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A SHORT CONSTRUCTION OF HIGHLY CHROMATIC DIGRAPHS WITHOUT SHORT CYCLES

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Abstract. A natural digraph analogue of the graph-theoretic concept of an 'independent set' is that of an 'acyclic set', namely a set of vertices not spanning a directed cycle. Hence a digraph analogue of a graph coloring is a decomposition of the vertex set into acyclic sets. In the spirit of a famous theorem of P. Erdős [Graph theory and probability, Canad. J. Math. **11** (1959), 34–38], it was shown probabilistically in [D. Bokal et al., The circular chromatic number of a digraph, J. Graph Theory **46** (2004), no. 3, 227–240] that there exist digraphs with arbitrarily large girth and chromatic number. Here we give a construction of such digraphs.

In [2], it is shown that the coloring theory for digraphs is similar to the coloring theory for graphs when stable sets are replaced by acyclic sets and homomorphisms are replaced by 'acyclic homomorphisms'. One of the results therein asserts the existence of digraphs with arbitrarily large girth and (digraph) chromatic number. This, of course, is analogous to the seminal theorem of Erdős [3] on graphs with arbitrarily large girth and chromatic number, and it is likewise proved probabilistically, whence non-constructively. It is worth noting that although many results about digraph coloring theory are generalizations of results about graphs, the aforementioned result in [2] is not a generalization of Erdős' theorem because the relationship between independent sets and cycles in graphs is different from the relationship between acyclic sets and directed cycles in digraphs.

In this note, we *construct* digraphs with arbitrarily large girth and chromatic number. In fact, the construction strengthens the result in [2] because it produces a digraph with girth k and chromatic number n for each pair k, n of integers exceeding one. It is also of interest that unlike the analogous graph constructions in [4, 5, 6], our construction is primitively recursive in n.

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Although basic terminology can be found in [1], we include the main definitions for completeness. The *girth* of a digraph D, denoted $g(D)$, is the length of its shortest directed cycle. Following [2], we define the *chromatic number* $\chi(D)$ of D to be the minimum number of parts in a partition of $V(D)$ into acyclic sets, and we say that D is *n-chromatic* if $\chi(D) = n$. An *acyclic homomorphism* from D to H is a mapping $\phi : V(D) \to V(H)$ such that $uv \in A(D)$ implies that either $\phi(u)\phi(v) \in A(H)$ or $\phi(u) = \phi(v)$, and for all vertices $x \in V(H)$, the *fiber* $\phi^{-1}(x)$ is acyclic. It is easy to check that the composition of two acyclic homomorphisms is again an acyclic homomorphism. We use the notation $D \to H$ to denote that there exists an acyclic homomorphism from D to H and define K_n^* to be the complete bidirected digraph with vertex set $[n]$. As in the case of the graph coloring analogue, an equivalent definition of the chromatic number is $\chi(D) = \min\{n \mid D \to K_n^*\}$. In order to confirm the correctness of our construction, we will need the fact that $D \to H$ implies that $g(D) \geq g(H)$, which is a direct consequence of Propositions 1.2 and 1.3 in [2]. It is worth noticing the subtle difference between the last statement and its graph analogue, which is true only for odd girth.

Theorem 1. *For any given integers k and n exceeding one, there exists an n-chromatic digraph D with $g(D) = k$.*

Proof. For $n = 2$, the directed k-cycle will suffice. For $n \geq 2$, we proceed by induction on n and suppose that we have already constructed a digraph D_n with chromatic number n, girth k, and $V(D_n) = \{d_1, d_2, ..., d_m\}$. We now define D_{n+1}.

For each $i \in [m]$ let D_n^i be a digraph with vertex set $V(D_n^i) = \{(d_1, i), (d_2, i), \ldots, (d_m, i)\}$, which is isomorphic to D_n in the natural way. Next, construct m directed paths P_{d_i}, for $1 \leq i \leq m$, each of length $k-2$, with vertex sets $\{(d_i, p_1), (d_i, p_2), \ldots, (d_i, p_{k-1})\}$ and arc sets $A(P_{d_i}) = \{\overrightarrow{(d_i, p_j)(d_i, p_{j+1})} \mid j \in [k-2]\}$. Now define m digraphs $H(n, i)$, for $1 \leq i \leq m$, with vertex sets $V(H(n, i)) \coloneqq V(D_n^i) \cup V(P_{d_i})$, and arc sets

$$\begin{aligned} A(H(n,i)) \coloneqq& A(D_n^i) \cup A(P_{d_i}) \\ &\cup \left\{\overrightarrow{(d,i)(d_i,p_1)} | d \in V(D_n)\right\} \cup \left\{\overrightarrow{(d_i,p_{k-1})(d,i)} | d \in V(D_n)\right\}. \end{aligned}$$

Finally, we define D_{n+1} to be the digraph with

$$V(D_{n+1}) \coloneqq \bigcup_{i=1}^{m} V(H(n,i))$$

and

$$A(D_{n+1}) \coloneqq \bigcup_{i=1}^{m} A(H(n,i)) \cup \left\{\overrightarrow{(d_i,p_\ell)(d_j,p_h)} | d_i d_j \in A(D_n), \ell, h \in [k-1]\right\}.$$

We first show that the girth of D_{n+1} is k. Since the girth of each $H(n, i)$ is k and there are no arcs from D_n^i to D_n^j for $j \neq i$, any cycle containing a vertex from some D_n^i has length exceeding $k-1$ provided that the subdigraph Σ of D_{n+1} induced by the vertices of the P_{d_i}'s has girth exceeding $k-1$. Since there exists an acyclic homomorphism $\psi : V(\Sigma) \to V(D_n)$ (sending every vertex in P_{d_i} to d_i), we have $g(\Sigma) \geq g(D_n) = k$. Therefore, $g(D_{n+1}) = k$.

It is clear that $\chi(D_{n+1}) \geq \chi(D_n) = n$, since D_n is isomorphic to a subgraph of D_{n+1}. If D_{n+1} is n-chromatic, then there exists an acyclic homomorphism $\sigma : V(D_{n+1}) \to V(K_n^*)$. To set up the contradiction we are about to derive, fix a σ 'color' $\alpha \in V(K_n^*)$. Since D_n^i is isomorphic to D_n, the function σ maps $V(D_n^i)$ onto $V(K_n^*)$, for all $i \in [m]$. Every vertex in D_n^i is in a cycle with the vertices of P_{d_i}, which implies that there exists a vertex $v_i \in P_{d_i}$ such that $\sigma(v_i) \neq \alpha$. The subgraph Λ of D_{n+1} induced by $\{v_1, v_2, \ldots, v_m\}$ is isomorphic to D_n. This contradicts the fact that D_n has chromatic number n, since σ, now seen to avoid α on $V(\Lambda)$, effectively maps $V(\Lambda)$ to $V(K_{n-1}^*)$ acyclically. Thus $\chi(D_{n+1}) \geq n+1$. We now show that $\chi(D_{n+1}) = n+1$ by giving an acyclic homomorphism from D_{n+1} to K_{n+1}^*. Let ζ be an acyclic homomorphism from D_n to K_n^*. Define a mapping $\phi : V(D_{n+1}) \to V(K_{n+1}^*)$ as follows. For vertices $(d_i, p) \in V(P_{d_i})$, define $\phi((d_i,p)) = \zeta(d_i)$ and for vertices $(d_j, i) \in V(D_n^i)$, let

$$\phi((d_j, i)) = \begin{cases} \zeta(d_j), & \text{if } \zeta(d_j) \neq \zeta(d_i), \\ n+1, & \text{otherwise.} \end{cases}$$

As the target digraph of ϕ is complete, to show that ϕ is an acyclic homomorphism, it will suffice to show that each fiber of ϕ is acyclic. The fibers of $\phi|_{D_n^i}$ are acyclic, for all $i \in [m]$ because they are identical, up to relabeling, to the fibers of ζ. This implies that the fibers of $\phi|_{H(n,i)}$ are acyclic since $\phi(V(P_{d_i})) \cap \phi(V(D_n^i)) = \varnothing$. Hence it suffices to show that the restriction of ϕ to $\Sigma\,(= D_{n+1}\,[\cup_{i=1}^m V(P_{d_i})])$ is an acyclic homomorphism. Let $\psi : V(\Sigma) \to V(D_n)$ be defined as above and notice that $\phi|_\Sigma = \zeta \circ \psi$, since $\phi((d_i,p)) = \zeta(d_i)$, for all vertices $(d_i, p) \in V(P_{d_i})$. As ζ and ψ are acyclic homomorphisms, so too is their composition $\phi|_\Sigma$. Therefore, ϕ is an acyclic homomorphism which finally implies that $\chi(D_{n+1}) = n+1$. □

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