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CYCLES, WHEELS, AND GEARS IN FINITE PLANES

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Abstract. The existence of a primitive element of $GF(q)$ with certain properties is used to prove that all cycles that could theoretically be embedded in $AG(2,q)$ and $PG(2,q)$ can, in fact, be embedded there (i.e. these planes are 'pancyclic'). We also study embeddings of wheel and gear graphs in arbitrary projective planes.

1. Introduction

In this article, a graph will be understood to be simple, finite, and undirected. Since we will mostly focus on cycles and cycle-related graphs we define, for $k \geq 3$, a k-cycle as the graph C_k with $V = \{x_1, \dots, x_k\}$ and $E = \{x_1x_2, x_2x_3, \dots, x_{k-1}x_k, x_kx_1\}$. We refer the reader to [11] for any graph theoretical notion we use and fail to define.

Next, we define the concepts in finite geometry that we will need later on; any those missing concepts may be found in [2].

Definition 1.1. *Let $\pi = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ where $\mathcal{P}$ is a set of points, $\mathcal{L}$ is a set of lines, and $\mathcal{I}$ is an incidence relation. Then π is an affine plane if it satisfies the following conditions:*

(1) Given any two distinct points, there is exactly one line incident with both of them.
(2) For every line l and every point P not incident with l there is a unique line m that is incident with P and that does not intersect l.
(3) There are three points that do not lie on the same line.

We may obtain a projective plane Π from any given affine plane π by the addition of a line at infinity, denoted ℓ_∞. Furthermore, lines which were parallel with one another in π, meet at a point at infinity in Π. Finally, these points at infinity are all incident with the line at infinity. Conversely, deleting any line in a projective plane (and all points incident with that line) yields an affine plane.

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In this work we consider planes that contain only a finite number of points and lines. In this case, it is known that for every affine plane $\pi = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ there is a positive integer q, called the order of the plane, such that $|\mathcal{P}| = q^2$, $|\mathcal{L}| = q^2 + q$, each line contains exactly q points, and every point is incident with exactly $q+1$ lines. A similar result is also valid for projective planes. In this case, the addition of ℓ_∞ yields the following: $|\mathcal{P}| = |\mathcal{L}| = q^2 + q + 1$, every line contains $q+1$ points, and every point is incident with $q+1$ lines. All known examples of finite planes have order equal to the power of a prime number.

It is known that for every q, a power of a prime, there is only one affine/projective plane that may be coordinatized by $GF(q)$. For every fixed q, we denote this plane by $AG(2,q)$ (if affine) or $PG(2,q)$ (if projective).

Our objective is to study how cycles, and some cycle-related graphs can be embedded in finite planes (both affine and projective). For this, we must define what we understand an embedding of a graph into a finite plane.

Definition 1.2. *Let $G = (V, E)$ be a graph. An embedding of G into a plane (affine or projective) $\pi = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, is an injective function $\psi : V \to \mathcal{P}$ that induces naturally an injective function $\overline{\psi} : E \to \mathcal{L}$ by preserving incidence. We call ψ an* embedding of G in π. *If such a function exists, we say that* G embeds in π *and write $G \hookrightarrow \pi$.*

Note that since $\overline{\psi}$ is injective, we will identify edges in G with *whole* lines. That is, if a line has been used as an edge for a graph, this line cannot be used again in the same embedding.

Definition 1.3. *We say that $AG(2,q)$ is pancyclic if and only if $C_k \hookrightarrow AG(2,q)$, for all $3 \leq k \leq q^2$. Similarly, we say that $PG(2,q)$ is pancyclic if and only if $C_k \hookrightarrow PG(2,q)$, for all $3 \leq k \leq q^2 + q + 1$.*

The idea of embedding a graph into other structures has been present for a long time. For instance, the history of embeddings of graphs into linear spaces goes back to Hall [4], includes Erdős [3], and the more recent work by Moorhouse and Williford [7]. On the other hand, not much is known about embeddings of graphs in finite planes: most of what is known is on embeddings of cycles. This is likely because studying k-cycles embedded in a projective plane Π is equivalent to studying embeddings of $(2k)$-cycles in the Levi graph of Π. For instance, one can use the Singer cycle in $PG(2,q)$ to construct a (q^2+q+1)-cycle in $PG(2,q)$ (e.g., see [5]). Also, the constructions by Schmeichel [8] proved that $PG(2,p)$ is pancyclic for p prime. Moreover, Schmeichel's longest cycle is different from the one constructed using the Singer cycle (these cycles are constructed in [5]). Recently, in [5], one may find expressions for the number of k-cycles in a projective plane of order q, for $3 \leq k \leq 6$. This work has been extended by Voropaev [10] to $7 \leq k \leq 10$.

Our work may also be related to [6], as in that article embeddings of cycles in projective planes are also studied. However, the approach in [6] is purely geometrical, and our approach is an effort to bring an algebraic perspective to the problem. In fact, we will coordinatize $AG(2,q)$ using a field to give an algebraic characterization of the pancyclicity of $AG(2,q)$ and $PG(2,q)$. We refer the reader to [5] and [6] for a thorough historical narrative on this problem.

2. Cycles in $AG(2,q)$ and $PG(2,q)$

In this section we investigate pancyclicity in $AG(2,q)$ and $PG(2,q)$ by modifying the approach of constructing of cycles in [6].

Let $\mathbb{F} = GF(q)$ and let $\langle \alpha \rangle = \mathbb{F}^*$, that is, α is a primitive element of $\mathbb{F}$. We will consider the following coordinatization of $AG(2,q)$ using $\mathbb{F}\times\mathbb{F}$. The points on its axes will be labeled using 0 or powers of α. Next we label the lines through $\mathcal{O} = (0,0)$ as follows:

$$l_i : \begin{cases} x = 0, & \text{if } i = 0 \\ y = x\alpha^i, & \text{if } i = 1, 2, \ldots, q-1 \\ y = 0, & \text{if } i = q. \end{cases}$$

Also, for any point Q in the plane we denote the line parallel to l_i that passes through Q by $l_i + Q$.

Pick any point $P_0 \in l_0$, different from $\mathcal{O}$. We define $P_i = (l_{i+1} + P_{i-1}) \cap l_i$ for $i = 1, \ldots, q-1$ and $P_q = (l_0 + P_{q-1}) \cap l_q$. Next, we connect P_{i-1} with P_i using $l_{i+1} + P_{i-1}$ for $i = 1, \ldots, q-1$, and connect P_{q-1} with P_q using $l_0 + P_{q-1}$. In this way, we obtain a path of length $q+1$. We denote this path by $\mathcal{P}_{P_0}$.

In [6], it is shown that the $q-1$ paths constructed this way share no points or lines with P_0 being any point on l_0 different from $\mathcal{O}$. Hence, these paths partition the points of $AG(2,q) \setminus \{\mathcal{O}\}$. Moreover, a path starting at $(0,\beta)$ may be obtained from the path starting at $(0,\alpha)$ by using a translation T_v with $v = (0, \beta - \alpha)$.

Note that no line parallel to l_1 has been used in the construction of these paths. Hence, using the line $l_1 + P_q$ we connect P_q with a (uniquely determined) point Q_0 on l_0. If $Q_0 = P_0$, then we get a cycle of length $q+1$. On the other hand, if $Q_0 \neq P_0$ then we may concatenate $\mathcal{P}_{Q_0}$ to the path starting at P_0 and ending at Q_0 to form a longer path. It seems that when $P_0 \neq Q_0$ we are able to create long cycles. But, how long? To answer this question we need to study the case when $P_0 \neq Q_0$. We will do this algebraically by identifying $AG(2,q)$ with $\mathbb{F}\times\mathbb{F}$.

Lemma 2.1. *Let $P_0 = (0,\beta)$ be a point in $AG(2,q)$ then*

$$P_{i+1} = \left(y = \alpha^{i+2}x + \beta(1+\alpha)^i\right) \cap \left(y = \alpha^{i+1}x\right) = \left(\frac{\beta(1+\alpha)^i}{\alpha^{i+1}(1-\alpha)}, \frac{\beta(1+\alpha)^i}{1-\alpha}\right)$$

for all $0 \leq i \leq (q-2)$. Also,

$$P_q = \left(\frac{\beta}{(1+\alpha)^2(1-\alpha)}, 0\right), \quad Q_0 = \left(0, \frac{-\alpha\beta}{(1-\alpha)(1+\alpha)^2}\right).$$

Proof. Let $P_0 = (0, \beta)$, then

$$P_1 = (y = \alpha^2 + \beta) \cap (y = \alpha x) = \left(\frac{\beta}{\alpha(1-\alpha)}, \frac{\beta}{(1-\alpha)} \right).$$

In general,

$$P_{i+1} = (y = \alpha^{i+2} x + b) \cap (y = \alpha^{i+1} x).$$

We find b by substituting the coordinates of P_i into $y = \alpha^{i+2} x + b$ and get

$$b = \beta(1+\alpha)^{i-1} \left(\frac{\alpha^i - \alpha^{i+2}}{\alpha^i(1-\alpha)} \right) = \beta(1+\alpha)^i.$$

So,

$$P_{i+1} = \big(y = \alpha^{i+2} x + \beta(1+\alpha^i)\big) \cap (y = \alpha^{i+1} x).$$

We then isolate x to get

$$x = \frac{\beta(1+\alpha^i)}{\alpha^{i+1}(1-\alpha)},$$

and it follows that for $1 \leq i \leq (q-2)$,

$$P_{i+1} = \left(\frac{\beta(1+\alpha)^i}{\alpha^{i+1}(1-\alpha)}, \frac{\beta(1+\alpha)^i}{(1-\alpha)} \right).$$

Finally, P_q and Q_0 are obtained using similar procedures. $\square$

The previous lemma proves that each of the paths of the form $\mathcal{P}_{P_0}$ starts at $(0, \beta)$ and returns to l_0 at

$$\left(0, \frac{-\alpha\beta}{(1-\alpha)(1+\alpha)^2} \right).$$

Note that this behavior is being dictated by the action of the group $\mathbb{F}^*$ on itself defined by

$$\alpha \cdot \beta = \frac{-\alpha}{(1-\alpha)(1+\alpha)^2} \beta.$$

The following result is almost immediate.

Theorem 2.2. *Assume that there is a primitive element $\alpha \in \mathbb{F}$ such that*

$$\gamma = \frac{-\alpha}{(1-\alpha)(1+\alpha)^2}$$

is also primitive. Then, $C_{q^2-1} \hookrightarrow AG(2, q)$.

Proof. Having γ be primitive means that the sequential action of α on $\mathbb{F}^*$ yields the cycle

$$\beta \xrightarrow{\alpha} \gamma\beta \xrightarrow{\alpha} \gamma^2\beta \xrightarrow{\alpha} \dots \xrightarrow{\alpha} \gamma^{q-2}\beta \xrightarrow{\alpha} \gamma^{q-1}\beta = \beta$$

which runs through all the elements in $\mathbb{F}^*$. Hence, the paths of the form $\mathcal{P}_{P_0}$ create a cycle of length $q^2 - 1$ when connected using lines parallel to l_1. $\square$

We have not been able to prove that the hypothesis in Theorem 2.2 must hold. However, Mathematica was used to verify that the hypothesis holds for all finite fields of order at most 10^6 and Python was used to verify that it holds for fields of prime order $10^6 \leq p \leq 10^7$. Different labelings of the lines through $\mathcal{O}$ in $AG(2,q)$ might yield different possibilities for γ in Theorem 2.2. For instance,

$$l_i : \begin{cases} x = 0, & \text{if } i = 0 \\ y = \alpha^i x, & \text{if } 1 \leq i \leq q-2 \\ y = 0, & \text{if } i = q-1 \\ y = x, & \text{if } i = q \end{cases} \tag{2.1}$$

yields

$$\gamma' = \frac{\alpha - 1}{(\alpha+1)^3}.$$

We tried many different labelings on the lines through $\mathcal{O}$, no other interesting γ's were found.

If γ obtained in Theorem 2.2 were equal to γ' obtained from (2.1), we would get $3\alpha = 1$, which would mean that $\text{char}(\mathbb{F}) \neq 3$. Moreover, if we also assume $\gamma' = 1$, then $(\alpha+1)^3 = \alpha - 1$, which implies $\alpha^4 = -1$, and thus $\langle \alpha \rangle$ has order 8, forcing $\mathbb{F} = GF(9)$. However this is impossible because $\text{char}(\mathbb{F}) \neq 3$. Hence, $\gamma \neq 1$ or $\gamma' \neq 1$. It follows that one may choose a coordinatization of $AG(2,q)$ such that $P_0 \neq Q_0$, and thus our construction may always be assumed to create cycles of length at least $2(q+1)$.

Lemma 2.3. *Let $q = 2^a$, where $a \in \mathbb{N}$, $a > 1$, and γ' is the element from (2.1). Then there is a primitive element α of $\mathbb{F} = GF(q)$ such that γ' is also a primitive element.*

Proof. When q is even we obtain,

$$\gamma' = \frac{1}{(\alpha+1)^2}.$$

A conjecture of Golomb that has been verified asserts that, if q is even and larger than 2, then there are consecutive primitive elements α and $\alpha+1$ of $\mathbb{F}$ (see e.g., the survey [1]). Next, since $q-1$ is odd then $(\alpha+1)^2$ is a primitive element because $\alpha+1$ is a primitive element and $\gcd(2, q-1) = 1$. Finally, we use the fact that the inverse of a primitive element is also a primitive element to get that $1/(\alpha+1)^2$ is a primitive element. $\square$

Remark. We will say that *Hypothesis J* holds when q is a power of 2, or the hypothesis in Theorem 2.2 holds.
As of now, we know that *Hypothesis J* holds when either $q = 2^a$ for some $a \in \mathbb{N}$, q is an odd prime less than 10^7, or when q is a power of an odd prime that is less than 10^6.

The following corollary is immediate.

Corollary 2.4. *If* Hypothesis J *holds, then* $C_{q^2-1} \hookrightarrow AG(2,q)$.

The largest possible cycle that could be embedded in $AG(2,q)$ has length q^2. We will construct such a cycle in $AG(2,q)$ by noticing that the q^2-1 cycles already constructed do not use any of the lines through $\mathcal{O}$.

Corollary 2.5. *If* Hypothesis J *holds, then* $C_{q^2} \hookrightarrow AG(2,q)$.

Proof. Let P and Q be two points in $AG(2,q)$ that are adjacent in the embedding of C_{q^2-1} in $AG(2,q)$ described in Corollary 2.4. Without loss of generality, assume that $P \in l_0$ and $Q \in l_1$. We disconnect P and Q by eliminating the line that joins them, and then we connect each one of them with $\mathcal{O}$ by using l_0 and l_1. This new cycle has length q^2. □

As of now, we have proven the embedding of cycles of length q^2-1 and q^2. What about shorter cycles?

Theorem 2.6. *If* Hypothesis J *holds, then* $AG(2,q)$ *is pancyclic.*

Proof. We only need to prove $C_k \hookrightarrow AG(2,q)$ for all $3 \le k \le q^2-2$. We know (see [5]) that $K_{q+1} \hookrightarrow AG(2,q)$, and thus get $C_k \hookrightarrow AG(2,q)$ for all $3 \le k \le q+1$. For $q+2 \le k \le q^2-2$, let

$$P_0 \to P_1 \to P_2 \to \cdots \to P_{q^2-3} \to P_{q^2-2} \to P_0$$

be the (q^2-1)-cycle in Corollary 2.4.
Let $k-1 = (q+1)\lambda + r$, where $r, \lambda \in \mathbb{N}$ and $0 \le r < q+1$. We have two cases:

(1) If $r \neq 0$ then the vertices P_1 and P_{k-1} are not on the same line through $\mathcal{O}$. Hence the cycle

$$\mathcal{O} \to P_1 \to P_2 \to \cdots \to P_{k-2} \to P_{k-1} \to \mathcal{O}$$

is a k-cycle embedded in $AG(2,q)$.

(2) If $k-1 = (q+1)\lambda$ then $\mathcal{O}$ and the vertices P_1 and P_{k-1} are collinear. As $q > 1$ neither P_{k-3} nor P_{k-2} are on the line joining P_1 and $\mathcal{O}$. Note that $\mathcal{O}$, P_{k-2}, and $P_{(k-3)+(q+1)+1}$ are collinear.

Since $k \le q^2-2$, then $\lambda \le q-2$, and thus $(k-3)+(q+1)+1 \le q^2-1$. Hence, the cycle

$$\mathcal{O} \to P_1 \to \cdots \to P_{k-3} \to P_{(k-3)+(q+1)} \to P_{(k-3)+(q+1)+1} \to \mathcal{O}$$

is an embedding of C_k in $AG(2,q)$. □

We now prove a result equivalent to Theorem 2.6 for projective planes. Since $PG(2,q)$ is constructed from $AG(2,q)$, Theorem 2.6 also holds in $PG(2,q)$. It is also known that C_{q^2+q+1} embeds in $PG(2,q)$, this cycle is constructed from the Singer cycle of the plane (see [5] and [9]). For the pancyclicity of $PG(2,q)$, it remains to embed k-cycles with length $q^2 \le k \le q^2+q$. Our plan is to modify the embedding of C_{q^2-1} in $AG(2,q)$ described in Corollary 2.4.

First take the (q^2-1)-cycle embedded in $PG(2,q)$ described in Corollary 2.4 and shorten it to get the following path on q^2-q-1 vertices:

$$\mathcal{P} : P_1 \to P_2 \to \cdots \to P_{q^2-q-2} \to P_{q^2-q-1}.$$

Note that the $q+1$ affine points $P_{q^2-q}, P_{q^2-q+1}, \ldots, P_{q^2-2}$, and P_0 have not been used in this path. Since $P_{q^2-q} \in l_2$, $P_{q^2-q+1} \in l_3$, $\ldots$, $P_{q^2-2} \in l_q$, and $P_0 \in l_0$, we will re-label these points (to simplify the notation later) as follows:

$$P_{q^2-q} = Q_2,\ P_{q^2-q+1} = Q_3, \ldots,\ P_{q^2-2} = Q_q,\ P_0 = Q_0.$$

Hence, $Q_i \in l_i$ for all $i = 2, 3, \ldots, q, 0$. The points of $PG(2,q)$ not used in this path are the:

(a) $q+1$ points on ℓ_∞: $\{(0), (1), \ldots, (q-1), (q)\}$, where (i) is the point on ℓ_∞ incident with l_i, for all $i = 0, 1, \ldots, q$;
(b) $q+1$ affine points $Q_2, Q_3, \ldots, Q_q, Q_0$, and $\mathcal{O}$.

In terms of lines, we have not used the:

(i) line ℓ_∞;
(ii) $q+1$ lines through $\mathcal{O}$;
(iii) $q-1$ lines m_i, joining Q_i and $Q_{i+1 \bmod q+1}$, for all $i = 2, 3, \ldots, q$;
(iv) line m_0 connecting Q_0 with P_1, and the line m joining P_{q^2-q-1} and Q_2.

This yields $q+1$ lines not incident with $\mathcal{O}$ and we are now ready to prove pancyclicity in $PG(2,q)$.

Theorem 2.7. *If* Hypothesis J *holds, then $PG(2,q)$ is pancyclic.*

Proof. We will use the path and information just described. This is based on the cycle described in Corollary 2.4. Recall that we only need to construct k-cycles, for $q^2 \le k \le q^2+q$.

Note that m_i is parallel to $l_{i+2 \bmod q+1}$, and thus goes through $(i+2 \bmod q+1)$ for $i = 2, 3, \ldots, q$. Similarly, m_0 goes through (2), and the line joining m is incident with (3). Now, the following two paths:

$$Q_2 \xrightarrow{l_2} (2) \xrightarrow{\ell_\infty} (3) \xrightarrow{l_3} Q_3 \xrightarrow{m_2} (4) \xrightarrow{l_4} \ldots \xrightarrow{m_{q-1}} (q) \xrightarrow{l_q} Q_q \xrightarrow{m_q} Q_0$$

and

$$Q_0 \xrightarrow{m_0} \underbrace{P_1 \to \cdots \to P_{q^2-q-1}}_{\text{in } \mathcal{P}} \xrightarrow{m} Q_2$$

may be joined to create a (q^2+q-1)-cycle in $PG(2,q)$. Call this cycle $\mathcal{C}$.

Since l_0, l_1, and $\mathcal{O}$ have not been used in $\mathcal{C}$, a slight modification of $\mathcal{C}$ allows us to get a (q^2+q)-cycle. That cycle is:

$$Q_2 \xrightarrow{l_2} (2) \xrightarrow{\ell_\infty} (3) \xrightarrow{l_3} \ldots \xrightarrow{m_q} Q_0 \xrightarrow{l_0} \mathcal{O} \xrightarrow{l_1} \underbrace{P_1 \to \cdots \to P_{q^2-q-1}}_{\text{in } \mathcal{P}} \xrightarrow{m} Q_2.$$

It remains to construct k-cycles for $q^2 \le k \le q^2+q-2$.

First notice that if instead of the subpath

$$P_{q^2-q-1} \xrightarrow{m} Q_2 \xrightarrow{l_2} (2) \xrightarrow{\ell_\infty} (3) \xrightarrow{l_3} Q_3$$

in $\mathcal{C}$, we had

$$P_{q^2-q-1} \xrightarrow{m} Q_2 \xrightarrow{l_2} \mathcal{O} \xrightarrow{l_3} Q_3,$$

then we get a (q^2+q-2)-cycle in $PG(2,q)$. Similarly, notice that if instead of the subpath

$$P_{q^2-q-1} \xrightarrow{m} Q_2 \xrightarrow{l_2} (2) \xrightarrow{\ell_\infty} (3)$$

in $\mathcal{C}$, we had

$$P_{q^2-q-1} \xrightarrow{m} (3),$$

then we get a (q^2+q-3)-cycle in $PG(2,q)$ that does not use ℓ_∞. Call this cycle $\mathcal{C}'$.
Next, for $i = 4, \dots, q$, delete the path

$$(3) \xrightarrow{l_3} Q_3 \to \cdots \to (i) \xrightarrow{l_i} Q_i$$

from $\mathcal{C}'$ (keeping (3) and Q_i in $\mathcal{C}'$) and connect both (3) and Q_i with $\mathcal{O}$, using l_3 and l_i, to get the cycle

$$(3) \xrightarrow{l_3} \mathcal{O} \xrightarrow{l_i} \underbrace{Q_i \to \cdots \to P_{q^2-q-1}}_{\text{in } \mathcal{C}'} \xrightarrow{m} (3)$$

which has length $(q^2+q-3)-(2i-6)$. Since $i = 4, \dots, q$, this yields cycles of lengths $q^2+q-5, q^2+q-7, \dots, q^2-q+3$.
Now, for each of these cycles (for each $i = 4, \dots, q$), replace

$$(3) \xrightarrow{l_3} \mathcal{O} \xrightarrow{l_i} Q_i$$

by

$$(3) \xrightarrow{\ell_\infty} (2) \xrightarrow{l_2} O \xrightarrow{l_i} Q_i$$

to get cycles with length $q^2+q-4, \dots, q^2-q+4$. Therefore, the only cycle left to be constructed would be a q^2-cycle, in the case that $q = 3$. However, this case is easy to handle without using the arguments in this proof. $\square$

3. Wheels and Gears

The graphs studied in this section are all related to cycles in some way. Embeddings of these graphs will often rely on first finding an embedding of a specific cycle and then embedding the additional vertices and edges that make up the graph. Most of the cycles we will be interested in will be short but also have some additional desirable properties. We will focus on results for projective planes although many of the constructions are generalizable to affine planes.

Throughout this section, we will use the same notation as in the previous section, and use Π_q to denote a *generic* projective plane of order q (i.e. one that is not necessarily isomorphic to $PG(2,q)$).

3.1. **Wheel graphs.** We define the wheel graph W_n to be the graph on $n+1$ vertices formed by a cycle of length n and one additional vertex, called the *centre*, that is adjacent to every vertex in the cycle. Hence, the centre of the wheel has degree n. Since no vertex of a graph embedded in Π_q can contain a vertex of degree greater than $q+1$, we see immediately that $n \leq q+1$ if $W_n \hookrightarrow \Pi_q$. We now show by construction that W_{q+1} can indeed by embedded in Π_q.

Theorem 3.1. *$W_n \hookrightarrow \Pi_q$ if and only if $3 \leq n \leq q+1$.*

Proof. Since having $W_n \hookrightarrow \Pi_q$ implies $n \leq q+1$, we proceed to construct wheels for all $n \leq q+1$.

We know that $K_{q+1} \hookrightarrow \Pi_q$, when q is odd, and $K_{q+2} \hookrightarrow \Pi_q$, when q is even (see [5]). This implies that the result is obtained except, possibly when $n = q+1$ and q is odd.

Assume q is odd, let $\mathcal{O}$ be any point of Π_q, and $\ell = \{P_1, \ldots, P_{q+1}\}$ be any line in Π_q not incident with $\mathcal{O}$. Denote by ℓ_i the line joining $\mathcal{O}$ and P_i. The points $P_1, P_3, P_5, \ldots, P_q$ are vertices on the $(q+1)$-cycle of the wheel. The edges connecting these points and $\mathcal{O}$ are the corresponding ℓ_i's. The other $(q+1)/2$ vertices will be taken from the other ℓ_i's. Let m be a line through P_1, different from ℓ and ℓ_1. Now choose Q_{2i} to be the point on $\ell_{2i} \cap m$ for $i = 1, \ldots, (q+1)/2$. Note that none of the Q_i's can be on ℓ.

To create the path

$$P_1 \to Q_2 \to P_3 \to Q_4 \to \cdots \to P_q,$$

we need to show that the lines connecting P_i with $Q_{i+1 \bmod q+1}$ and Q_j with $P_{j+1 \bmod q+1}$ are all distinct. This is clear because if the line connecting P_i and $Q_{i+1 \bmod q+1}$ were equal to the line connecting P_j and $Q_{j+1 \bmod q+1}$, then P_i and P_j would be on this line. Therefore, either we have the trivial case, $i = j$, or this line must be ℓ, but ℓ does not contain any of the Q_i's. A similar argument shows that all the lines needed in this path are distinct. Now, if we extend this path into a cycle by joining P_q with Q_{q+1} as done above, we would encounter the problem of having the line joining Q_{q+1} with P_1 being m, which has already been used. So, we choose a point $T_{q+1} \in \ell_{q+1}$ such that the line t, through T_{q+1} and P_1, is different from m and ℓ. Since every line used in the path above goes through a point P_i, the lines t and s (joining T_{q+1} and P_q) are different from all others. We get the $(q+1)$-cycle

$$P_1 \to Q_2 \to P_3 \to Q_4 \to \cdots \to P_q \xrightarrow{s} T_{q+1} \xrightarrow{t} P_1.$$

Since, all the vertices in the cycle connect to $\mathcal{O}$ by using different lines, the vertices $\{P_1, Q_2, P_3, Q_4, \ldots, P_q, Q_{q+1}, \mathcal{O}\}$ form an embedding of W_{q+1}, as desired. □

3.2. **Gear graphs.** We define a *gear graph*, G_n, to be a graph on $2n+1$ vertices and $3n$ edges. The graph consists of a $2n$-cycle, and a centre vertex that is adjacent to every other vertex in the $(2n)$-cycle. Note that no gear graph can embed in Π_2, since the smallest gear graph has 9 edges and there are only 7 lines in such a plane. For $q = 3$ and $q = 4$, the only possible embeddings are $G_3 \hookrightarrow \Pi_3$, $G_3 \hookrightarrow \Pi_4$, $G_4 \hookrightarrow \Pi_4$, and $G_5 \hookrightarrow \Pi_4$. These are all easy to verify, and no further details will be provided. From now on, we assume $q > 4$.

Since the centre of G_n has degree n, we want to prove that $G_n \hookrightarrow \Pi_q$, for all $3 \leq n \leq q + 1$.

Lemma 3.2. *$G_n \hookrightarrow \Pi_q$, for all $3 \leq n \leq \lfloor (q + 1)/2 \rfloor$.*

Proof. If n is even, then $G_{n/2}$ is a subgraph of W_n. The result follows from the fact that $W_n \hookrightarrow \Pi_q$ for all $3 \leq n \leq q + 1$ (Theorem 3.1). □

To embed larger gears in Π_q, we need to construct a very specific family of cycles and the corresponding gears. This will all described in the proof of our next result.

Theorem 3.3. *$G_n \hookrightarrow \Pi_q$, for all $3 \leq n \leq q$.*

Proof. The discussion for $q = 2, 3, 4$ was settled at the beginning of this subsection. For $q > 4$, Lemma 3.2 proves the theorem for all $3 \leq n \leq \lfloor (q + 1)/2 \rfloor$. Also, it is easy to verify that $G_4 \hookrightarrow \Pi_5$. For $n > \lfloor (q + 1)/2 \rfloor$ we will give explicit constructions. First, recall that two distinct paths

$$\mathcal{P}_{P_0} : P_0 \to P_1 \to \cdots \to P_q$$
$$\mathcal{P}_{Q_0} : Q_1 \to Q_2 \to \cdots \to Q_q$$

constructed at the beginning of Section 2 are disjoint in terms of both points and lines as long as $P_0 \neq Q_0$ (two points different from $\mathcal{O}$ on l_0). From now on, fix P_0 and given $n > 4$, we will choose an appropriate Q_0 that will allow us to create a $(2n)$-cycle out of $\mathcal{P}_{P_0}$ and $\mathcal{P}_{Q_0}$.

Let n be even, then shorten $\mathcal{P}_{P_0}$ to

$$\mathcal{P}'_{P_0} : P_0 \to P_1 \to \cdots \to P_{n-2}.$$

Note that since $n - 2 \leq q - 2$, no lines parallel to l_0 or l_1 have been used in the construction of this path. Since $q > 3$, there are at least 2 lines through (0) different from $l_0 + P_{n-2}$, $l_0 + P_1$, and ℓ_∞, each of these lines intersect l_1 at a point different from P_1. Now, there are $q - 1$ lines through (1) different from $l_1 + P_0$ and ℓ_∞, each of these lines intersect l_{n-1} at a point different from $l_{n-1} \cap (l_1 + P_0)$. Choose $Q_0 \in l_0$ so that (the points on $\mathcal{P}_{Q_0}$) $Q_{n-1} \neq l_{n-1} \cap (l_1 + P_0)$, $Q_2 \neq P_2$, and $Q_2 \notin l_0 + P_{n-2}$. We get the following $(2n)$-cycle.

$$(1) \xrightarrow{l_1 + P_0} \underbrace{P_0 \to \cdots \to P_{n-2}}_{\text{in } \mathcal{P}_{P_0}} \xrightarrow{l_0 + P_{n-2}} (0) \ldots$$

$$\ldots \xrightarrow{l_0 + Q_1} \underbrace{Q_1 \to \cdots \to Q_{n-1}}_{\text{in } \mathcal{P}_{Q_0}} \xrightarrow{l_1 + Q_{n-1}} (1).$$

To create G_n, join $\mathcal{O}$ with $P_0, P_2, \ldots, P_{n-2}, Q_1, Q_3, \ldots, Q_{n-1}$.

When n is odd we proceed similarly. Consider the path,

$$\mathcal{P}'_{P_0} : P_0 \to P_1 \to \cdots \to P_{n-1}.$$

Notice that no lines through (0) or (1) have been used and that P_{n-3} must be different from one of $(l_1 + P_0) \cap l_{n-3}$ and $(l_n + P_0) \cap l_{n-3}$. We will say that $P_{n-3} \neq (l_k + P_0) \cap l_{n-3}$, where k is either 0 or n. So, since $P_{n-3} \neq (l_k + P_0) \cap l_{n-3}$ we choose Q_0 so that $Q_{n-3} = (l_k + P_0) \cap l_{n-3}$. Now observe neither $\mathcal{P}_{P_0}$ or $\mathcal{P}_{Q_0}$ can go through Q_0. Because of l_0, there are at least two lines through Q_0 that have not been used so far. Let T be a point on l_n such that T is on one of the lines through Q_0 that are available, and $T \neq (l_0 + P_{n-1}) \cap l_n$. Note that

$$m = \overleftrightarrow{Q_0 T}$$

cannot be equal to the lines $l_k + P_0$, or $l_0 + T$. We get the following $(2n)$-cycle:

$$Q_{n-3} \xrightarrow{l_k + P_0} \underbrace{P_0 \to \cdots \to P_{n-1}}_{\text{in } \mathcal{P}_{P_0}} \xrightarrow{l_0 + P_{n-1}} (0) \xrightarrow{l_0 + T} T \xrightarrow{m} \underbrace{Q_0 \to \cdots \to Q_{n-3}}_{\text{in } \mathcal{P}_{Q_0}}$$

where $l_0 + T$ could be ℓ_∞.

To create G_n we join $\mathcal{O}$ with $P_0, P_2, \ldots, P_{n-1}, T, Q_1, \ldots, Q_{n-4}$. We used $n > 4$ in order to get that $1 \leq n - 4$, and thus that the last selection of vertices (connected with $\mathcal{O}$) is well defined. $\square$

Thus far, we know we can embed gear graphs, G_n, where n is any integer between 3 and q. The next step in embeddings of gear graphs is to determine the largest gear that can be embedded.

Theorem 3.4. *Let $q > 4$. Then $G_{q+1} \hookrightarrow \Pi_q$. Furthermore, this is the largest gear that can be embedded in Π_q.*

Proof. First, notice that G_{q+1} is the largest possible gear that can be embedded in Π_q because the degree of the centre of G_{q+1} is $q+1$, which is the largest allowed in Π_q.

To show that G_{q+1} actually embeds in Π_q, we need to construct a cycle of length $2(q+1)$ without using $\mathcal{O}$ and any of the $q+1$ lines through $\mathcal{O}$. The cycle needs to be constructed in such a way that we are able to connect every other vertex of the cycle to the point $\mathcal{O}$. The construction of this cycle depends on the parity of the order of the plane.

Case 1: (q is even).

Assume q is even. Choose $q+1$ points $P_i \in l_i \setminus \{\mathcal{O}(i)\}$ for $i = 0, 1, \ldots, q$. Choose P_1 arbitrarily, next choose P_3 such that $P_3 \notin l_2 + P_1$, then choose P_5 such that $P_5 \notin l_4 + P_3$, etc. In general, choose $P_i \notin l_{i-1} + P_{i-2}$, for all $i = 1, 3, \ldots, q-1$ (only for i odd). Now choose P_0 such that $P_0 \notin l_q + P_{q-1}$. As done before, we choose $P_i \notin l_{i-1} + P_{i-2}$ for $i = 2, 3, \ldots, q$ (now only for i even) taking care with the choice of P_q that $P_1 \notin l_0 + P_q$. If for the chosen P_q, $P_1 \in l_0 + P_q$ then choose a different P_q. We can do this because the only condition to choose P_q was that $P_q \notin l_{q-1} + P_{q-2}$. It follows that we get the $(2q+2)$-cycle:

$$(0) \xrightarrow{l_0+P_1} P_1 \xrightarrow{l_2+P_1} (2) \to \cdots \to P_{q-1} \xrightarrow{l_q+P_{q-1}} (q) \xrightarrow{l_q+P_0} P_0 \xrightarrow{l_1+P_0} (1) \ldots$$
$$\cdots \to (1) \xrightarrow{l_1+P_2} P_2 \to \cdots \to (q-1) \xrightarrow{l_{q-1}+P_q} P_q \xrightarrow{l_0+P_q} (0).$$

We obtain G_{q+1} by joining $\mathcal{O}$ with $(0), (2), (4), \ldots (q), (1), (3), \ldots, (q-1)$ using the $q+1$ lines through $\mathcal{O}$.

Note that this proof also works for $q = 4$.

Case 2: (q is odd).

We perform a similar construction to the one for q even. Choose P_1, $P_3, \ldots, P_q$ as above, then begin by arbitrarily choosing P_2, and continue in the same fashion to get $P_2, P_4, \ldots, P_{q-1}$. In case $P_1 \in l_q + P_{q-1}$, choose a different P_{q-1}. Hence the line $l_q + P_1$ has not been used yet. Now we want to find a point T such that $T \notin l_0 + P_q$, $T \notin l_1 + P_2$, $T \notin l_0$, $T \notin l_1$, and that $T \neq P_i$ for $i = 1, 2, \ldots, q$. This point will be connected to (0) and (1) to create the cycle we need.

There are $q-2$ lines through (0) different from $l_0 + P_q$, l_0, and ℓ_∞. These $q-2$ lines may be used to connect (0) with $q(q-2)$ distinct points of Π_q. Similarly, there are $q-2$ lines through (1) different from $l_1 + P_2$, l_1, and ℓ_∞. It follows that there are at least

$$q(q-2) - 2(q-1) = q(q-4) + 2 \geq q + 2$$

points that can be reached simultaneously by lines through (0) or (1), different from the six lines to be avoided. Of these points, at most q could be a P_i. Hence, there are at least two possibilities to choose T from and we get the following $(2q+2)$-cycle:

$$(1) \xrightarrow{l_1+P_2} P_2 \xrightarrow{l_3+P_2} (3) \to \cdots \to P_{q-1} \xrightarrow{l_q+P_{q-1}} (q) \xrightarrow{l_q+P_1} P_1 \xrightarrow{l_2+P_1} (2) \ldots$$
$$\cdots \to (2) \xrightarrow{l_2+P_3} P_3 \to \cdots \to (q-1) \xrightarrow{l_{q-1}+P_q} P_q \xrightarrow{l_0+P_q} (0) \xrightarrow{l_0+T} T \xrightarrow{l_1+T} (1).$$

We obtain G_{q+1} by joining $\mathcal{O}$ with $(1), (3), \ldots, (q), (2), (4), \ldots (q-1), (0)$ using the $q+1$ lines through $\mathcal{O}$.

□

We have also invstigated using the techniques to study embeddings of other cycle-related families of graphs such as helm graphs and prism graphs, obtaining similar results. Current work focuses on developing a theory of embeddings in finite projective spaces.

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