} \in \Omega_{\bar{n}}(G)$, the orbit of any $x \in \mathbf{K}$ under $\langle \bar{g} \rangle$ is finite. Using the Baire category theorem, and the fact that for a fixed x the set $A_x = \{\bar{g} \in \Omega_{\bar{n}}(G) \colon \langle \bar{g} \rangle \cdot x \text{ is finite}\}$ is open, it is enough to show that for fixed $x \in \mathbf{K}$ the set A_x is dense. To see this, fix a nonempty open subset V of $\Omega_{\bar{n}}(G)$; without loss of generality, we may assume that there exist finite substructures $B_1, \dots, B_k \subset \mathbf{K}$ and automorphisms $g_1, \dots, g_k$ such that $g_i^{n_i} = 1$, $g_i(B_i) = B_i$ and

$$\forall \bar{h} \in \Omega_{\bar{n}}(G) \ (\forall i \ h_{i|B_i} = g_{i|B_i}) \Rightarrow \bar{h} \in V \ .$$

Set $B = \{x\} \cup \cup_{i=1}^k B_i$. Applying the isomorphic extension property to B, we may find a finite $C \subseteq \mathbf{K}$ containing B and automorphisms $h_1, \dots, h_k$ of C such that h_i coincides with g_i on B_i, and $h_i^{n_i} = 1$. Using the usual back-and-forth construction and the isomorphic extension property, we may extend each h_i to an automorphism of $\mathbf{K}$ such that $h_i^{n_i} = 1$. By construction, $\bar{h} = (h_1, \dots, h_k) \in V$ and $\langle \bar{h} \rangle \cdot x \subseteq C$ is finite. Now, we show that, for a generic $\bar{g} \in \Omega_{\bar{n}}(G)$, the following holds:

> For any finite $\langle \bar{g} \rangle$-invariant $B \subset \mathbf{K}$, any finite $C \in \mathcal{K}$ such that $B \leq C$ and any $\bar{h} \in \Omega_{\bar{n}}(C)$ fixing B setwise and coinciding with $\bar{g}$ on B , there exists $\widetilde{C} \subseteq \mathbf{K}$ such that $(\widetilde{C}, \bar{g}_{|\widetilde{C}}) \cong (C, \bar{h}_{|C})$.

To see this, let Σ denote the set of $\bar{g} \in \Omega_{\bar{n}}(G)$ satisfying the above condition. It is clear that Σ is G_δ; to see that it is dense, fix $B \leq C$ and $\bar{p} \in \Omega_{\bar{n}}(B)$ (the first part of the proof shows that the general case reduces to this one). Pick some nonempty open set O in $\Omega_{\bar{n}}(G)$. We may assume without loss of generality that there exists a finite $D \subset \mathbf{K}$ containing B and a tuple of partial isomorphisms $\bar{q} \in \Omega_{\bar{n}}(D)$ such that O consists of all tuples extending $\bar{q}$. We need to find $\bar{g}$ extending $\bar{q}$ and such that either some g_i does not coincide with p_i on B, or there exists $\widetilde{C} \subseteq \mathbf{K}$ such that $(\widetilde{C}, \bar{g}_{|\widetilde{C}}) \cong (C, \bar{h}_{|C})$. If some q_i does not coincide with p_i on $E = C \cap D$, we have nothing to prove. In the other case, E must be fixed by $q_1, \dots, q_n$, hence by $p_1, \dots, p_n$. Then, we may use the (IAP) and (IEP) to amalgamate D and C over E, obtaining a substructure $F \in \mathcal{K}$ and automorphisms $g_1, \dots, g_n$ of F with $g_i^{n_i} = 1$ for all i, g_i coinciding with q_i on D and with p_i on B. We may assume that

$D \subseteq F \subseteq \mathbf{K}$. Then we may extend each f_i to an element of $\mathrm{Aut}(\mathbf{K})$ still denoted by g_i and such that $g_i^{n_i} = 1$; this is the tuple we were looking for.

Thus, the set of elements $\bar{g}$ satisfying the following two conditions is dense G_δ in $\Omega_{\bar{n}}(G)$:

(i) The orbit of any $x \in \mathbf{K}$ under $\langle \bar{g} \rangle$ is finite.

(ii) For any finite $\langle \bar{g} \rangle$-invariant $B \subseteq \mathbf{K}$, any finite $C \in \mathcal{K}$ such that $B \leq C$ and any $\bar{h} \in \Omega_{\bar{n}}(C)$ fixing B setwise and coinciding with $\bar{g}$ on B , there exists $\widetilde{C} \subseteq \mathbf{K}$ such that $(\widetilde{C}, \bar{g}_{|\widetilde{C}}) \cong (C, \bar{h}_{|C})$.

Since any two tuples satisfying both conditions above are conjugate, we are done. □

Note that essentially the same proof as above would enable one to show that if $\mathcal{K}$ is a Fraïssé class with the (IAP) and (EP), then the automorphism group G of its limit has *ample generics*, i.e. there is a comeager diagonal conjugacy class in G^n for each integer n. This would also follow easily from the results of [14] or [19]. Below we will only use that, under the above conditions on $\mathcal{K}$, G has a dense conjugacy class.

Remark. In the case $\mathcal{K}$ is the class of Q-metric spaces for some metric value set Q, the conclusion of Proposition 4.9 follows from [24, Theorem 5], since a tuple of elements of finite order $n_1, \ldots, n_k$ is the same thing as a homomorphism from $\mathbf{Z}_{n_1} * \ldots * \mathbf{Z}_{n_k}$ to the isometry group of $\mathbf{U}_Q$, and $\mathbf{Z}_{n_1} * \ldots * \mathbf{Z}_{n_k}$ has property (RZ). Rosendal's proof also uses a version of the Herwig–Lascar–Solecki theorem.

From now on, we assume that, whenever $\mathcal{K}$ is a Fraïssé class with the (IAP), $A, B, C \in \mathcal{K}$ and $i \colon A \to B$, $j \colon A \to C$ are embeddings, the isomorphism type of the triple (D, α, β) produced by the (IAP) only depends on the isomorphism type of (A, B, C, i, j). For a class with the free amalgamation property, we choose D to be the free amalgam of B and C over A; for a class of metric spaces, we choose for D the metric amalgam as presented in Section 4.2. Slightly abusing notation, we will often call D the I-amalgam of B and C over A.

Definition 4.10. Let $\mathcal{K}$ be a Fraïssé class with the (IAP), and n an integer. We say that n *allows extensions* if whenever $B \in \mathcal{K}$, ϕ a partial isomorphism of B and $A \leq B$ are such that $\phi(A) = A$, $\phi_{|A}^n = 1$, and C is the I-amalgam of $\mathrm{dom}(\phi)$ and $\phi(\mathrm{dom}(\phi))$ over A, then there exists $D \in \mathcal{K}$ containing C and ϕ_D an automorphism of D extending ϕ and such that $\phi_D^n = 1$.

Lemma 4.11. *If $\mathcal{K}$ is a Fraïssé class with the free amalgamation property, or the class of finite Q-metric spaces for some metric value set Q, then any integer $n \geq 2$ allows extensions. If $\mathcal{K}$ is the class of all finite tournaments, then 3 allows extensions but 2 does not.*

Proof. We begin with the case when $\mathcal{K}$ has the free amalgamation property or is the class of finite $\mathbf{Q}$-metric spaces. Pick A, B, ϕ as in the definition of numbers allowing extensions. We may assume that B is the I-amalgam of $\mathrm{dom}(\phi)$ and $\phi(\mathrm{dom}(\phi))$ over A.

Enumerate B as $\{b_1, \dots, b_m\}$ with $A = \{b_1, \dots, b_p\}$. Fix $j \in \{0, \dots, n-1\}$. We define a structure $B^{(j)}$ with universe a m-element set $\{b_1^{(j)}, \dots, b_m^{(j)}\}$ as follows: if R is a q-ary relational symbol of the language of $\mathcal{K}$, and $(b_{k_1}^{(j)}, \dots, b_{k_q}^{(j)})$ is a q-tuple of elements of $B^{(j)}$, we first define $(x_1, \dots, x_q) \in B^q$ by setting

$$\forall i \leq q,\ x_i = \begin{cases} \phi^{-j}(b_{k_i}) & \text{if } k_i \leq p, \\ b_{k_i} & \text{otherwise.} \end{cases}$$

Then we set

$$B^{(j)} \models R(b_{k_1}^{(j)}, \dots, b_{k_q}^{(j)}) \Leftrightarrow B \models R(x_1, \dots, x_q) \ .$$

Then the map $\phi^{(j)} \colon B \mapsto B^{(j)}$ defined by

$$\phi^{(j)}(b_i) = \begin{cases} b_k^{(j)} & \text{if } i \leq p \text{ and } \phi^j(b_i) = b_k, \\ b_i^{(j)} & \text{otherwise} \end{cases}$$

is an isomorphism from B onto $B^{(j)}$, so each $B^{(j)}$ belongs to $\mathcal{K}$. Each $\phi^{(j)}$ naturally induces an embedding i_j of A into $B^{(j)}$. Using these identifications of A with a substructure of $B^{(j)}$, we may form the I-amalgam D of the $B^{(j)}$'s over A. It is straightforward to check that ϕ extends to an automorphism of D of order n.

For tournaments, we should first explain our amalgamation procedure: if B, C are tournaments with a common subtournament A, we amalgamate them by saying that any element of $B \setminus A$ loses to any element of $C \setminus A$. This amalgamation procedure satisfies the (IAP) and, using the same reasoning as above, one sees from Definition 4.10 that 3 allows extensions. The fact that 2 does not allow extensions is obvious, as the automorphism group of any tournament cannot contain an element of order 2. □

We are almost ready to state, and prove, the main result of this section. Before that, we need to recall a definition and a well-known lemma.

Definition 4.12. Let X, Y be Polish spaces, $f \colon X \to Y$ a continuous map and $x \in X$. Say that x is *locally dense for* f if, whenever U is a neighborhood of x, $\overline{f(U)}$ is a neighborhood of $f(x)$.

Lemma 4.13 ("Dougherty's lemma", see e.g. [20])**.** *Assume that X, Y are Polish spaces, $f \colon X \to Y$ is continuous and the set of points which are locally dense for f is dense. Then $f(X)$ is not meager.*

Proof. Assume $f(X)$ is meager, and let F_n be a countable family of closed subsets of Y with empty interior such that $f(X) \subseteq \cup_n F_n$. Then $X = \cup f^{-1}(F_n)$, so some $f^{-1}(F_n)$ must have nonempty interior, hence contain a point of local density for f. Then $\overline{f(f^{-1}(F_n))} \subseteq F_n$ has nonempty interior, a contradiction. □

Proposition 4.14. *Let $\mathcal{K}$ be a Fraïssé class with the (IAP) and (IEP), and n, m be two integers allowing extensions. Define $\pi\colon \Omega_{(n,m)}(G) \to G$ by $\pi(\sigma,\tau) = \sigma\tau$. Then any generic element of $\Omega_{(n,m)}(G)$ is locally dense for π.*

Proof. Let (σ,τ) be a generic element of $\Omega_{(n,m)}(G)$, and U be a neighborhood of (σ,τ). Using Proposition 4.9, we know that there exists a finite $A \subseteq \mathbf{K}$ which is $\langle\sigma,\tau\rangle$-invariant and such that

$$\forall(\phi,\psi) \in \Omega_{(n,m)}(G)\ (\phi_{|A} = \sigma_{|A},\ \psi_{|A} = \tau_{|A}) \Rightarrow (\phi,\psi) \in U\ .$$

We claim that

$$\overline{\pi(U)} \supseteq O = \{g \in G\colon g_{|A} = \sigma\tau_{|A}\}\ .$$

Proving this will yield the desired result. To see that it is true, let V be a nonempty open subset of O; we may assume without loss of generality that there exists a finite $B \subseteq \mathbf{K}$ containing A, and an automorphism g of $\mathbf{K}$ such that $g(B) = B$ and

$$\forall h \in G\ h_{|B} = g_{|B} \Rightarrow h \in V\ .$$

Let $B = \{b_1,\dots,b_M\}$ be enumerated in such a way that $A = \{b_1,\dots,b_p\}$ (with $p < M$ otherwise there is nothing to prove). As in the proof of Lemma 4.11, we pick an abstract M-point set $C = \{c_1,\dots,c_M\}$ and turn it into a $\mathcal{L}$-structure as follows. Given R a q-ary relational symbol of $\mathcal{L}$, and $(c_{k_1},\dots,c_{k_q})$ a q-tuple of elements of C, we first define $(x_1,\dots,x_q) \in B$ by setting

$$\forall i \le q\ x_i = \begin{cases} \tau^{-1}(b_{k_i}) & \text{if } k_i \le p,\\ b_{k_i} & \text{otherwise.}\end{cases}$$

Then we set

$$C \models R(c_{k_1},\dots,c_{k_q}) \Leftrightarrow B \models R(x_1,\dots,x_q)\ .$$

By construction, τ induces an isomorphism $\tilde\tau\colon B \to C$, defined by

$$\tilde\tau(b_i) = \begin{cases} c_j & \text{if } i \le p \text{ and } \tau(b_i) = b_j,\\ c_i & \text{otherwise.}\end{cases}$$

We may also use σ to define $\tilde\sigma\colon C \to B$ by setting

$$\tilde\sigma(c_i) = \begin{cases} \sigma(b_i) & \text{if } i \le p,\\ g(b_i) & \text{otherwise.}\end{cases}$$

It is straightforward to check from the definition (and the fact that $\sigma\tau = g$ on A) that $\tilde\sigma$ is an isomorphism from C to B, and $\tilde\sigma\tilde\tau = g$.

We may form the I-amalgam D of B, C over A, using the map $\tilde\tau_{|A}$ for our embedding from A to C. Then, $(D,\tilde\tau)$ fits our setup for numbers allowing extensions, so by assumption on m we can find E containing D such that $\tilde\tau$ extends to an isomorphism of $\Omega_m(E)$, still denoted by $\tilde\tau$. Applying the same reasoning to $(D, \tilde{\sigma}^{-1})$ and n, we find F containing D and such that $\tilde{\sigma}$ extends to an isomorphism in $\Omega_n(F)$.

Amalgamating E, F over D, and applying the (IEP) one last time, we finally find $H \in \mathcal{K}$ containing D and $(\tilde{\sigma}, \tilde{\tau})$ extending the original $(\tilde{\sigma}, \tilde{\tau})$ such that $\tilde{\sigma}^n = 1$, $\tilde{\tau}^m = 1$. We may assume that $H \leq \mathbf{K}$ and extend $\tilde{\sigma}$, $\tilde{\tau}$ to automorphisms of $\mathbf{K}$; by construction $(\tilde{\sigma}, \tilde{\tau}) \in \Omega_{n,m}(G)$. The construction ensures that $(\tilde{\sigma}, \tilde{\tau}) \in U$ and $\tilde{\sigma}\tilde{\tau}_{|B} = g_{|B}$, showing that $\tilde{\sigma}\tilde{\tau} \in V$, and we are done. $\square$

Theorem 4.15. *Let $\mathcal{K}$ be a Fraïssé class with the (IAP) and (IEP), and g_i be a generic element of order i. Then, for any quadruple $\bar{i}$ of integers each allowing extensions, and any $g \in G$, there exist $h_1, \dots, h_4$ such that each h_j is conjugate to g_{i_j} and $g = h_1, \dots, h_4$.*

Proof. Applying Proposition 4.14 (whose notations we reuse here), we know that $\pi(\Omega_{(n_1,n_2)}(G))$ is not meager. Since this set is analytic and conjugacy-invariant, and there is a dense conjugacy class in G, the $0-1$-topological law ([17, 8.46]) implies that $\pi(\Omega_{(n_1,n_2)}(G))$ is comeager, and an easy Baire category argument using the fact that points of local density are dense shows that for any comeager subset A of $\Omega_{(n_1,n_2)}(G)$ the set $\pi(A)$ is comeager. The same argument works for (n_3, n_4).

Now, let $B_{(n_1,n_2)}$ be the set of elements which are a product of a conjugate of g_1 and a conjugate of g_2, and $B_{(n_3,n_4)}$ be the set of elements which are a product of a conjugate of g_3 and a conjugate of g_4. Since each of those sets is comeager, we may apply Pettis' lemma ([17] 8.9) and obtain that $G = B_{(n_1,n_2)} \cdot B_{(n_3,n_4)}$. $\square$

Remark. When i is a single integer allowing extensions, this shows that every element of G is a product of four conjugates of g_i. Similar arguments would show, for instance, that in the automorphism group of the random tournament, there exists an element g of order 3 such that every other element is a product of four conjugates of g.

5. Applications to the isometry group of the Urysohn space.

We turn to applying our results to the isometry group of the Urysohn space $\mathbf{U}$.

Notation. We say that a metric value set Q is *divisible* if $q/n \in Q$ for all $q \in Q$ and all $n \in \mathbf{N}^*$. We recall that metric value sets are by definition countable; also, whenever Q is divisible, it must be dense in $[0, \sup(Q)]$.

We denote by $\mathbf{U}$ the (unbounded) Urysohn space and by $\mathbf{U}_R$ the Urysohn space of diameter R.

Note that if Q is divisible and unbounded, then the completion of $\mathbf{U}_Q$ is isometric to $\mathbf{U}$, while if Q is divisible and bounded with $\sup(Q) = R$ then the completion of $\mathbf{U}_Q$ is isometric to $\mathbf{U}_R$.

Lemma 5.1. *Let $g_1, \dots, g_n$ be a finite family of isometries of* $\mathbf{U}$*. Then there exists a divisible countable metric value set Q and an isometric copy X of $\mathbf{U}_Q$ which is dense in $\mathbf{U}$ and such that $g_i(X) = X$ for all $i \in \{1, \dots, n\}$.*

The same lemma, with obvious modifications, would also be true for $\mathbf{U}_1$ instead of $\mathbf{U}$; this lemma was proved, earlier and independently, by Tent and Ziegler [28]. Their proof used a model-theoretic argument based on the Löwenheim–Skolem theorem, which is essentially the same idea as the proof below.

Proof. We proceed by induction: we build an increasing sequence of countable dense subspaces (Y_i) of $\mathbf{U}$ and divisible metric value sets Q_i such that:

(i) For all i, Y_i is a Q_i-metric space and Y_i is $\langle \bar{g} \rangle$-invariant.

(ii) For any finite subset A of Y_i, and any one-point Q_i-metric extension $A \cup \{z\}$ of A, there exists $\tilde{z} \in Y_{i+1}$ such that $d(\tilde{z}, a) = d(z, a)$ for all $a \in A$.

To begin the construction, let X_0 be any countable dense subspace of $\mathbf{U}$, Y_0 be the countable set $\langle \bar{g} \rangle \cdot X_0$, and Q_0 the smallest divisible metric value set containing the values of the distance on Y_0 (this is indeed a countable set).

Now, assume that (Y_i, Q_i) has been built. Then, there are only countably many one-point Q_i-extensions of finite subsets of Y_i, and each of them is realized in $\mathbf{U}$, so we may add a countable set $\{x_j\}$ to Y_i so that all one-point Q_i-metric extensions of Y_i are realized in $X_{i+1} = Y_i \cup \{x_j\}$. Let $Y_{i+1} = \langle \bar{g} \rangle \cdot X_{i+1}$, and Q_{i+1} be the smallest divisible metric value set containing all the values of the distance on Y_{i+1}.

At the end of this construction, $Q = \bigcup Q_i$ is a countable, divisible metric value set, and $\bigcup Y_i$ is a Q-metric space which is $\langle \bar{g} \rangle$-invariant, isometric to $\mathbf{U}_Q$ and dense in $\mathbf{U}$. $\square$

Theorem 5.2. *Every element of* $\mathrm{Iso}(\mathbf{U})$ *is a commutator and a product of at most four elements of order n for all $n \geq 2$. The same result is true for* $\mathrm{Iso}(\mathbf{U}_1)$*.*

Proof. Pick $g \in \mathrm{Iso}(\mathbf{U})$, and apply Lemma 5.1 to find a dense, countable metric value set Q and a dense copy X of $\mathbf{U}_Q$ such that $g(X) = X$. Since $\mathrm{Iso}(\mathbf{U}_Q)$ has a comeager conjugacy class, every element of $\mathrm{Iso}(X)$ is a commutator; thus there exist $a, b \in \mathrm{Iso}(X)$ such that $g_{|X} = aba^{-1}b^{-1}$. These elements a, b uniquely extend to isometries of $\mathbf{U}$ (still denoted by a, b) and we obtain $g = aba^{-1}b^{-1}$.

The proof that every element of $\mathrm{Iso}(\mathbf{U})$ is a product of at most four elements of order n for all $n \geq 2$ follows similarly from Lemma 5.1 and Theorem 4.15.

It is clear that exactly the same argument works for $\mathbf{U}_1$. $\square$

Below, we explain how to obtain a stronger result: for any n there exists an element g of order n in $\mathrm{Iso}(\mathbf{U})$ such that every element of $\mathrm{Iso}(\mathbf{U})$ is a product of at most four conjugates of g (and the same fact holds for $\mathrm{Iso}(\mathbf{U}_1)$;

to avoid unecessary repetitions, we focus on the case Iso($\mathbf{U}$) below). For this, we first need to show that, as in the discrete case, there exists a generic element of order n in Iso($\mathbf{U}$). This might be a bit surprising, since Kechris proved that, as opposed to the discrete case, conjugacy classes are meager in Iso($\mathbf{U}$); this fact was published in Glasner and Weiss' paper [8].

Notation. In what follows we let G denote Iso($\mathbf{U}$) and, for any integer n, set

$$\Omega_n(G) = \{g \in G \colon g^n = 1\} \ .$$

Definition 5.3. Let $n \geq 2$ and $\sigma \in \Omega_n(G)$. We say that σ has the *n-approximate extension property* if for any $\epsilon > 0$, any finite σ-invariant subset $A = \{a_1, \dots, a_m\}$ of $\mathbf{U}$ and any (B, d_B, τ) such that

$$B = \{a_1, \dots, a_m, b, \tau(b), \dots, \tau^{n-1}(b)\}$$

is a metric space containing A, τ is an isometry coinciding with σ on A, and $\tau^n = 1$, there exists $\tilde{b} \in \mathbf{U}$ such that :

(i) $\forall i \in \{1, \dots, m\}\ \forall j \in \{0, \dots, n-1\}\ |d_B(a_i, \tau^j(b)) - d(a_i, \sigma^j(\tilde{b}))| \leq \epsilon$.
(ii) $\forall i, j \in \{0, \dots, n-1\}\ |d(\sigma^i(\tilde{b}), \sigma^j(\tilde{b})) - d(\tau^i(b), \tau^j(b))| \leq \epsilon$.

If the condition above is satisfied for $\epsilon = 0$, we say that σ has the *n-extension property.*

Note that, in the definition of the n-approximate extension property, we allow the possibility that $\tau^q(b) = b$ for some strict divisor q of n; in that case $(\{\tau(b), \dots, \tau^{n-1}(b)\}, d_B)$ is a pseudometric space rather than a metric space, which is why we will have to manipulate pseudometrics below.

We turn to proving that the n-approximate extension property and the n-extension property are equivalent; as a first step for that proof, we need the following lemma. In the case where A below is empty, this lemma is due to Uspenskij [31, Proposition 7.1]; the proof we give is essentially the same as Uspenskij's. The lemma below was also known for a long time to Henson, who never published it, and the second author heard it from him.

Lemma 5.4. *Let $A = \{a_1, \dots, a_n\}$ be an enumerated finite metric space. Let $B = A \cup \{b_1 \dots, b_p\}$ and $C = A \cup \{c_1, \dots, c_p\}$ be two enumerated finite pseudometric spaces containing A, and $\epsilon > 0$ be such that*

(i) $\forall i, j \in \{1, \dots, p\}\ \ |d_B(b_i, b_j) - d_C(c_i, c_j)| \leq 2\epsilon$.
(ii) $\forall i \in \{1, \dots, n\}\ \forall j \in \{1, \dots, p\}\ \ |d_B(a_i, b_j) - d_C(a_i, c_j)| \leq \epsilon$.

Then there exists a pseudometric ρ on $X = B \cup C$ such that:

(1) *ρ extends d_B, d_C.*
(2) $\forall i \in \{1, \dots, p\}\ \rho(b_i, c_i) \leq \epsilon$.
(3) *If $f \colon A \to A$ is an isometry, and π is a permutation of $\{1, \dots, p\}$ such that setting $f(b_i) = b_{\pi(i)}$ extends f to an isometry of (B, d_B), and setting $f(c_i) = c_{\pi(i)}$ extends f to an isometry of C, then setting $f(b_i) = b_{\pi(i)}$ and $f(c_i) = c_{\pi(i)}$ extends f to an isometry of (X, ρ).*

This pseudometric is such that $\rho(b, c) > 0$ for all $b \in B \setminus A$, $c \in C \setminus A$.

Proof. We use the same idea as in [31]. First, define a partial function ω on X by the following conditions:

- $\forall b, b' \in B\ \omega(b, b') = d_B(b, b')$.
- $\forall c, c' \in C\ \omega(c, c') = d_B(c, c')$.
- $\forall i \in \{1, \dots, p\}\ \omega(b_i, c_i) = \epsilon$.

Note that, if a, a' belong to A then the first two conditions both give $\omega(a, a') = d(a, a')$. If $x, y \in X$, we say that a finite sequence $x_0, \dots, x_k$ of elements of X is a *path from x to y* if $x_0 = x$, $y = x_k$, and $\omega(x_i, x_{i+1})$ is defined for all $i \in \{0, \dots, k-1\}$. Then we set

$$\rho(x, y) = \inf\left\{\sum_{i=0}^{k-1} \omega(x_i, x_{i+1})\colon (x_0, \dots, x_k) \text{ is a path from } x \text{ to } y\right\}.$$

It is clear that ρ is a pseudometric, and conditions (2) and (3) follow immediately from the definition. The only fact that remains to be checked is (1); this is straightforward but a bit tedious, reducing to cases. □

Lemma 5.5. *Let $n \geq 2$ and $\sigma \in \Omega_n(G)$. Then σ has the n-extension property if, and only if, it has the n-approximate extension property.*

Proof. Fix $A = \{a_1, \dots, a_m\}$, (B, d_B, τ) as in the definition of the n-approximate extension property. Using the completeness of $\mathbf{U}$, it is enough to build a sequence (c_q) of elements of $\mathbf{U}$ such that:

(1) $\forall q,\ d(c_q, c_{q+1}) \leq 2^{1-q}$.
(2) $\forall q,\ \forall i \in \{1, \dots, m\},\ \forall j \in \{0, \dots, n-1\}$,
$$\left|d(\sigma^j(c_q), a_i) - d_B(\tau^j(b), a_i)\right| \leq 2^{-q}.$$
(3) $\forall q,\ \forall i, j \in \{0, \dots, n-1\}$,
$$\left|d(\sigma^i(c_q), \sigma^j(c_q)) - d_B(\tau^i(b), \tau^j(b))\right| \leq 2^{-q}.$$

Use the approximate extension property of σ to define c_0, then assume that c_q has been defined. Let $C_q = A \cup \{c_q, \dots, \sigma^{n-1}(c_q)\}$. Using Lemma 5.4 applied to the extensions B, C_q of A, one may find a metric ρ on $X = B \cup C_q$ extending the original metrics, such that $\rho(b, c_q) \leq 2^{-q}$ and the map $f\colon X \to X$ defined by

$$f(x) = \begin{cases} \sigma(x) & \text{if } x \in A, \\ \tau^{j+1}(b) & \text{if } x = \tau^j(b), \\ \sigma^{j+1}(c_q) & \text{if } x = \sigma^j(c_q) \end{cases}$$

is an isometry of X. We may see (X, f) as an abstract extension of C_q, and apply the approximate extension property of σ with $\epsilon = 2^{-q-1}$ to find c_{q+1}. All the desired properties are straightforward to check from the definitions and the triangle inequality. □

There remains one last piece of bookkeeping ahead of us.

Lemma 5.6. *Let* $\mathbf{Q}$ *be a divisible metric value set, and see* $\mathbf{U}_Q$ *as a dense subset of* $\mathbf{U}$. *Let* $n \geq 2$ *be an integer and* σ *be a generic element of* $\Omega_n(\mathrm{Iso}(\mathbf{U}_Q))$. *Then the extension of* σ *to an element of* $\Omega_n(G)$ *has the* n*-almost extension property (hence also the* n*-extension property).*

Proof. We still denote by σ the extension of σ to $\mathbf{U}$. To see that it has the n-approximate extension property, fix a finite, σ-invariant set $A = \{a_1, \dots, a_m\}$, a finite enumerated pseudometric space $B = (\{a_1, \dots, a_m\} \cup \{b, \dots, \tau^{n-1}(b)\}, d_B)$ containing A, an isometry τ of B that extends $\sigma_{|A}$ and such that $\tau^n = 1$, and $\epsilon > 0$. We may assume that $\epsilon < \mathrm{diam}(B)$. To simplify notation, we write $b_i = \tau^i(b)$ for $i = 0, \dots, n-1$. Our proof is in three steps.

First step: Let ρ denote the metric on B defined by:

- $\forall a, a' \in A,\ \rho(a, a') = d(a, a')$.
- $\forall b \in B \setminus A,\ \forall a \in A,\ \rho(a, b) = \mathrm{diam}(B)$.
- $\forall b \neq b' \in B,\ \rho(b, b') = \mathrm{diam}(B)$.

It is easy to check that ρ is a metric on B, and that τ is an isometry of (B, ρ). Define $\tilde{d}$ on B by setting, for all $b, b' \in B$,

$$\tilde{d}(b, b') = \Big(1 - \frac{\epsilon}{\mathrm{diam}(B)}\Big) d_B(b, b') + \frac{\epsilon}{\mathrm{diam}(B)} \rho(b, b') \ .$$

Then $\tilde{d}$ is a metric on B, coinciding with d on A, and we have:

$$\forall i, j \in \{0, \dots, n-1\},\ |\tilde{d}(b_i, b_j) - d_B(b_i, b_j)| \leq 2\epsilon \ . \tag{5.1}$$

$$\forall i \in \{1, \dots, m\},\ \forall j \in \{1, \dots, p\},\ |\tilde{d}(a_i, b_j) - d_B(a_i, b_j)| \leq 2\epsilon \ . \tag{5.2}$$

What we have gained by introducing this new metric is that now there exists some $\delta > 0$ such that, for any triple $\{x, y, z\}$ of distinct elements of B not all contained in A, one has

$$\tilde{d}(x, y) + \tilde{d}(y, z) \geq \tilde{d}(x, z) + 3\delta \ . \tag{5.3}$$

We may, and do, assume that $\delta < \epsilon$ and fix such a δ for the remainder of the proof. Now we approximate our original metric space by a Q-metric space in order to use the genericity of σ in $\Omega_n(G)$ to prove that σ has the n-almost extension property.

Second step: Pick $a'_1, \dots, a'_m \in \mathbf{U}_Q$ such that $d(a_i, a'_i) \leq \delta$. Define

$$A' = \{\sigma^k(a'_i) \colon k \in \{0, \dots, n-1\},\ i \in \{1, \dots, m\}\} \ .$$

We may extend $\tilde{d}$ to a metric on $X = B \cup A'$, still denoted by $\tilde{d}$, and extending d on $A \cup A'$ by setting, for all $b \in B, a' \in A'$:

$$\tilde{d}(b, a') = \min\{\tilde{d}(b, a) + d(a, a') \colon a \in A\} \ .$$

For any triple $\{x, y, z\}$ of distinct elements of X not contained in $A \cup A'$, (5.3) ensures that we still have

$$d(x, y) + d(y, z) \geq d(x, z) + 3\delta \ . \tag{5.4}$$

The metric $\tilde{d}$ may not have be Q-valued. We modify it as follows: let $r_1, \ldots, r_j$ denote the values of $\tilde{d}$ which do not belong to Q. For any $i \in \{1, \ldots, j\}$, find $q_i \in Q$ such that $|q_i - r_i| \le \delta$.

Then define a new metric D on $Y = \{b_0, \ldots, b_{n-1}\} \cup A'$ by setting $D(x,x') = q_i$ whenever $\tilde{d}(x,x') = r_i$, $D(x,x') = \tilde{d}(x,x')$ otherwise. The values taken by D belong to Q, and we claim that it is indeed a metric. To see this, pick a triple $\{x,y,z\}$ of distinct elements of Y. If $\{x,y,z\} \subseteq A'$ then D coincides with $\tilde{d}$ on $\{x,y,z\}$ so the triangle inequality holds. Otherwise, we have:

$$\begin{aligned} D(x,y) + D(y,z) &\ge \tilde{d}(x,y) + \tilde{d}(y,z) - 2\delta \\ &\ge \tilde{d}(x,z) + 3\delta - 2\delta \qquad \text{(by (5.4))} \\ &\ge \tilde{d}(x,z) + \delta \\ &\ge D(x,z) \ . \end{aligned}$$

Also, for any $i,j \in \{0, \ldots, n-1\}$ we have

$$|D(b_i,b_j) - d_B(b_i,b_j)| \le \delta + |\tilde{d}(b_i,b_j) - d_B(b_i,b_j)| \le \delta + 2\epsilon \le 3\epsilon \ .$$

Define a map ϕ on Y by setting $\phi(b_i) = \tau(b_i)$ for $i \in \{0, \ldots, n-1\}$, $\phi(a') = \sigma(a')$ for $a' \in A'$. We have $\phi^n = 1$; of course, ϕ does not need to be an isometry, and this is what we have to take care of in the last step of the proof.

Third step: For $x,y \in Y$, we set

$$\widetilde{D}(x,y) = \frac{1}{n} \sum_{k=0}^{n-1} D(\phi^k(x), \phi^k(y)) \ .$$

Then $\widetilde{D}$ is a metric on Y, taking its values in Q because Q is divisible, and ϕ is an isometry of $(Y, \widetilde{D})$, extending $\sigma_{|A'}$. Also, $\widetilde{D}(b_i,b_j) = D(b_i,b_j)$ for all $i,j \in \{0, \ldots, n-1\}$. Using the fact that σ is a generic element of $\Omega_n(G)$, we can find $c \in \mathbf{U}_Q$ such that

$$\big(A' \cup \{c, \sigma(c), \ldots, \sigma^{n-1}(c)\}, \sigma, d\big) \cong \big(A' \cup \{b, \tau(b), \ldots, \tau^{n-1}(b)\}, \phi, \tilde{D}\big) \ .$$

We already know that for all $i,j \in \{0, \ldots, n-1\}$,

$$|d(\sigma^i(c), \sigma^j(c)) - d_B(\tau^i(b), \tau^j(b))| = |D(b_i,b_j) - d_B(b_i,b_j)| \le 3\epsilon \ .$$

Letting $c_i = \sigma^i(c)$, the only thing that remains to be checked in order to prove that σ has the n-almost extension property is whether $|d(c_i,a_j) - d_B(b_i,a_j)|$ is small for all $i \in \{0, \ldots, n-1\}, j \in \{1, \ldots, m\}$. Since $d(c_i,a'_j) = \tilde{D}(b_i,a'_j)$, we have that

$$|d(c_i,a'_j) - d_B(b_i,a_j)| \le \frac{1}{n} \sum_{k=0}^{n-1} \big| D(\tau^k(b_i), \sigma^k(a_j) - d_B(b_i,a_j) \big| \ . \tag{5.5}$$

By definition of D, for all $k \in \{0, \dots, n-1\}$,

$$|D(\tau^k(b_i), \sigma^k(a'_j)) - \tilde{d}(\tau^k(b_i), \sigma^k(a'_j))| \leq \delta \ .$$

Since $\tilde{d}$ was a metric on $B \cup A \cup A'$ coinciding with d on $A \cup A'$, we have, for all $k \in \{0, \dots, n-1\}$,

$$\big|\tilde{d}(\tau^k(b_i), \sigma^k(a'_j) - \tilde{d}(\tau^k(b_i), \sigma^k(a_j)\big| \leq \tilde{d}\big(\sigma^k(a'_j), \sigma^k(a_j)\big) = d(a_j, a'_j) \leq \delta \ .$$

Using these inequalities, and the fact that τ is a $\tilde{d}$-isometry extending σ, it follows from (5.1) and (5.5) that, for all $i \in \{0, \dots, n-1\}$ and all $j \in \{1, \dots, m\}$,

$$|d(c_i, a'_j) - d_B(b_i, a_j)| \leq \frac{1}{n} \sum_{k=0}^{n-1} \big(2\delta + |\tilde{d}(b_i, a_j) - d(b_i, a_j)|\big) \leq 2\epsilon + 2\delta \leq 4\epsilon \ .$$

Finally, since $d(a'_j, a_j) \leq \delta \leq \epsilon$, we obtain $|d(c_i, a_j) - d_B(b_i, a_j)| \leq 5\epsilon$ and, since ϵ was arbitrary, this concludes the proof. □

Essentially the same reasoning as above enables one to check that the n-approximate extension property is equivalent to the following condition:

For any $\epsilon > 0$, any finite σ-invariant subset $A = \{a_1, \dots, a_m\}$ of $\mathbf{U_Q}$ and any (B, d_B, τ) such that $B = \{a_1, \dots, a_m, b, \tau(b), \dots, \tau^{n-1}(b)\}$ is a $\mathbf{Q}$-metric space containing A, τ coincides with σ on A, and $\tau^n = 1$, there exists $\tilde{b} \in \mathbf{U}$ such that :

(i) $\forall i \in \{1, \dots, m\}\ \forall j \in \{0, \dots, n-1\}\ |d_B(a_i, \tau^j(b)) - d(a_i, \sigma^j(\tilde{b}))| < \epsilon$.
(ii) $\forall i, j \in \{0, \dots, n-1\}\ |d(\sigma^i(\tilde{b}), \sigma^j(\tilde{b})) - d(\tau^i(b), \tau^j(b))| < \epsilon$.

Thus, the set of $\sigma \in \Omega_n(G)$ having the n-approximate extension property is G_δ. It is easy to check from Lemmas 5.1 and 5.6 that this set is dense in $\Omega_n(G)$. Since any two elements of $\Omega_n(G)$ with the n-approximate extension property are conjugate (because they actually have the n-extension property), we have finally obtained the following result.

Theorem 5.7. *For any integer n there exists an element g_n whose conjugacy class is comeager in $\{g \in \mathrm{Iso}(\mathbf{U})\colon g^n = 1\}$. Any $g \in \mathrm{Iso}(\mathbf{U})$ is a product of four conjugates of g_n.*

Proof. We have already explained why the first sentence is true. The second one follows as in the proof of Theorem 5.2 from the fact that whenever Q is a countable, divisible metric value set and $\mathbf{U}_Q$ is densely embedded in $\mathbf{U}$, the extension to $\mathbf{U}$ of a generic element of $\mathrm{Iso}(\mathbf{U}_Q)$ is a generic element of $\Omega_n(\mathrm{Iso}(\mathbf{U}))$ (because it has the n-extension property). □

We conclude the paper by discussing an open problem and a partial answer. It is known that there are elements of $\mathrm{Iso}(\mathbf{U})$ without roots of order n for any $n \geq 2$. However it is unknown whether a generic element must have roots of any order (or even, square roots). Note that by the $0-1$ topological law, the existence of dense conjugacy classes implies that the set of elements admitting a n-th root is either meager or comeager for all n.

It is also unknown whether a stronger condition holds, namely whether a generic element g may be embedded in a *flow*, i.e. whether there exists a continuous homomorphism F from $(\mathbf{R},+)$ to $\mathrm{Iso}(\mathbf{U})$ such that $g = F(1)$. We can answer this question in the particular case of generic elements of order n, and obtain a stronger result than expected in the bounded case.

Definition 5.8. Let X be a bounded metric space. We let d_u denote the uniform metric on $\mathrm{Iso}(X)$, defined by

$$d_u(g,h) = \sup\{d(g(x),h(x)) \colon x \in X\} \ .$$

The uniform metric d_u is bi-invariant, complete, and is not separable in general.

Lemma 5.9. *Let $n \geq 2$ be an integer, X a compact metric space of diameter at most 1, and Y a dense countable $\mathbf{Q}$-metric subspace of X. Assume that $F = (F_t)$ is a flow of isometries of X such that $F_1 = \sigma$ fixes Y, $\sigma^n = 1$, and $d_u(F_t, F_s) \leq n|t-s|$ for all $t, s \in \mathbf{R}$.*

Let $Z = Y \cup A$ be a $\mathbf{Q}$-metric space containing Y with A finite, and τ an isometry of Z extending $\sigma_{|Y}$ such that $\tau^n = 1$. Then there exists a compact metric space $\widetilde{X}$ containing X of diameter at most 1, a dense countable $\mathbf{Q}$-metric subspace $\widetilde{Y}$ of $\widetilde{X}$ containing $Y \cup A$, and a flow $G = (G_t)$ of isometries of $\widetilde{X}$, such that:

(i) $G_{|X} = F$.
(ii) $G_{1|A} = \tau$ *and* $G_1(\widetilde{Y}) = \widetilde{Y}$.
(iii) $\forall t, s \in \mathbf{R}\ d_u(G_t, G_s) \leq n|t-s|$.

Proof. By induction, we may assume that $A = \{a, \tau(a), \dots, \tau^{n-1}(a)\}$ for some $a \in A$. For $t, s \in \mathbf{R}$ we set $\delta(t,s) = n|t-s|$; for $t \in \mathbf{R}$ and $x \in X$ we denote $t \cdot x = F_t(x)$. Our assumptions imply that we have, for all $t \in \mathbf{R}$:

$$\delta(t,0) \geq \sup_{x \in X} d(t \cdot x, x) \ . \tag{5.6}$$

We define a map ω on $(\mathbf{R} \cup X)^2$ by the following conditions:

- For all $t, s \in \mathbf{R}$, $\omega(t,s) = \min\{d(a, \tau^i(a)) + \delta(t+i, s)\}$.
- For all $t \in \mathbf{R}, x \in X$, $\omega(t,x) = d(a, (-t) \cdot x)$.
- For all $x, y \in X$, $\omega(x,y) = d(x,y)$.

We let ρ be the pseudometric on $\mathbf{R} \cup X$ associated to ω, i.e.

$$\forall b, c \in \mathbf{R} \cup X\ \ \rho(b,c) = \inf\left\{\sum_{i=0}^{n} \omega(x_i, x_{i+1}) \colon x_0 = b, x_{n+1} = c\right\} \ .$$

Then one may check the following facts (the verifications, if done in the same order as below, are completely straightforward so we omit them):

(i) $\omega_{|\mathbf{R}^2}$ is an invariant pseudometric.

(ii) For any triple x, y, z with at least two of its elements in X one has $\omega(x,y) + \omega(y,z) \geq \omega(x,z)$.
(iii) For any $s, t \in \mathbf{R}$, for any $x \in X$, $\omega(t,s) + \omega(s,x) \geq \omega(t,x)$.
(iv) On X, ρ and d coincide.
(v) For all $i \in \mathbf{Z}$, $\rho(0,i) = \omega(0,i) = d(a, \tau^i(a))$.
(vi) The infimum in the definition of ρ is actually a minimum, so $\rho(a,b) \in \mathbf{Q}$ for all $a, b \in \mathbf{Q} \cup Y$.
(vii) For all $t, s \in \mathbf{R}$, $\rho(t,s) \leq n|t-s|$.

Once all these things are checked, we are essentially done: let $\widetilde{X}$ be the metric space obtained by identifying points b, c such that $\rho(b,c) = 0$. Then $\widetilde{X}$ is naturally isometric to $X \cup (\mathbf{R}/n\mathbf{Z}, \rho)$, with ρ coinciding with d on X; $\mathbf{R}$ acts isometrically on $\widetilde{X}$, extending the action given by F, the action on $\mathbf{R}/n\mathbf{Z}$ being the quotient of the translation action of $(\mathbf{R},+)$ on itself. We may identify $Y \cup A$ with $Y \cup \{0, \dots, n-1\}$, and under this identification we have $j \cdot a = \tau^j(a)$ for all $j \in \mathbf{Z}$. Define $\widetilde{Y}$ as $Y \cup \mathbf{Q}/n\mathbf{Z}$, which is a dense $\mathbf{Q}$-metric subspace of $\widetilde{X}$ on which ρ takes only rational values. Replacing ρ by $\min(\rho, 1)$, completes the proof. □

Theorem 5.10. *Let n be an integer. Then a generic element of $\Omega_n(\mathrm{Iso}(\mathbf{U}))$ embeds in a flow. A generic element of $\Omega_n(\mathrm{Iso}(\mathbf{U}_1))$ embeds in a flow which is n-Lipschitz from $(\mathbf{R},+)$ to $(\mathrm{Iso}(\mathbf{U}_1), d_u)$.*

Before the proof, we need to introduce some notation.

Notation. Let Y be a countable $\mathbf{Q}$-metric space of diameter at most 1, n be an integer and σ an element of $\mathrm{Iso}(Y)$ such that $\sigma^n = 1$. For any finite, σ-invariant subset $A \subseteq Y$, any finite metric space B containing A and any isometry τ of B coinciding with σ on A and such that $\tau^n = 1$, one may form the metric amalgam (for spaces of diameter at most 1) Z of Y and B over A and then extend τ to an isometry of Z.

If (Z, τ) has been obtained by the procedure above, we say that (Z, τ) is a *finite extension of* (Y, σ) *attached to* $(A, B, \tau_{|B})$; we denote by $E(Y, \sigma)$ the countable set of all (isomorphism types of) (Z, τ) which can be obtained by this procedure.

Proof of Theorem 5.10. We do not give the proof for $\mathrm{Iso}(\mathbf{U})$, which is based on an easy modification of Lemma 5.9 and the construction below. Fix a bijection $f \colon \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ such that $f(p,q) > p$ for all (p,q).

Then, using Lemma 5.9, we may build a sequence of compact metric spaces (X_m), with dense $\mathbf{Q}$-metric subspaces Y_m, and a sequence of isometric flows F_m such that:

(i) For all m, $F_m(1) = \sigma_m$ is such that $\sigma_m^n = 1$, $\sigma_m(Y_m) = Y_m$ and σ_m has only finite orbits.
(ii) For all m, for all $x \in X_m$, for all $t \in \mathbf{R}$, $d(F_m(t)(x), x) \leq n|t|$.
(iii) For all m, X_{m+1} contains X_m, Y_{m+1} contains Y_m, and F_{m+1} extends F_m.

(iv) For all m, let $\{(Z_{m,i},\tau_{m,i})\}_{i\in\mathbf{N}}$ be an enumeration of $E(Y_m,\sigma_m)$, and for all i let $A_{m,i}$, $B_{m,i}$, $\tau_{m,i}$ be such that $(Z_{m,i},\tau_{m,i})$ is a finite extension of (Y_m,σ) attached to $(A_{m,i},B_{m,i},\tau_{m,i})$. Then if $m=f(p,q)$, there exists $B\subseteq X_{m+1}$ containing $A_{p,q}$ such that $(B,A_{p,q},\sigma_m)\cong(B_{p,q},A_{p,q},\tau_{p,q})$.

Let $X_\infty=\cup X_m$, $Y_\infty=\cup Y_m$, F_∞ denote the flow on X_∞ produced by the above construction, and $\sigma_\infty=F_\infty(1)$. Then the construction (the last condition in particular) implies that Y_∞ is isometric to $\mathbf{U}_{Q_1}$ and σ_∞ is a generic element of $\mathrm{Iso}(\mathbf{U}_{Q_1})$. The completion of X_∞ contains Y_∞ as a dense subspace, so it is isometric to the completion of $\mathbf{U}_{Q_1}$, i.e. to $\mathbf{U}_1$. Thus σ_∞ extends to a generic element of $\mathrm{Iso}(\mathbf{U}_1)$ and the flow F_∞ extends to a flow such that $F_\infty(1)=\sigma_\infty$, $d_u(F_\infty(t),1)\le n|t|$ for all $t\in\mathbf{R}$. □

Corollary. $(\mathrm{Iso}(\mathbf{U}_1),d_u)$ *is path-connected.*

Proof. Let σ be a generic element in $\Omega_2(\mathrm{Iso}(\mathbf{U}_1))$, and $g\in G$. We know that there is a 2-Lipschitz flow $F\colon(\mathbf{R},+)\to(\mathrm{Iso}(\mathbf{U}_1),d_u)$ such that $\sigma=F(1)$, and elements $k_1,\dots,k_4$ such that $g=k_1\sigma k_1^{-1}\cdots k_4\sigma k_4^{-1}$. Define $\phi\colon[0,1]\to\mathrm{Iso}(\mathbf{U}_1)$ by

$$\phi(t)=k_1F(t)k_1^{-1}\cdots k_4F(t)k_4^{-1}\ .$$

Then $\phi(0)=id_{\mathbf{U}}$, $\phi(1)=g$, and ϕ is continuous from $[0,1]$ to $(\mathrm{Iso}(\mathbf{U}_1),d_u)$ (actually, $d_u(\phi(t),\phi(s))\le 8|t-s|$ for all t,s). □

To close the paper, we note that the analogue of Corollary 5.11 for $(\mathrm{Iso}(\mathbf{U}),d_u)$ is false, since for a generic $g\in\mathrm{Iso}(\mathbf{U})$ one has $d_u(g,1)=+\infty$, hence g is not in the path-connected component of 1. Similarly, the generic element g_n of order n in $\mathrm{Iso}(\mathbf{U})$ is such that $d_u(g_n,1)=+\infty$, so one cannot improve the statement of Theorem 5.10 to obtain d_u-continuity in the case of $\mathrm{Iso}(\mathbf{U})$.

Also, note that Corollary 5.11 is an immediate corollary of the fact that the isometry group of $\mathbf{U}_1$ is simple, a fact which was announced by Tent–Ziegler [27] after this paper was submitted. Their proof is model-theoretic, based on ideas from stability theory, and very different from what our techniques in this paper, even though in both cases the Baire category theorem plays a major role.

Acknowledgements

Several conversations with Alexander Berenstein on the topic of this article were very helpful to us, and this paper benefited from a careful reading by the anonymous referee, to whom we are grateful. The work presented here is part of the first author's Ph.D thesis [2], which was supervised by Éric Jaligot and the second author; Éric sadly passed away around the time the final version of this article was completed. He was a dear colleague and friend, and we dedicate this article to his memory.

References

1. Howard Becker and Alexander S. Kechris, *The descriptive set theory of Polish group actions*, London Mathematical Society Lecture Note Series, vol. 232, Cambridge University Press, Cambridge, 1996. MR MR1425877 (98d:54068)
2. Doğan Bilge, *Automorphism groups of homogeneous structures*, Ph.D. thesis, Université Claude Bernard Lyon 1, 2012, Available at `http://math.univ-lyon1.fr/~melleray/DoganThesis.pdf`.
3. Samuel Coskey, Paul Ellis, and Scott Schneider, *The conjugacy problem for the automorphism group of the random graph*, Arch. Math. Logic **50** (2011), no. 1-2, 215–221. MR 2765643
4. Roland Fraïssé, *Sur l'extension aux relations de quelques propriétés des ordres*, Ann. Sci. Ecole Norm. Sup. (3) **71** (1954), 363–388. MR MR0069239 (16,1006b)
5. Harvey Friedman and Lee Stanley, *A Borel reducibility theory for classes of countable structures*, J. Symbolic Logic **54** (1989), no. 3, 894–914. MR MR1011177 (91f:03062)
6. Su Gao, *Invariant descriptive set theory*, Pure and Applied Mathematics (Boca Raton), vol. 293, CRC Press, Boca Raton, FL, 2009. MR 2455198 (2011b:03001)
7. Eli Glasner and Jonathan L. King, *A zero-one law for dynamical properties*, Topological dynamics and applications (Minneapolis, MN, 1995), Contemp. Math., vol. 215, Amer. Math. Soc., Providence, RI, 1998, pp. 231–242. MR MR1603201 (99d:28039)
8. Eli Glasner and Benjamin Weiss, *Topological groups with Rokhlin properties*, Colloq. Math. **110** (2008), no. 1, 51–80. MR 2353899 (2008k:37006)
9. C. Ward Henson, *A family of countable homogeneous graphs*, Pacific J. Math. **38** (1971), 69–83. MR 0304242 (46 #3377)
10. Bernhard Herwig and Daniel Lascar, *Extending partial automorphisms and the profinite topology on free groups*, Trans. Amer. Math. Soc. **352** (2000), no. 5, 1985–2021. MR MR1621745 (2000j:20036)
11. Greg Hjorth, *Classification and orbit equivalence relations*, Mathematical Surveys and Monographs, vol. 75, American Mathematical Society, Providence, RI, 2000. MR 1725642 (2000k:03097)
12. Wilfrid Hodges, *Model theory*, Encyclopedia of Mathematics and its Applications, vol. 42, Cambridge University Press, Cambridge, 1993. MR MR1221741 (94e:03002)
13. Wilfrid Hodges, Ian Hodkinson, Daniel Lascar, and Saharon Shelah, *The small index property for ω-stable ω-categorical structures and for the random graph*, J. London Math. Soc. (2) **48** (1993), no. 2, 204–218. MR MR1231710 (94d:03063)
14. A. A. Ivanov, *Generic expansions of ω-categorical structures and semantics of generalized quantifiers*, J. Symbolic Logic **64** (1999), no. 2, 775–789. MR 1777786 (2001i:03081)
15. Eric Jaligot, *On stabilizers of some moieties of the random tournament*, Combinatorica **27** (2007), no. 1, 129–133. MR 2310790 (2008a:05118)
16. Miroslav Katětov, *On universal metric spaces*, General topology and its relations to modern analysis and algebra, VI (Prague, 1986), Res. Exp. Math., vol. 16, Heldermann, Berlin, 1988, pp. 323–330. MR 952617 (89k:54066)
17. Alexander S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995. MR MR1321597 (96e:03057)
18. ______, *Actions of Polish groups and classification problems*, Analysis and logic (Mons, 1997), London Math. Soc. Lecture Note Ser., vol. 262, Cambridge Univ. Press, Cambridge, 2002, pp. 115–187. MR MR1967835 (2004b:03070)
19. Alexander S. Kechris and Christian Rosendal, *Turbulence, amalgamation and generic automorphisms of homogeneous structures*, Proc. Lond. Math. Soc. **94** (2007), no. 2, 302–350.

20. Jonathan King, *The commutant is the weak closure of the powers, for rank-1 transformations*, Ergodic Theory Dynam. Systems **6** (1986), no. 3, 363–384. MR 863200 (88a:28021)
21. D. MacPherson and K. Tent, *Simplicity of some automorphism groups*, Journal of Algebra **342** (2011), no. 1, 40–52.
22. Dugald Macpherson, *A survey of homogeneous structures*, Discrete Math. **311** (2011), no. 15, 1599–1634. MR 2800979
23. Julien Melleray, *On the geometry of Urysohn's universal metric space*, Topology Appl. **154** (2007), no. 2, 384–403. MR 2278687 (2007h:54033)
24. Christian Rosendal, *Finitely approximable groups and actions part II: Generic representations*, Journal of Symbolic Logic, to appear, 2010.
25. Sławomir Solecki, *Extending partial isometries*, Israel J. Math. **150** (2005), 315–331. MR MR2255813 (2007h:54034)
26. ——, *Notes on a strengthening of the Herwig–Lascar extension theorem*, Unpublished note, available at `http://www.math.uiuc.edu/~ssolecki/papers/HervLascfin.pdf`, 2009.
27. Katrin Tent and Martin Ziegler, *The isometry group of the bounded Urysohn space is simple*, Bulletin of the London Mathematical Society (2013).
28. ——, *On the isometry group of the Urysohn space*, 2013. MR 3022717
29. Pavel Urysohn, *Sur un espace métrique universel.*, C. R. **180** (1925), 803–806 (French).
30. Vladimir V. Uspenskij, *On the group of isometries of the Urysohn universal metric space*, Comment. Math. Univ. Carolin. **31** (1990), no. 1, 181–182. MR 1056185 (91f:22003)
31. ——, *On subgroups of minimal topological groups*, Topology Appl. **155** (2008), no. 14, 1580–1606. MR MR2435151 (2009d:22003)

Université de Lyon
CNRS UMR 5208
Université Lyon 1
Institut Camille Jordan
43 blvd. du 11 novembre 1918
F-69622 Villeurbanne Cedex
France
E-mail address: dobilge@ku.edu.tr

Université de Lyon
CNRS UMR 5208
Université Lyon 1
Institut Camille Jordan
43 blvd. du 11 novembre 1918
F-69622 Villeurbanne Cedex
France
E-mail address: melleray@math.univ-lyon1.fr