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NORMALITY OF ONE-MATCHING SEMI-CAYLEY GRAPHS OVER FINITE ABELIAN GROUPS WITH MAXIMUM DEGREE THREE

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Abstract. A graph Γ is said to be a semi-Cayley graph over a group G if it admits G as a semiregular automorphism group with two orbits of equal size. We say that Γ is normal if G is a normal subgroup of $\mathrm{Aut}(\Gamma)$. We prove that every connected intransitive one-matching semi-Cayley graph, with maximum degree three, over a finite abelian group is normal and characterize all such nonnormal graphs.

1. Introduction

Throughout this paper, groups are finite and graphs are finite, connected, simple, and undirected. For the graph-theoretic and group-theoretic terminology not defined here, we refer the reader to [7, 24]. Let G be a permutation group on Ω and $\alpha \in \Omega$. Denote by G_α the stabilizer of α in G, that is, the subgroup of G fixing the point α. We say that G is semiregular on Ω if $G_\alpha = 1$ for every $\alpha \in \Omega$ and regular if G is transitive and semiregular. Let G be a group and S a subset of G not containing the identity element 1_G. The Cayley digraph $\Gamma = \mathrm{Cay(G,S)}$ of G with respect to S has vertex set G and arc set $\{(g, sg) \mid g \in G, s \in S\}$. If $S = S^{-1}$ then $\mathrm{Cay(G,S)}$ can be viewed as an undirected graph, identifying an undirected edge with two directed edges (g, h) and (h, g). This graph is called the Cayley graph of G with

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respect to S. By a theorem of Sabidussi [22], a graph Γ is a Cayley graph over a group G if and only if there exists a regular subgroup of $\mathrm{Aut}(\Gamma)$ isomorphic to G.

There is a natural generalization of Sabidussi's Theorem. A graph Γ is called an n-Cayley graph over a group G if there exists an n-orbit semiregular subgroup of $\mathrm{Aut}(\Gamma)$ isomorphic to G. Undirected and loop-free 2-Cayley graphs are called *semi-Cayley* [4, 9], and also *bi-Cayley* by some authors [25]. n-Cayley graphs have played an important role in many classical fields of graph theory, such as strongly regular graphs [17, 20, 21, 9], Hamiltonian graphs [23], n-extendable graphs [11, 19], the spectrum of graphs [3, 2, 12], automorphisms [5, 1, 14, 25], and the connectivity of graphs [8, 18].

A graph Γ is called a *semi-Cayley graph* over a group G if $\mathrm{Aut}(\Gamma)$ admits a semiregular subgroup R_G isomorphic to G with two orbits (of equal size). Let Γ be a semi-Cayley graph over a group G. Then there exists subsets R, L, and S of G such that $R = R^{-1}$, $L = L^{-1}$, and $1 \notin R \cup L$ such that $\Gamma \cong \mathrm{SC}(G; R, L, S)$, where $\mathrm{SC}(G; R, L, S)$ is an undirected graph with vertices $G \times \{1, 2\}$ and its edge set consists of three sets (see [9, Lemma 2.1]):

$$\begin{aligned}
&\{\{(x,1),(y,1)\} \mid yx^{-1} \in R\} \quad \text{(right edges)},\\
&\{\{(x,2),(y,2)\} \mid yx^{-1} \in L\} \quad \text{(left edges)},\\
&\{\{(x,1),(y,2)\} \mid yx^{-1} \in S\} \quad \text{(spoke edges)}.
\end{aligned}$$

Furthermore, $R_G := \{\rho_g \mid g \in G\}$, where $\rho_g : G \times \{1,2\} \to G \times \{1,2\}$ and $(x,i)^{\rho_g} = (xg, i)$, $i = 1, 2$, is a semiregular subgroup of $\mathrm{Aut}(\mathrm{SC}(G; R, L, S))$ isomorphic to G with two orbits $G \times \{1\}$ and $G \times \{2\}$. A semi-Cayley graph $\Gamma = \mathrm{SC}(G; R, L, S)$ over a group G is called *normal* over G if R_G is a normal subgroup of $\mathrm{Aut}(\Gamma)$ (see [5, p. 42]) and it is called *one-matching* if $S = \{1\}$ (see [16, p. 603]). In this paper, we prove:

Theorem 1.1. *Let $\Gamma = \mathrm{SC}(G; R, L, \{1\})$ be a connected one-matching semi-Cayley graph over a finite abelian group $G \neq 1$ with $|R|, |L| \leq 2$. Then Γ is normal if and only if none of the following are satisfied (even after interchanging R and L)*

1. $|R| = |L| = 1$ *(so $G \cong \mathbb{Z}_2$ or $\mathbb{Z}_2^2$),*
2. $|R| = |L| = 2$ *and* $|R \cap L| = 1$ *(so $G \cong \mathbb{Z}_2^2$ or $\mathbb{Z}_2^3$),*
3. $R = L = \{a, a^{-1}\}$, *where* $o(a) = 4$ *(so $G = \langle a \rangle \cong \mathbb{Z}_4$),*
4. $R = \{a, b\}$, $L = \{c, c^{-1}\}$, *where* $o(a) = o(b) = 2$, $o(c) = 4$ *and* $G = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \cong \mathbb{Z}_2^2 \times \mathbb{Z}_4$,
5. $R = \{a, a^{-1}\}$, $L = \{b, ba^2\}$, *where* $o(a) = 4$, $o(b) = 2$ *and* $G = \langle a \rangle \times \langle b \rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$,

(6) $R = \{a, a^{-1}\}$, $L = \{a^k, a^{-k}\}$, *where* $o(a) = n$ *and* (n, k) *is one of the pairs* $(5, 2)$, $(8, 3)$, $(10, 2)$, $(10, 3)$, $(12, 5)$ *or* $(24, 5)$ *(so* $G \cong \mathbb{Z}_n$*),*

(7) $R = \{a, a^{-1}\}$, $L = \{a^3b, a^{-3}b\}$ *or* $L = \{a^2b, a^{-2}b\}$, *where* $o(a) = 10$, $o(b) = 2$ *and* $G = \langle a\rangle \times \langle b\rangle \cong \mathbb{Z}_{10} \times \mathbb{Z}_2$,

(8) $R = \{a, a^{-1}\}$, $L = \{ab, a^{-1}b\}$, *where* $o(a) = 4$, $o(b) = 2$ *and* $G = \langle a\rangle \times \langle b\rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$.

Furthermore, in all of the above cases, Γ *is transitive.*

For a graph Γ, we use $V(\Gamma)$, $E(\Gamma)$, $A(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ to denote its vertex set, edge set, arc set and its full automorphism group respectively. For $v \in V(\Gamma)$, $N(u)$ is the neighborhood of u in Γ, that is, the set of vertices adjacent to u in Γ. A graph Γ is called transitive if $\mathrm{Aut}(\Gamma)$ is transitive on $V(\Gamma)$, otherwise it is called intransitive. Also a graph Γ is said to be edge-transitive and arc-transitive (or symmetric) if $\mathrm{Aut}(\Gamma)$ acts transitively on $\mathrm{E}(\Gamma)$ and $\mathrm{A}(\Gamma)$, respectively.

2. Preliminaries

Let $\Gamma = \mathrm{SC}(G; R, L, \{1\})$ be a one-matching semi-Cayley graph over a finite group $G \neq 1$. Let $\Gamma_0 = \mathrm{SC}(G; L, R, \{1\})$ the graph obtained from interchanging the left and right edges of Γ. Then $\Gamma \cong \Gamma_0$. Furthermore, $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\Gamma_0)$ and also $R_G \trianglelefteq \mathrm{Aut}(\Gamma)$ if and only if $R_G \trianglelefteq \mathrm{Aut}(\Gamma_0)$. Hence, in studying the normality of Γ, we may assume that $|L| \leq |R|$. Moreover, since Γ is a normal over a group G if and if its complement Γ^c is normal over G, we may assume that Γ is connected or equivalently $G = \langle R \cup L\rangle$.

Let $\Gamma = \mathrm{SC}(G; R, L, \{1\})$ be a connected semi-Cayley graph over a finite abelian group G, and let A and V, be its automorphism group and vertex set, respectively. For each $\sigma \in \mathrm{Aut}(G)$ we define two maps

$$\begin{aligned} \varphi_\sigma &: V(\Gamma) \to V(\Gamma);\ (x, 1)^{\varphi_\sigma} = (x^\sigma, 1),\ (x, 2)^{\varphi_\sigma} = (x^\sigma, 2), \\ \psi_\sigma &: V(\Gamma) \to V(\Gamma);\ (x, 1)^{\psi_\sigma} = (x^\sigma, 2),\ (x, 2)^{\psi_\sigma} = (x^\sigma, 1). \end{aligned}$$

Set

$$\begin{aligned} X &:= \{\varphi_\sigma \mid \sigma \in \mathrm{Aut}(G), R^\sigma = R, L^\sigma = L\}, \\ Y &:= \{\psi_\sigma \mid \sigma \in \mathrm{Aut}(G), R^\sigma = L, L^\sigma = R\}, \end{aligned}$$

and let us denote $X \cup Y$ by $\mathrm{Aut}(G; R, L)$. Then $N_A(R_G) = R_G \rtimes \mathrm{Aut}(G; R, L)$ by [5, Theorem 1]. So $R_G \trianglelefteq A$ if and only if $A = R_G \rtimes \mathrm{Aut}(G; R, L)$ [5, Proposition 2 (1)]. Moreover, if $R_G \trianglelefteq A$, then $A_{(1,1)} = X$ and the converse holds if Γ is intransitive [5, Proposition 2 (2)]. Also if $R_G \trianglelefteq A$ then Γ is intransitive if and only if $A_{(1,1)} = \mathrm{Aut}(G; R, L)$ [5, Corollary 2.9]. Note that if $Y \neq \emptyset$, then Γ is transitive. So if $R_G \trianglelefteq A$

then Γ is transitive if and only if $Y \neq \emptyset$. Also, by the following lemma and above results, if Γ is intransitive or $Y \neq \emptyset$, then Γ is normal if and only if $A_{(1,1)} = X$. It is easy to see that $A_{(1,1)} \cap N_A(R_G) = A_{(1,2)} \cap N_A(R_G) = X$. In particular, if $R_G \trianglelefteq A$ then $A_{(1,1)} = A_{(1,2)} = X$. In what follows, unless otherwise stated, we keep the above notations and use the above results without referring them.

Lemma 2.1. *Let $Y \neq \emptyset$. Then Γ is normal if and only if $A_{(1,1)} = X$.*

Proof. If Γ is normal then $A_{(1,1)} = X$. Conversely, suppose that $A_{(1,1)} = X$. Let $\beta \in A$ be arbitrary. We have to show that $\beta \in N_A(R_G)$. Since $Y \neq \emptyset$ (and $Y \subseteq N_A(R_G)$), we may assume that $(1,1)^\beta \in G \times \{1\}$ (if $(1,1)^\beta \in G \times \{2\}$, then we replace β with βy for some $y \in Y$). Then after multiplying by an element of R_G, we may assume that $(1,1)^\beta = (1,1)$. So $\beta \in A_{(1,1)} = X \subseteq N_A(R_G)$. □

3. Proof of Theorem 1.1

Keeping the notations of previous section, recall that $\Gamma = \mathrm{SC}(G; R, L, \{1\})$ is a connected semi-Cayley graph over a finite abelian group $G \neq 1$ with $|L| \leq |R| \leq 2$, and A denotes the automorphism group of Γ. To prove Theorem 1.1, we consider all the possibilities for the orders of R and L and their intersection.

Let us start with the following lemma:

Lemma 3.1. *Let Γ be edge-transitive. Then it is nonnormal. Also if Γ is arc-transitive then Γ is nonnormal.*

Proof. It is enough to note that any element of the normalizer of R_G must map G-orbits to G-orbits but an element of A that takes a right edge or left edge to a spoke edge does not do this. Since every connected arc-transitive graph is edge-transitive, the second part is clear. □

Lemma 3.2. *Let $L = \emptyset$, $R \neq \emptyset$. Then Γ is intransitive and normal, and*

(1) *if $|R| = 1$ then $G \cong \mathbb{Z}_2$, $\Gamma \cong P_4$, $A \cong \mathbb{Z}_2$,*
(2) *if $|R| = 2$ then $G \cong \mathbb{Z}_n$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$, $n \geq 3$, and $A \cong D_{2|G|}$.*

Proof. (1) It is clear.

(2) Since Γ is connected and $L = \emptyset$, we have $G = \langle R \rangle \cong \mathbb{Z}_n$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$, for some $n \geq 3$. Hence $\mathrm{Cay}(G, R)$ is a $|G|$-cycle. By [5, Lemma 4.1], $A \cong \mathrm{Aut}(\mathrm{Cay}(G, R)) \cong D_{2|G|}$. □

Lemma 3.3. *Let $R = \{a\}$ and $L = \{b\}$. Then Γ is transitive and nonnormal and one of the following holds:*

(1) *$G \cong \mathbb{Z}_2$, $A \cong D_8$,*

(2) $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, $A \cong D_{16}$.

Proof. If $a = b$ then $G \cong \mathbb{Z}_2$ and otherwise $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. In both cases, Γ is a $2|G|$-cycle and so $A \cong D_{4|G|}$. Furthermore, in both cases $A_{(1,1)} \neq X$, which implies that both are nonnormal. $\square$

Lemma 3.4. *Let Γ be intransitive, $R \cap L = \emptyset$, and Γ_Ω be the quotient graph of Γ with respect to the one-matching set $\Omega = \{\{(g,1),(g,2)\} \mid g \in G\}$. Then $A \leq \mathrm{Aut}(\Gamma_\Omega)$, where Γ_Ω is a Cayley graph of R_G with respect to $S = \{\rho_r, \rho_l \mid r \in R, l \in L\}$ of valency $|R|+|L|$. In particular, if Γ_Ω is a normal Cayley graph of R_G then Γ is a normal semi-Cayley graph of R_G.*

Proof. We consider the action of A on Ω. Let K be the kernel of this action. Since Γ is intransitive, it implies that $K = 1$ and so $A \leq \mathrm{Aut}(\Gamma_\Omega)$. Clearly R_G acts transitively on $\mathrm{V}(\Gamma_\Omega)$. Now suppose that $\rho_h \in R_G$ and $\{(g,1),(g,2)\}^{\rho_h} = \{(g,1),(g,2)\}$. Therefore $(g,1)^{\rho_h} = (g,1)$ and $(g,2)^{\rho_h} = (g,2)$ and so $(gh,1) = (g,1)$. Thus $\rho_h = 1$ and R_G acts regularly on $\mathrm{V}(\Gamma_\Omega)$ and so Γ_Ω is a Cayley graph on R_G with respect to S. Also since $R \cap L = \emptyset$, it implies that Γ_Ω has valency $|R| + |L|$. $\square$

Lemma 3.5. *If $|R| = 2$ and $|L| = 1$ then Γ is normal.*

Proof. Let $R = \{a,b\}$ and $L = \{c\}$. If $c = a$ or $c = b$ then $a^2 = b^2 = 1$, $A = R_G \cong \mathbb{Z}_2^2$ and so Γ is normal. Hence, we may assume that $c \neq a, b$. Suppose, towards a contradiction, that Γ is nonnormal. Then $R \cap L = \emptyset$. Let $\Omega = \{\{(g,1),(g,2)\} \mid g \in G\}$ and Γ_Ω be the Cayley graph of R_G with respect to $S = \{\rho_a, \rho_b, \rho_c\}$. Since Γ is nonnormal, Lemma 3.4 and [6, Theorem 1.2] imply that one of the following occurs:

(1) $o(a) = 4$, $b = a^{-1}$ and $c = a^2$,
(2) $o(a) = 4$, $b = a^{-1}$, $c^2 = 1$ and $c \notin \langle a \rangle$,
(3) $o(a) = 6$, $b = a^{-1}$ and $c = a^3$.

In the first case, $A \cong D_8$ and Γ is normal, in the second case $A \cong \mathbb{Z}_2 \times D_8$ and Γ is normal, and in the last case, $A \cong D_{12}$ and Γ is normal. Hence we get a contradiction. $\square$

Lemma 3.6. *Let $R = L$, $|R| = 2$. Then Γ is transitive and the following are equivalent:*

(1) *Γ is normal,*
(2) *Γ is not arc-transitive,*
(3) *$R = \{a, a^{-1}\}$, where a is of order $k > 2$ and $k \neq 4$.*

Proof. It is easy to see that Γ is isomorphic to the n-prism graph, the cartesian product of an n-cycle with a path with two vertices, where

$n = |G|$, which is isomorphic to a Cayley graph on the dihedral group $D_{2n} = \langle s, t \mid s^n = t^2 = (st)^2 = 1\rangle$, with respect to $S = \{s, s^{-1}, t\}$. Hence Γ is transitive.

By Lemma 3.1, (1) implies (2). Now suppose that (2) holds. If $R = \{b, c\}$, where $b^2 = c^2 = 1$, then Γ is isomorphic to the three dimensional hypercube, which is arc-transitive, a contradiction. Hence $R = \{a, a^{-1}\}$, where a is of order $k > 2$. Hence $G = \langle a\rangle \cong \mathbb{Z}_k$. Hence, by [16, Theorem 1.1], $k \neq 4$. Thus (2) implies (3). To complete the proof, it is enough to prove that (3) implies (1). Suppose (3) holds. Then $G \cong \mathbb{Z}_k$ and it is easy to see that Γ is isomorphic to the generalized Petersen graph $GP(k, 1)$ (see [10]). Also by [10, Theorems 1 and 2] $GP(k, 1)$ is vertex transitive and $A \cong D_{2k} \times \mathbb{Z}_2$. Hence Γ is vertex transitive and so $|A| = |A_{(1,1)}|2k$. This shows that $|A_{(1,1)}| = 2$. Since $R = L = \{a, a^{-1}\}$, $Y \neq \emptyset$ and $|X| \geq 2$. Since $X \leq A_{(1,1)}$, we have $X = A_{(1,1)}$. Hence Γ is normal, i.e. (1) holds. This completes the proof. $\square$

Lemma 3.7. *Let $|R| = |L| = 2$, $|R \cap L| = 1$. Then Γ is transitive and nonnormal. Also one of the following holds:*

(1) $G = \langle a, b\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, $R = \{a, b\}$ *and* $L = \{ab, b\}$,
(2) $G = \langle a, b, c\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $R = \{a, b\}$ *and* $L = \{b, c\}$.

Proof. Since $R = R^{-1}$ and $L = L^{-1}$, both R and L consist of two involutions. Assume that $R = \{a, b\}$ and $L = \{b, c\}$. Since $G = \langle a, b, c\rangle$, if $c = ab$, $G = \langle a, b\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, otherwise $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. In the former case,

$$\sigma = ((ab, 1), (b, 2))((a, 2), (ab, 2))((a, 1), (1, 2)) \in A_{(1,1)}$$

but $\sigma \notin X$. Therefore Γ is not normal. In the latter,

$$\begin{aligned}\sigma = {} & ((a, 1), (1, 2))((a, 2), (c, 2))((ab, 1), (b, 2)) \\ & ((ab, 2), (bc, 2))((ac, 2), (c, 1)) \in A_{(1,1)}\end{aligned}$$

but $\sigma \notin X$. So Γ is not normal. Also in both cases we see that Γ is transitive. $\square$

Lemma 3.8. *Let $|R| = |L| = 2$, $R \cap L = \emptyset$. If $R = \{a, b\}$, where $a^2 = b^2 = 1$, then one of the following holds:*

(1) $L = \{ab, c\}$, *where $c^2 = 1$. In this case $G \cong \mathbb{Z}_2^3$, Γ is intransitive and normal.*
(2) $L = \{c, d\}$, *where $c^2 = d^2 = 1$. In this case $G \cong \mathbb{Z}_2^4$ and Γ is transitive and normal.*

(3) *$L = \{c, c^{-1}\}$, where c is of order $n > 2$. In this case, $G \cong \mathbb{Z}_2^2 \times \mathbb{Z}_n$, and Γ is normal if and only if Γ is intransitive if and only if $n \neq 4$.*
(4) *$L = \{c, c^{-1}\}$, where $o(c) = n > 2$ is even, and $b = c^{n/2}$. In this case, $G \cong \mathbb{Z}_n \times \mathbb{Z}_2$, and Γ is normal and intransitive.*
(5) *$L = \{c, c^{-1}\}$, where $o(c) = n > 2$ is even, and $b = ac^{n/2}$. In this case, $G \cong \mathbb{Z}_n \times \mathbb{Z}_2$ and Γ is normal if and only if Γ is intransitive if and only if $n \neq 4$.*

Proof. It is obvious that the possibilities of L are exactly the same given in (1)–(5).

(1) In this case, $G = \langle a, b, c\rangle \cong \mathbb{Z}_2^3$. Then, by GAP [13], Γ is intransitive and normal.

(2) In this case, $G \cong \mathbb{Z}_2^4$ and by GAP, $A \cong (D_8 \times D_8) \rtimes \mathbb{Z}_2$, Γ is transitive and normal.

(3) Suppose that $L = \{c, c^{-1}\}$, where c is an element of order $n > 2$. Then $G \cong \mathbb{Z}_2^2 \times \mathbb{Z}_n$. We prove that Γ is normal if and only if it is intransitive if and only if $n \neq 4$.

If $n = 4$ then by GAP, Γ is transitive. Conversely, suppose that Γ is transitive. Then there exists $\alpha \in \mathrm{Aut}(\Gamma)$ such that $(1,1)^\alpha = (1,2)$. Then α maps the 4-cycle

$$(1,1), (b,1), (ab,1), (a,1), (1,1)$$

to a 4-cycle including the point $(1,2)$. Since $R \cap L = \emptyset$, we have $(a,1)^\alpha, (b,1)^\alpha \neq (1,1)$. Hence $(a,1)^\alpha, (b,1)^\alpha \in \{(c,2), (c^{-1},2)\}$, which implies that $(ab,1)^\alpha = (c^2,2) = (c^{-2},2)$. This means that $n = 4$.

Let Γ is normal. Then, since $Y = \emptyset$, Γ is intransitive. Conversely, suppose that Γ is intransitive. So $n \neq 4$, by the above discussion. Now [6, Theorem 1.2] and Lemma 3.4, imply that Γ is normal.

(4) In this case $G \cong \mathbb{Z}_n \times \mathbb{Z}_2$. If $n = 4$, then by GAP, Γ is intransitive and normal. Hence, we may assume that $n \neq 4$. So, by a similar argument of the previous case, Γ is intransitive. Suppose, towards a contradiction, that Γ is nonnormal. Then, by Lemma 3.4 and [6, Theorem 1.2], $n = 6$. Now, by GAP, $A \cong \mathbb{Z}_2^2 \times S_3$ which implies that Γ is normal, a contradiction.

(5) In this case $G \cong \mathbb{Z}_n \times \mathbb{Z}_2$. If $n = 4$ then Γ is transitive and nonnormal. Let $n \neq 4$. Then by a similar argument of the case (3), Γ is intransitive. By the same argument in case (4), if Γ is nonnormal, then $n = 6$, which implies that, by GAP, $A \cong D_8 \times \mathbb{Z}_3$ and Γ is normal. $\square$

Let S be an inverse-closed subset of a group G not containing the identity element of G. Recall that a permutation φ of G is a colour-preserving automorphism of $\mathrm{Cay}(G,S)$ if and only if we have $(xs)^\varphi \in \{x^\varphi s^{\pm 1}\}$ for each $x \in G$ and $s \in S$ [15, p. 190].

Lemma 3.9. *Let $R = \{a, a^{-1}\}$ and $L = \{b, b^{-1}\}$, $o(a), o(b) \geq 3$, and $R \cap L = \emptyset$. If Γ is intransitive then it is normal.*

Proof. Suppose, towards a contradiction, that there exists $\alpha \in A$ that does not normalize R_G. Since Γ is intransitive, there is a permutation σ of G such that $(g,i)^\alpha = (g^\sigma, i)$ for all $g \in G$ and $i = 1, 2$. There is a natural colouring of $\mathrm{Cay}(G, \{a^{\pm 1}, b^{\pm 1}\})$ with two colours, where a-edges have one colour and b-edges have the other colour. Then σ is a colour-preserving automorphism of $\mathrm{Cay}(G, \{a^{\pm 1}, b^{\pm 1}\})$ because α is an automorphism of Γ, which means that $(ga)^\sigma \in \{ga^{\pm 1}\}$ and $(gb)^\sigma \in \{gb^{\pm 1}\}$.

Since α does not normalize R_G (and G is 2-generated), we know from [15, Proposition 4.1] that G has a direct factor that is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4$. So $o(a)$ and $o(b)$ are even. Therefore $o(a), o(b) \neq 3$ and so $o(a), o(b) \geq 4$. If $o(a) = o(b) = 4$ then, by GAP, Γ is a transitive graph which is a contradiction. So we may assume that $o(b) > 4$.

By composing with a translation, we may assume that σ fixes 1. We may also assume that σ fixes a by composing with inversion if necessary. Then $(a^k)^\sigma = a^k$ for all k.

We claim that we may assume $b^\sigma = b$. Suppose $b^\sigma \neq b$, so $b^\sigma = b^{-1}$. Then σ is the identity on $\langle a \rangle$ but inverts $\langle b \rangle$, which implies that $|\langle a \rangle \cap \langle b \rangle| \leq 2$. Therefore there is an automorphism of G agrees with σ on $\langle a \rangle \cup \langle b \rangle$. By composing with this automorphism, we have $b^\sigma = b$ as desired.

Since σ does not normalize R_G, we know that σ is not the identity permutation and so there is some minimal $k > 0$ such that $(a^k b)^\sigma = a^k b^{-1}$. Since $a^{k-1}b$ is adjacent to $a^k b$ via an a-edge, we have $a^{k-1}ba = a^k b^{-1}$ or $a^{k-1}ba^{-1} = a^k b^{-1}$. The first implies that $b^2 = 1$ which contradicts the fact that $o(b) \geq 4$. The second implies $a^2 = b^2$. Since $|\langle a \rangle \cap \langle b \rangle| \leq 2$, we have $o(b) \leq 4$ which contradicts the fact $o(b) > 4$. □

Corollary 3.10. *Let $R = \{a, a^{-1}\}$ and $L = \{b, b^{-1}\}$, where $o(a) \neq o(b)$ and $R \cap L = \emptyset$, then Γ is normal if and only if Γ is intransitive.*

Proof. One direction is clear by Lemma 3.9. Let Γ be normal and suppose, towards a contradiction, that Γ is transitive. Then there exists $\alpha \in A$ such that $(1,1)^\alpha = (1,2)$. Since Γ is normal, there exists $\sigma \in \mathrm{Aut}(G)$ such that $\alpha = \psi_\sigma$, $R^\sigma = L$ and $L^\sigma = R$, which implies that $o(a) = o(b)$, a contradiction. □

Lemma 3.11. *Let $|R| = |L| = 2$ and $R \cap L = \emptyset$. If $R = \{a, a^{-1}\}$, where $o(a) = n \geq 3$, then, perhaps after interchanging R and L, one of the following holds:*

(1) *$L = \{a^{n/2}, b\}$, where n is even, $b^2 = 1$ and $b \notin \langle a \rangle$. In this case $G \cong \mathbb{Z}_n \times \mathbb{Z}_2$, Γ is normal and intransitive.*
(2) *$L = \{b, ba^{n/2}\}$, where n is even, $b^2 = 1$ and $b \notin \langle a \rangle$. In this case $G \cong \mathbb{Z}_n \times \mathbb{Z}_2$, if $n = 4$ then Γ is transitive and nonnormal and otherwise Γ is normal and intransitive.*
(3) *$L = \{b, c\}$, where $b^2 = c^2 = 1$, $\langle a \rangle \cap \langle b, c \rangle = 1$ and $b, c \notin \langle a \rangle$. In this case $G \cong \mathbb{Z}_n \times \mathbb{Z}_2^2$, if $n = 4$ then Γ is transitive and nonnormal and otherwise Γ is normal and intransitive.*
(4) *$L = \{a^k, a^{-k}\}$, for some $k \geq 2$. In this case, $G \cong \mathbb{Z}_n$. Furthermore, Γ is nonnormal if and only if $(n, k) = (5, 2), (8, 3), (10, 2), (10, 3), (12, 5), (24, 5)$. Also if Γ is nonnormal then Γ is transitive.*
(5) *$L = \{b, b^{-1}\}$, $b \notin \langle a \rangle$, and $\langle b \rangle \cap \langle a \rangle \neq 1$. In this case, Γ is nonnormal if and only if $L = \{a^3y, a^{-3}y\}$ or $L = \{a^2y, a^{-2}y\}$, where $o(a) = 10$, $o(y) = 2$, and $G = \langle a \rangle \times \langle y \rangle \cong \mathbb{Z}_{10} \times \mathbb{Z}_2$, or $L = \{ay, a^{-1}y\}$, where $o(a) = 4$, $o(y) = 2$, and $G = \langle a \rangle \times \langle y \rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$. Also if Γ is nonnormal then Γ is transitive.*
(6) *$L = \{b, b^{-1}\}$, and $\langle b \rangle \cap \langle a \rangle = 1$. In this case, $G \cong \mathbb{Z}_n \times \mathbb{Z}_l$, where $l = o(b)$ and Γ is normal. Furthermore, Γ is transitive if and only if $o(a) = o(b)$.*

Proof. It is easy to see that the only possibilities of L are the cases (1)–(6). Since $\mathrm{SC}(G; R, L, \{1\}) \cong \mathrm{SC}(G; L, R, \{1\})$, by the last three cases of Lemma 3.8, cases (1), (2), and (3) are clear.

(4) In this case, Γ is isomorphic to the generalized Petersen graph $GP(n, k)$. Let Γ is nonnormal and suppose, for the contrary,

$$(n, k) \notin \{(5, 2), (8, 3), (10, 2), (10, 3), (12, 5), (24, 5)\}.$$

Then Γ is not edge-transitive by [10, Lemma 3 and Theorem 2]. Hence $|A| = 4n$ [10, Theorem 1 and Theorem 2]. Since Γ is nonnormal, Lemma 3.9 and [7, p. 105] imply that $k^2 \equiv \pm 1 \pmod{n}$. Hence $(k, n) = 1$. Let $\sigma_1, \sigma_2, \sigma_3 : G \to G$ be the maps by the rules $(a^i)^{\sigma_1} = a^{ik}$, $(a^i)^{\sigma_2} = a^{-ik}$, and $(a^i)^{\sigma_3} = a^{-i}$. Then these three maps are automorphisms of G. Furthermore, $\psi_{\sigma_1}, \psi_{\sigma_2}, \varphi_{\sigma_3} \in \mathrm{Aut}(G; R, L)$. So $|\mathrm{Aut}(G; R, L)| \geq 4$, which implies that $A = N_A(R_G)$ i.e., Γ is normal, a contradiction.

Conversely, suppose that

$$(n, k) \in \{(5, 2), (8, 3), (10, 2), (10, 3), (12, 5), (24, 5)\}.$$

Then Γ is arc-transitive by [7, p. 105] and so it is nonnormal by Lemma 3.1. If Γ is nonnormal then it is transitive by Lemma 3.9.

(5) Let $L = \{b, b^{-1}\}$, $b \notin \langle a\rangle$, $a \notin \langle b\rangle$, and $\langle b\rangle \cap \langle a\rangle \neq 1$. If Γ is intransitive, then by Lemma 3.9, Γ is normal. Hence, we may assume that Γ is transitive. If Γ is arc-transitive then by [26, Proposition 5.1(2)], it is the unique arc-transitive cubic graph of order 40, denoted by F040A in the Foster Census, $G = \langle x\rangle \times \langle y\rangle \cong \mathbb{Z}_{10} \times \mathbb{Z}_2$, and we may assume that $a = x$ and $b \in \{x^3y, x^2y\}$. Then Γ is nonnormal by Lemma 3.1. So we may now assume that Γ is not arc-transitive. Then by [26, Theorem 1.1], up to isomorphism, one of the following occurs:

(1) $G = \langle x\rangle \times \langle y\rangle \cong \mathbb{Z}_{mk} \times \mathbb{Z}_m$, $k \geq 3, m \geq 1$, where $(m, k, t) = (1, 10, 2)$ or $(t, mk) = 1$ and $t^2 \equiv -1 \pmod k$ and we may assume that $a = x$, $b = x^t y$. Clearly $m = 1$ is impossible, because $b \notin \langle a\rangle$. Also $\mathrm{Aut}(\Gamma) \cong R_G \rtimes \mathbb{Z}_4$ [26, Theorem 5.5(3)], which implies that Γ is normal.

(2) $G = \langle x\rangle \times \langle y\rangle \cong \mathbb{Z}_{mk} \times \mathbb{Z}_m$, $km \geq 3$, and $m \geq 1$, where $(t, mk) = 1$, $t^2 \equiv 1 \pmod k$ and Γ is a Cayley graph over $G \rtimes \langle z\rangle$ for some involution z, [26, Theorem 5.2(5)]. Furthermore, we may assume that $a = x$, $b = x^t y$. Clearly $m = 1$ is impossible because $b \notin \langle a\rangle$.

Since Γ is connected and transitive but not edge-transitive, every automorphism of Γ maps G-orbits to G-orbits. If $(m, k) = (2, 2)$, then $L = \{ay, a^{-1}y\}$, $G \cong \mathbb{Z}_4 \times \mathbb{Z}_2$, and Γ is nonnormal over G by GAP. Hence, we may assume that $(m, k) \neq (2, 2)$. Then we claim that Γ is normal. Suppose, towards a contradiction, that Γ is not normal. So, there exists a colour-preserving automorphism σ of $\Gamma_0 = \mathrm{Cay}(G, \{a, a^{-1}, b, b^{-1}\})$ which fixes 1 but is not a group automorphism of G (see [15, Remark 2.1]). Since the map $x \mapsto x^{-1}$ is an automorphism of G, we may assume that $a^\sigma = a$.

We may assume that σ is not the identity. Then there is some i such that $(a^i b)^\sigma = a^i b^{-1}$. By composing with a translation, we may assume that $i = 0$ and $b^\sigma = b^{-1}$. Then we have $(b^m)^\sigma = b^{-m}$. But $b^m \in \langle a\rangle$ and σ is the identity on $\langle a\rangle$. So b^m must have order two which means that $b^m = a^{km/2}$. So $tm \equiv km/2 \pmod{km}$ which means that $t \equiv k/2 \pmod k$. Since $t^2 \equiv 1 \pmod k$ this implies that $k = 2$. So, by [15, Corollary 4.2], $|G| = 2m^2$ is divisible by 8. Thus m is even and $m \geq 4$ because $(m, k) \neq (2, 2)$. Since $\langle a\rangle \cap \langle b\rangle \neq 1$ and $(b^m)^{-1} = b^m$, the map $\varphi : a^i b^j \mapsto a^i b^{-j}$ is a well-defined automorphism of G that is also an automorphism of Γ_0. Furthermore, $\psi = \sigma\varphi$ is a colour-preserving automorphism of Γ_0 which fixes all powers of a (including 1) and b, but is not a group automorphism of G.

Since $mk, m \geq 4$, it is easy to see that for all $g \in G$, g and gab are the only common neighbours of ga and gb in Γ_0. Putting $g = 1$, we get $(ab)^\psi = ab$. Now putting $g = a$ we get $(a^2b)^\psi = a^2b$. By continuing this procedure we get $(a^ib)^\psi = a^ib$ for all i. Since $m \geq 4$, we have $b^2 \neq 1$. So, for all i we have $(a^ib^2)^\psi = a^ib^2$. This implies that $(a^ib^3)^\psi = a^ib^3$ for all i. By continuing this procedure, we get $(a^ib^j)^\psi = a^ib^j$ for all i, j. This means that ψ is the trivial automorphism of G. Hence σ is an automorphism of group G, a contradiction.

(6) Let $L = \{b, b^{-1}\}$, where $\langle a \rangle \cap \langle b \rangle = 1$. We claim that Γ is normal. If Γ is intransitive, then by Lemma 3.9, Γ is normal. Hence, we may assume that Γ is transitive. Then, by [26, Theorem 1.1], $G = \langle x \rangle \times \langle y \rangle$, $o(x) = mk$, $o(y) = m$, for some $m, k \geq 1$, where $mk \geq 3$. Furthermore $a = x$ and $b = a^ty$ for some integer t with $(t, mk) = 1$ and $t^2 \equiv 1 \pmod{k}$, or $(m, k, t) = (1, 10, 2)$, or $(t, mk) = 1$ and $t^2 \equiv -1 \pmod{k}$. Clearly $(m, k, t) = (1, 10, 2)$ is impossible, because $\langle a \rangle \cap \langle b \rangle = 1$. So we have $b^m = a^{tm} \in \langle a \rangle \cap \langle b \rangle = 1$. Thus $o(b)$ divides m, and k divides t. The latter implies that $k = t = 1$. Thus $b = ay$ and $o(b) = o(a) = m$.

Since Γ is connected, [26, Proposition 5.1] implies that Γ is not edge-transitive. So every automorphism of Γ maps G-orbits to G-orbits. Suppose, towards a contradiction, that Γ is not normal. Similar to the previous case, there exists a colour-preserving automorphism σ of $\Gamma_0 = \mathrm{Cay}(G, \{a, a^{-1}, b, b^{-1}\})$ which fixes 1 but is not a group autmorphism of G and we may assume that $a^\sigma = a$ and $b^\sigma = b^{-1}$. Then [15, Theorem 1.3(ii)] implies that 8 divides $|G| = m^2$. So 4 divides m. Since $\langle a \rangle \cap \langle b \rangle = 1$, $\varphi : a^ib^j \mapsto a^ib^{-j}$ is a well-defined automorphism of G that is also an automorphism of Γ_0. Again, by the same argument in the last paragraph of the proof of previous case, we get σ is an automorphism of G which is a contradiction. So we have proved that Γ is normal.

As we saw above, if Γ is transitive, then $o(a) = o(b)$. Conversely, suppose that $o(a) = o(b)$. Then $\sigma : a^ib^j \mapsto a^jb^i$ is a group automorphism of G and $\langle R_G, \psi_\sigma \rangle$, where ψ is defined by the rule $(g, 1)^\psi = (g^\sigma, 2), (g, 2)^\psi = (g^\sigma, 1)$ for all $g \in G$, is a transitive subgroup of $\mathrm{Aut}(\Gamma)$. This completes the proof. □

Proof of Theorem 1.1. It is a direct consequence of Lemmas 3.2, 3.3, 3.5–3.8, and 3.11. □

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