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GRAPHS EMBEDDED INTO FINITE PROJECTIVE PLANES

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Abstract. We study embeddings of graphs in finite projective planes, and present related results for some families of graphs including complete graphs and complete bipartite graphs. We also make connections between embeddings of graphs and the existence of certain substructures in a plane, such as Baer subplanes and arcs.

1. Introduction

We are interested in studying how graphs embed into finite planes, as one may learn about the structure of the plane by knowing what graphs can be embedded into it, and in what quantities. In this article, we get a few results of this type, namely Remarks 2, 4 and 6.

Graph embeddings into finite projective planes has a short history. Although the original inspiration for the study of cycles in projective planes takes us back many decades (see [5] and [6] for a thorough historical narrative), considering the embedding of more complicated graphs has not been done in a serious and systematic way. Hence, this article intends to answer a few questions that naturally arise, and to present results that show unexpected connections between embeddings of complete bipartite graphs and well-known substructures of some projective planes.

We summarize the short literature on this subject as follows: [5] and [9] deal with counting how many k-cycles can be embedded in a given projective plane, for small values of k. Meanwhile, the pancyclicity (admitting embeddings of all possible cycles) of every projective plane is proved in [6]. All these results are obtained by using purely geometric arguments. For a more algebraic approach, the reader may want to read [7], where embeddings of cycles into $PG(2,q)$ are studied.

We now introduce necessary terminology and notation. Most of the content of this section is 'folklore'; any notions not found here may be found in [1] or [4].

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We consider only simple and undirected graphs. As usual, we denote a graph by $G = (V(G), E(G))$, where $V(G)$ and $E(G)$ (or V and E if the context allows it) are the set of vertices and edges of G, respectively. We denote the edge connecting $v, w \in V(G)$ by vw, in this case we will say that v and w are adjacent, and that v, w are incident with vw. Given two graphs G and G', a homomorphism between them is a map $\alpha : G \to G'$ preserving incidence. If α is one-to-one then it is called an isomorphism between G and G', and if also $G = G'$ then it is called an automorphism. The group of automorphisms of G is $Aut(G)$.

If $G = (V, E)$ is a bipartite graph such that $V = V_1 \cup V_2$ and $vw \in E$ implies $v \in V_1$ and $w \in V_2$, then we will say that V_1 and V_2 are the classes of G.

Definition 1. *A projective plane $\pi = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ is a collection of points $\mathcal{P}$ and lines $\mathcal{L}$, paired with a symmetric incidence relation $\mathcal{I}$ such that:*
(a) Given $P_1, P_2 \in \mathcal{P}$, there exists a unique $\ell \in \mathcal{L}$ that is incident with both points.
(b) Every line is incident with at least three points.
(c) Given $\ell_1, \ell_2 \in \mathcal{L}$, there exists a unique $P \in \mathcal{P}$ that is incident with both lines.
(d) There are four points, no three of them incident with the same line.

It is known that every finite projective plane π is associated to a natural number q, called the *order* of π, such that $|\mathcal{P}| = |\mathcal{L}| = q^2 + q + 1$, and that the number of points on each line is exactly $q + 1$, as well as the number of lines through any point.

All known examples of projective planes have order equal to the power of some prime. In fact, for every power of a prime there are examples of planes having that order, and if this number is at least 9, and not a prime number, then non-isomorphic planes having that order exist.

Definition 2. *Let $\pi = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a projective plane. If there exists a subset $\mathcal{P}_0$ of points and a subset $\mathcal{L}_0$ of lines in π such that $(\mathcal{P}_0, \mathcal{L}_0, \mathcal{I})$ is a projective plane, then we say that $\pi_0 = (\mathcal{P}_0, \mathcal{L}_0, \mathcal{I})$ is a subplane of π and denote this relation by $\pi_0 \subseteq \pi$.*

Theorem 1 (Bruck [2])**.** *Let π be a projective plane of order q. If π_0 is a subplane of π with order n, then either $n^2 = q$ or $q \geq n^2 + n$.*

In the case when $n^2 = q$ we say that π_0 is a Baer subplane of π. It is known that every line of π must intersect π_0 in at least one point. Similarly, every point of π is incident with at least one line of π_0.

We now define our notion of embedding.

Definition 3. *An embedding ϕ of a graph $G = (V, E)$ into a projective plane $\pi = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ is an injective function*

$$\phi : V \to \mathcal{P}$$

that induces, by preserving incidence, an injective function

$$\overline{\phi} : E \to \mathcal{L}.$$

If such an embedding exists we say that G embeds in π, and write $G \hookrightarrow \pi$. When the context is clear, we will denote an embedding ϕ of G into π as $\phi : G \hookrightarrow \pi$.

Remark 1. *If $G \hookrightarrow \pi$ then $H \hookrightarrow \pi$, for all $H \subset G$. Similarly, if $G \hookrightarrow \pi_0$, and $\pi_0 \subseteq \pi$, then $G \hookrightarrow \pi$.*

2. General results on embeddings of graphs

Note that if G is a graph, $\varphi \in Aut(G)$, and $\phi : G \hookrightarrow \pi$, then $\phi \circ \varphi$ is also an embedding of G into π. However, these two embeddings 'look the same' in π, as they use the same points and lines, and the incidence between the embedded vertices and edges is also the same. This inspires the following definitions.

Definition 4. *Let G be a graph, π a projective plane, and ϕ, ψ two embeddings $G \hookrightarrow \pi$.*
(a) If $\psi = \phi \circ \varphi$, for some $\varphi \in Aut(G)$, then we will say that ϕ and ψ are equivalent embeddings of G into π.
(b) The number of embeddings of G in π is denoted by $N_\pi(G)$. The number of non-equivalent embeddings of G into π is denoted by $n_\pi(G)$.
(c) The incidence structure $(\phi(V(G)), \phi(E(G)), \mathcal{I})$, where $\mathcal{I}$ is naturally induced by ϕ and G, will be said to be an image of G in π.

Theorem 2. *Let G be a graph embedded in a projective plane π. Then,*

$$N_\pi(G) = n_\pi(G)|Aut(G)|$$

Moreover, $n_\pi(G)$ is equal to the number of images of G in π.

Proof. Using Definition 4 (a) we obtain a free action of $Aut(G)$ on the set of embeddings of G into π. It follows that,

$$N_\pi(G) = n_\pi(G)|Aut(G)|.$$

Moreover, if ϕ and ψ are two embeddings of a graph G into π yielding the same image then $\phi^{-1}\psi$ is an automorphism of G. Hence, the number of images of G in π is equal to $n_\pi(G)$. □

It is easy to see that the action of a collineation group of a plane π may create new embeddings of a graph G out of a known one. This observation yields the following generalization of Lemma 8 in [5].

Lemma 1. *Let π be a projective plane of order q that admits a cyclic collineation group H of order m acting freely on the points of π. Assume that $G \hookrightarrow \pi$. Then, $(m/d) \mid n_\pi(G)$, where $\gcd(|V(G)|, m) = d$.*

The same result holds for a group acting freely on the lines of π, under the condition $\gcd(|E(G)|, m) = d$.

Corollary 1. *Assume that a projective plane π of order q admits a Singer cycle, and that G is a graph, embedded in π, such that* $\gcd(|V(G)|, q^2+q+1) = 1$ *or* $\gcd(|E(G)|, q^2+q+1) = 1$, *then* $(q^2+q+1) \mid n_\pi(G)$.

Since not every plane admits a Singer cycle, it is interesting to ask whether some type of converse of Corollary 1 is true. The work of Voropaev [9] provides a counterexample for cycles, as it says that all finite projective planes of order 9 have the same number of cycles, including those that do not contain Singer cycles. As of now, the best we can do is explained in the following remark.

Remark 2. *If $G \hookrightarrow \pi$, and $\pi \cong \Pi$, then $G \hookrightarrow \Pi$ and $n_\pi(G) = n_\Pi(G)$. Hence, if $n_\pi(G) \neq n_\Pi(G)$, for two projective planes π and Π and some graph G, then $\pi \ncong \Pi$.*

Note that Remark 2 gives us a tool to prove that two planes are not isomorphic; we *just* need to find a graph that embeds differently into each of the two planes. Unfortunately, finding such a graph does not seem to be an easy task.

Obvious bounds for the embedding of a graph G into a projective plane π of order q are:

- $|V(G)| \leq q^2+q+1$,
- $|E(G)| \leq q^2+q+1$, and
- $deg(v) \leq q+1$, for all $v \in V(G)$.

These bounds are sharp. The first two bounds depend on the fact that a cycle of length q^2+q+1 can be embedded in π (see [6]), while the third bound is obtained from the existence of $q+1$ lines through any point of π.

We now start striving for a 'Kuratowski-like' result, in which we prove that every graph is planar, given a large enough plane. In order to do this we look at complete graphs and realize that this problem boils down, quite naturally, to a well-studied object in finite geometry.

Definition 5. *Let π be a projective plane of order q. An n-arc in π is a set of n points of π such that no three of them are collinear. A $(q+1)$-arc in π is said to be an oval, and a $(q+2)$-arc in π is called a hyperoval.*

Remark 3. *A well-known result states that hyperovals may only exist in finite projective planes of even order. No arc can have cardinality larger than that of a hyperoval. Not all projective planes contain ovals. For instance, there are exactly four finite projective planes of order* 16 *containing no hyperovals (see* [8]*).*

Theorem 3. *Let π be a projective plane of order q, and $n \in \mathbb{N}$. Then, $K_n \hookrightarrow \pi$ if and only if there is an n-arc in π. Moreover, $n_\pi(K_n)$ is equal to the number of n-arcs in π.*

Proof. Let P, Q, R be any three points of π corresponding to vertices of an embedded K_n. Then, since the lines joining any 2 of these points must correspond to edges in K_n, it is necessary that P, Q, R are not on a common line. Conversely, if there is an n arc in π, then we can connect all possible pairs of these points using distinct lines, yielding an embedding of K_n into π. The natural one-to-one correspondence between images of K_n and n-arcs in π yields the result. $\square$

Corollary 2. *Let π be a projective plane of order q, and $n \in \mathbb{N}$. If $K_n \hookrightarrow \pi$, then q is even and $n \leq q+2$, or q is odd and $n \leq q+1$.*

Remark 4. *The converse of Corollary 2 is not true, as there are planes where ovals/hyperovals do not exist (see Remark 3). This yields examples of planes that, although they have the same order do not admit embeddings of the same graphs. This situation is similar to that in Remark 2, but now restricted to complete graphs.*

We now look at conditions that would guarantee the embedding of a K_n in a plane of order q. We want this condition to be independent of whether or not the plane contains ovals/hyperovals.

Lemma 2. *Let $n \in \mathbb{N}$, and $q \geq n(n-3)/2$ be the order of a projective plane π. Then, $K_n \hookrightarrow \pi$.*

Proof. We will prove this by induction on n. The first interesting case is $n = 4$, which holds because every projective plane has a quadrangle. We assume the result is true for all $m < n$ and consider a projective plane π having order $q \geq n(n-3)/2$. But,

$$q \geq \frac{n(n-3)}{2} > \frac{(n-1)(n-4)}{2},$$

and so, by induction, $K_{n-1} \hookrightarrow \pi$. Since the number of lines used in this embedding of K_{n-1} is equal to

$$\frac{(n-1)(n-2)}{2} = \frac{n(n-3)}{2} + 1 \leq q+1,$$

these lines cover at most

$$(q-1)\frac{(n-1)(n-2)}{2} + (n-1) \leq (q-1)(q+1) + (n-1) = q^2 + (n-2) < q^2 + q + 1$$

points of π. If P is a point in π that is not in the union of these lines, then any line connecting P with a point of the embedding of K_{n-1} cannot have been used as an edge in the embedding of K_{n-1}. Hence, by adjoining P to the embedding of K_{n-1}, we get that $K_n \hookrightarrow \pi$. $\square$

The following theorem is a consequence of Lemma 2 and Remark 1.

Theorem 4. *Every graph G can be embedded in any projective plane of order $q \geq |V(G)|(|V(G)|-3)/2$.*

It turns out that we can tighten these arguments up quite a bit for specific families of graphs.

3. Complete bipartite graphs

We begin this section with a result about subgraphs of complete bipartite graphs that is similar to Theorem 4.

Lemma 3. *Let π be a plane of order q, then all subgraphs of $K_{q,q}$ embed into π_q.*

Proof. Choose a point P_0 of π and any two distinct lines ℓ and m in π, both through P_0. Let $\{P_1, P_2, \ldots, P_n\}$ be a set of n distinct points in $\ell \setminus \{P_0\}$ and let $\{Q_1, Q_2, \ldots, Q_n\}$ be a set of n distinct points in $m \setminus \{P_0\}$. Let $l_{i,j}$ be the line of π incident with P_i and Q_j. Since in a projective plane there is a unique line through any two points, the set $\{l_{i,j};\ i,j = 1,2,\ldots,n\}$ has exactly n^2 elements. By embedding vertices in one class to the points in $\ell \setminus \{P_0\}$ and the vertices in the other class to the points in $m \setminus \{P_0\}$, it follows that $K_{n,n} \hookrightarrow \pi$. Each edge in $K_{n,n}$ is then naturally embedded into the corresponding unique line through each pair of points. □

Using ideas already discussed in Lemma 3, we can give a complete characterization for when complete bipartite graphs are embeddable in π.

Theorem 5. *The only complete bipartite graphs embedded in a projective plane π of order q are:*
(a) $K_{1,q+1}$, or
(b) $K_{n,m}$, where $n, m \leq q$.

Proof. The graph, $G = K_{1,q+1}$ may be embedded in π by mapping the vertex in the class of size 1 to an arbitrary point P, then mapping the $q+1$ edges of G to each of the $q+1$ lines through P. The vertices of the class of size $q+1$ may then be mapped to points on the lines corresponding to the mapped edges.

Suppose $\phi : K_{2,q+1} \hookrightarrow \pi$ is an embedding of $G = K_{2,q+1}$. Let u_1, u_2 be the vertices in the class of size 2. Since $q+1$ edges are embedded to lines through $\phi(u_1)$, an edge e_1 containing vertex u_1 is embedded to the line $\phi(u_1)\phi(u_2)$ by the pigeonhole principle. Similarly, an edge e_2 containing vertex u_2 is embedded to the line $\phi(u_1)\phi(u_2)$. Since no edge in G is incident with both u_1 and u_2, $e_1 \neq e_2$ and a contradiction is reached.

Since $K_{2,q+1}$ is a subgraph of all $K_{n,m}$ with $n \geq 2, m \geq q+1$, an embedding of such a $K_{n,m}$ into a plane of order q would imply an embedding of $K_{2,q+1}$, which is a contradiction.

Lemma 3 guarantees that $K_{n,m}$, where $n, m \leq q$, can always be embedded in π. Thus all cases have been considered and the theorem is proved. □

We now present results concerning the number of embeddings of complete bipartite graphs into planes. Most of these results will be in extremal cases, and will eventually lead us to consider subplanes in the following section.

Lemma 4. *Let π be a projective plane of order q, and $1 \le n \le q+1$. Then,*

$$n_\pi(K_{1,n}) = \binom{q+1}{n} q^n (q^2+q+1).$$

Proof. Let $G = K_{1,n}$, and $\phi : G \hookrightarrow \pi$. Let $v \in V(G)$ be the vertex in the singleton class. An embedding of $K_{1,n}$ is uniquely determined by the choice of $\phi(v)$, of which there are q^2+q+1 choices, and the choice of exactly one point on n of the $q+1$ lines through $\phi(v)$. Once the n lines to be used in the embedding are chosen, there are exactly q^n ways to choose the n additional points in $\phi(V(G))$. □

Lemma 5. *Let π be a projective plane of order q. Consider $\phi : K_{n,q} \hookrightarrow \pi$, for $2 \le n \le q$, and let U and V, containing n and q vertices respectively, be the two classes of $K_{n,q}$. Then, all points in the embedding of U are collinear.*

Proof. Note that any given $P \in \phi(U)$ is adjacent to the elements in $\phi(V)$ via q of the $q+1$ lines through P, forcing that none of the other $n-1$ elements in $\phi(U)$ is incident with any of these lines. Hence, $\phi(U) \subseteq m$, where m is the only line through P that does not intersect $\phi(V)$. □

Theorem 6. *Let π be a projective plane of order q. Consider $G = K_{n,q} \hookrightarrow \pi$, for $2 \le n \le q$, where the classes of G are U and V, with $|U| = n$ and $|V| = q$. Then,*
(a) For $n = q$ we get:

$$n_\pi(K_{q,q}) = \binom{q^2+q+1}{2}.$$

(b) If $n < q$, and every embedding of V lies on a line, then

$$n_\pi(K_{n,q}) = 2\binom{q^2+q+1}{2}\binom{q}{n}.$$

(c) If $n = q-1$, then the points in the embedding of V are collinear, and

$$n_\pi(K_{q-1,q}) = q^2(q+1)(q^2+q+1).$$

Proof. Let $\phi : G \hookrightarrow \pi$, $2 \le n \le q$, and U, V as described above.
(a) Lemma 5 implies that the $2q$ points in the embedding of G must lie on two lines. It follows that the number of embeddings of G into π is given by the number of pairs of intersecting lines in π.
(b) We first choose two lines of π, and then choose one of them to be the one containing $\phi(U)$. The result then follows by choosing n points, from the q available points on this line, to create $\phi(U)$.
(c) First notice that the result is valid for $q = 2$ (and consistent with Lemma 4). For $q > 2$, the value of $n_\pi(K_{q-1,q})$ will follow from (b), as soon as we prove that the points of $\phi(G)$ lie on two lines.

Consider $P, Q \in \phi(V)$, and note that the remaining $q-2$ points of $\phi(V)$ cannot be on lines that connect P or Q with the $q-1$ points of $\phi(U)$. It follows that the number of distinct points on these lines is exactly

$$2(q(q-1)+1)-(q-1)^2 = q^2+1.$$

Also, the two unused points on the line containing $\phi(U)$ cannot be used, as this line passes through two points that are not neighbors in G. So, the remaining $q-2$ points of $\phi(V)$ have to be located among the

$$(q^2+q+1)-(q^2+1)-2 = q-2$$

points remaining in π. These are exactly the points on PQ, different from P and Q, not intersecting the line containing $\phi(U)$. $\square$

In the previous two proofs we have used lines connecting points in the same class of the embedding of a complete bipartite graph. We formalize this idea, as it will help us to obtain several interesting results.

Definition 6. *Let π be a projective plane and G a graph such that $\phi : G \hookrightarrow \pi$. If $P = \phi(v)$, $Q = \phi(w)$, where $v, w \in V(G)$ and $vw \notin E(G)$, then the line PQ is a complement line of the embedding.*

Lemma 6. *Assume that $\phi : K_{m,n} \hookrightarrow \pi$. Then the set of complement lines is disjoint from the set of embedded edges.*

Proof. Let U and V be the classes of $K_{m,n}$, and suppose $uv \in E(K_{m,n})$ is embedded to a complement line $\ell = \phi(uu_0)$, where $u_0 \in U$. Then the edge u_0v is embedded to ℓ. Thus two edges are embedded to the same line which is a contradiction. $\square$

Lemma 7. *Assume that $\phi : K_{m,n} \hookrightarrow \pi$, and that the classes of $K_{m,n}$ are U and V. Then $\phi(U)$ together with the set of complement lines of $\phi(U)$ form a linear space under the incidence relation of π.*

Proof. By definition, every complement line must contain at least two points of $\phi(U)$. Since we embed into a projective plane, every pair of points may have at most one line through them. $\square$

Theorem 6 (b) can be used to obtain $n_\pi(K_{q-n,q})$, as long as we know that all embeddings of $K_{q-n,q}$ lie on two lines. Our next theorem provides conditions for this to happen.

Theorem 7. *Let π be a projective plane of order q, and $2 \leq q-n \leq q$. Consider $G = K_{q-n,q} \hookrightarrow \pi$, where the classes of G are U and V, with $|U| = q-n$ and $|V| = q$. Then, $q > n^2$ implies that the vertices of the embedding of G lie on two lines.*

Proof. Lemma 5 tells us that $|V| = q$ forces the existence of a line ℓ such that $\phi(U) \subseteq \ell$. We let $A_1, A_2, \ldots, A_{n+1}$ be the $n+1$ points on $\ell \setminus \phi(U)$.

Let P and Q be two distinct points in $\phi(V)$. Since PQ intersects ℓ we assume, without loss of generality, that $A_{n+1} \in PQ$. It follows that the elements of $\phi(V)$ must lie on PQ, or be one of the points of intersection of the lines PA_i, and QA_j, for $1 \leq i, j \leq n$. However, $A_i \notin \phi(V)$, for all $1 \leq i \leq n$, and thus at most $n(n-1)$ vertices of $\phi(V)$ are not on PQ. Note that every complement line through P must contain at most n points of $\phi(V)$.

Now assume that there is a point $R \in \phi(V)$ not collinear with P and Q. The argument used above (now with R instead of Q) forces PQ to contain at most n points of $\phi(V)$. Hence, $q = |\phi(V)| \leq n(n-1) + n = n^2$, which contradicts our hypothesis. So, the points of $\phi(V)$ must be collinear. □

Remark 5. *The bound $q > n^2$ in Theorem 7 is sharp. For example, if $n = q - 2$ and $q \geq 5$, then we get*

$$n_\pi(K_{q-2,q}) = \frac{q^2(q^2-1)(q^2+q+1)}{2}.$$

But if $q = 3$ we get $n_\pi(K_{1,3}) = 468$, and for $q = 4$ we obtain $n_\pi(K_{2,4}) = 5040$. These two values are not consistent with Theorem 7.

Moreover, if $q = n^2$ and π has a Baer subplane $\mathcal{B}$ then we consider a line ℓ of $\mathcal{B}$ (ℓ is also seen as a 'longer' line in π). We let $\phi(U)$ to be the points on $\ell \setminus \mathcal{B}$, and $\phi(V)$ to be the points on $\mathcal{B} \setminus \ell$. This selection of $\phi(U)$ and $\phi(V)$ yields an embedding of $K_{q-n,q}$ into π, with $q = n^2$, not having the points on $\phi(V)$ lying on a line.

Remark 6. *Not all projective planes have Baer subplanes. Hence, the number of embeddings of $K_{q-n,q}$ in a given plane gives us information about the structure of the plane that hosts it.*

4. Subplanes

We know, by Lemma 7, that the set of points in the embedding of a class of $K_{m,n}$ together with all its complement lines forms a partial plane. We are interested in studying whether this structure could ever be a subplane. Our hopes are, firstly, based on the following remark, which reminds us of the Erdős–de Brujin theorem.

Remark 7. *Consider π a projective plane of order q, and assume $G = K_{m,n} \hookrightarrow \pi$, where $n = s^2 + s + 1$ and $m = q - s$, for some $s \in \mathbb{N}$. There are at most $q + 1 - (q - s) = s + 1$ complement lines per point in the class of $K_{m,n}$ of size $s^2 + s + 1$.*

Theorem 8 (Erdős–de Brujin [3])**.** *Let $S = (P, L)$ be a finite linear space with $|P| = v$ and $|L| = b > 1$. Then*
(a) $b \geq v$.
(b) if $b = v$, any two lines of S intersect at a point in S.
In case (b), either one line has $v - 1$ points and all others have two points, or S is a finite projective plane.

The following lemma will help us in the study of these potential subplanes.

Lemma 8. *Let π be a finite projective plane of order q, and $A = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ a set of points and lines of π satisfying the axioms of a linear space. Assume that there is an integer $1 < n \leq q$ such that no point of A is incident with more than $n+1$ lines in A. Then,*
(a) If $|\mathcal{P}| = n^2 + n + 1$, then either $|\mathcal{L}| = 1$, or A is a subplane of π.
(b) If $|\mathcal{P}| > n^2 + n + 1$, then $|\mathcal{L}| = 1$.

Proof. (a) If $|\mathcal{L}| \neq 1$, by Theorem 8, A is either a finite projective plane or contains a line with $n^2 + n$ points, with a single point not on the line. Suppose the latter, then there would be $n^2 + n$ lines through the point not on the line, contradicting the assumption that each point may have at most $n + 1$ lines of A incident with it. Hence, A is a finite projective plane if $|\mathcal{L}| \neq 1$. If $|\mathcal{L}| = 1$, all points must be collinear, since every two points are collinear in a linear space.
(b) Follows from Theorem 8 (a). □

The next result shows when we obtain subplanes for a class of an embedded complete bipartite graph.

Theorem 9. *Let π be a projective plane of order q and let $n > 1$ be a natural number. Then, any embedding of $K_{q-n,\ n^2+n+1}$ maps the class of size $n^2 + n + 1$ to either a subplane of order n, or to points on a line.*

Proof. Note that, implicitly, we are assuming $q > n^2 + n$, as this is needed for $K_{q-n,\ n^2+n+1}$ to embed into π.

Let P be an embedded vertex of the class of size n^2+n+1. From Remark 7, P is incident with at most $n + 1$ complement lines. It follows that the set of embedded vertices of the class of size $n^2 + n + 1$, together with their complement lines and the incidence relation inherited from π satisfies the hypothesis of Lemma 8. The result follows. □

Corollary 3. *Let π be a projective plane of order q and let $n > 1$ be a natural number. Then any embedding of $K_{s,t}$, where $s \geq q - n$, and $t > n^2 + n + 1$ maps the class of size t to points on a line.*

Proof. Since $t > n^2+n+1$, there are at least two embeddings of $K_{q-n,\ n^2+n+1}$ into π, sharing a $K_{q-n,\ n^2+n}$. We denote these embeddings G_1 and G_2. We now apply Theorem 9 for each of these graphs. If the two classes of size n^2+n+1 are subplanes we get a contradiction, as the partial plane on n^2+n points (the intersection of the two subplanes) can be extended in a unique way to a projective plane, as the missing point must be the intersection of two lines that are parallel in the subplane. It follows that, without loss of generality, G_1 has its class of cardinality $n^2 + n + 1$ contained in a line. Hence, G_2 must have the same property, as G_1 and G_2 share a $K_{q-n,\ n^2+n}$. Moreover, since these lines will share at least two points, they are the same line. □

Corollary 4. *Let π be a projective plane of order q, and let $n, s, t \in \mathbb{N}$ be such that $q = (n+1)^2$, and $q \geq s, t > q - n$. Then,*

$$n_\pi(K_{s,t}) = 2\binom{q^2+q+1}{2}\binom{q}{s}\binom{q}{t},$$

for $s \neq t$, and

$$n_\pi(K_{s,s}) = \binom{q^2+q+1}{2}\binom{q}{s}^2.$$

Proof. If $q = (n+1)^2$, then $q - n = n^2 + n + 1$. It follows that $q \geq s, t > n^2+n+1$, and thus each class of $K_{s,t}$ must be embedded into a line. Hence, the number of images of $K_{s,t}$ in π is obtained by counting how to choose two distinct lines in π and then how to choose points on them. $\square$

Remark 8. *If t is a Mersenne prime, or $t+1$ is a Fermat prime, then the pair $(t, (t+1)^2)$ could serve as the (n, q) pair required in the hypothesis of Corollary 4. An example of a pair not covered already would be $(n, q) = (8, 3^4)$.*

Finally, we look at other possibilities for classes of embedded complete bipartite graphs to be Baer subplanes.

Lemma 9. *Let Π be a projective plane of order q^2, and π be a Baer subplane of Π. Then, an embedding of a maximal complete graph K_{q^2,q^2} into Π can contain at most either:*
(a) q^2 points of a Baer subplane, for $q > 2$, or
(b) q^2+1 points of a Baer subplane, for $q = 2$.

Proof. Let $q > 2$ and $\phi : K_{q^2,q^2} \hookrightarrow \Pi$. Let K be the subgraph of K_{q^2,q^2} induced by the set of vertices which are embedded to points of π.

Suppose $\phi(K_{q^2,q^2})$ contains q^2+1 points of a Baer subplane. It follows that K must be a complete bipartite graph, and thus Theorem 5 can be applied to K and π. Hence, since $q^2 = q+1$ is impossible, K must be of the form $K_{s,t}$ with $2 \leq s, t \leq q$. Then, there is an integer n such that $2 \leq n \leq q^2 - 1$ and K is isomorphic to K_{n,q^2+1-n}. It follows that this graph contains exactly $n(q^2+1-n)$ edges. This number must minimize at the endpoints of the interval $[2, q^2-1]$, as $n(q^2+1-n)$ is quadratic in n with negative leading coefficient. Hence, the minimum number of edges is $2(q^2-1) = (q^2+q+1)+(q^2-q-3)$, but this number is larger than q^2+q+1 because $q > 2$, a contradiction. If $q = 2$ the result follows similarly. $\square$

Lemma 9 yields a string of corollaries, the first one of them being the most immediate.

Corollary 5. *Any complete bipartite graph $K_{m,n}$ embedded in a plane of order q^2 can contain at most q^2+1 points of a Baer subplane.*

Corollary 6. *Let Π be a plane of order q. If a subplane π of order n contains a cycle of an embedded $K_{q,q}$, then it contains an embedded $K_{n,n}$.*

Proof. If such a cycle exists, at least two points from either class of $K_{q,q}$ must be in π. Since $K_{q,q}$ is maximal, we know that points in each partition are on the same line. Hence, the cycle implies that π contains both such lines and thus contains the full sub-embedding of $K_{q,q}$ □

Corollary 7. *Let π be a projective plane of order q and $\phi : K_{q,q} \hookrightarrow \pi$. Then, $\phi(K_{q,q})$ can contain at most $2n$ points of a subplane of π of order n.*

Proof. Since the image of $K_{q,q}$ under an embedding is on two lines (Theorem 6), any intersection of the subplane and two lines of the parent plane contains at most $2n+1$ points, but in the maximal case, the point of intersection of the two lines is not an embedded vertex, and thus there may at most be $2n$ vertices embedded into the subplane. □

Corollary 8. *An embedding of K_{q^2,q^2} into a projective plane Π of order q^2 must contain at least q lines of every Baer subplane of Π.*

Proof. Let π be an arbitrary Baer subplane of Π. Since there are q^4+q^2+1 lines in Π, and q^2+q+1 lines in π, there are q^4-q lines of Π not in π. Since K_{q^2,q^2} contains q^4 edges, there must be q lines which are in π. □

The ideas in this article have the potential to be very useful in the study of finite projective planes. This claim is based on Remarks 2, 4 and 6, and on the fact that most of the results in Section 4 depend on the existence of Baer subplanes, which are known to be present in certain (but not all) planes, and occur in different numbers depending on the plane.

We plan to continue studying embeddings of graphs into projective planes by looking for connections between the existence of certain configurations (e.g., Pappus, Desargues, etc) in a given plane and the types of graphs that can be embedded in it.

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