Contributions to Discrete Mathematics

Volume 10, Number 1, Pages 126–133
ISSN 1715-0868

ON UNIFORMLY RESOLVABLE $\{K_2, P_k\}$-DESIGNS WITH $k = 3, 4$

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Abstract. Given a collection of graphs $\mathcal{H}$, a uniformly resolvable $\mathcal{H}$-*design* of order v is a decomposition of the edges of K_v into isomorphic copies of graphs from $\mathcal{H}$ (also called *blocks*) in such a way that all blocks in a given parallel class are isomorphic to the same graph from $\mathcal{H}$. We consider the case $\mathcal{H} = \{K_2, P_k\}$ with $k = 3, 4$, and prove that the necessary conditions on the existence of such designs are also sufficient.

1. Introduction

Given a collection of graphs $\mathcal{H}$, an $\mathcal{H}$-*design* of order v is a decomposition of the edges of K_v into isomorphic copies of graphs from $\mathcal{H}$, the copies of $H \in \mathcal{H}$ in the decomposition are called *blocks*. An $\mathcal{H}$-design is called *resolvable* if it is possible to partition the blocks into *classes* $\mathcal{P}_i$ such that every point of K_v appears exactly once in some block of each $\mathcal{P}_i$.

A resolvable $\mathcal{H}$-decomposition of K_v is sometimes also referred to as a $\mathcal{H}$-*factorization of* K_v, a class can be called an $\mathcal{H}$-*factor of* K_v. The case where $\mathcal{H}$ is a single edge (K_2) is known as a 1-*factorization of* K_v and it is well known to exist if and only if v is even. A single class of a 1-factorization, a pairing of all points, is also known as a 1-*factor* or a *perfect matching*. A resolvable $\mathcal{H}$-design is called *uniform* if every block of the class is isomorphic to the same graph from $\mathcal{H}$. Of particular note is the result of Rees [10] which finds necessary and sufficient conditions for the existence of uniformly resolvable $\{K_2, K_3\}$-designs of order v. Uniformly resolvable decompositions of K_v have also been studied in [2, 3, 4, 5, 6, 7, 8, 9, 12, 11, 14, 13]. In what follows, we will denote by $[a_1, \ldots, a_k]$, $k \geq 2$, the path P_k having vertex set $\{a_1, \ldots, a_k\}$ and edge set $\{\{a_1, a_2\}, \{a_2, a_3\}, \ldots, \{a_{k-1}, a_k\}\}$. If v is even and $k \in \{3, 4\}$, let (K_2, P_k)-URD$(v; r, s)$ denote a uniformly resolvable decomposition of K_v into r classes containing only copies of 1-factors and s classes containing only copies of paths P_k. Let URD$(v; K_2, P_k)$ denote the set of all pairs (r, s) such that there exists a (K_2, P_k)-URD$(v; r, s)$.

Received by the editors September 12, 2014, and in revised form January 10, 2015.
2010 *Mathematics Subject Classification.* 05B05.
Key words and phrases. Resolvable graph decomposition; uniformly resolvable designs; paths.
Research supported by MIUR-PRIN 2012 (Italy) and INDAM-GNSAGA (Italy).

Given $v \equiv 0 \pmod 6$, define $J_1(v)$ according to the following table:

v	$J_1(v)$
$0 \pmod{12}$	$\{(v-1-4x, 3x), x = 0, 1, \ldots, (v-4)/4\}$
$6 \pmod{12}$	$\{(v-1-4x, 3x), x = 0, 1, \ldots, (v-2)/4\}$

TABLE 1. The set $J_1(v)$.

Given $v \equiv 0 \pmod 4$, define $J_2(v)$ according to the following table:

v	$J_2(v)$
$0 \pmod{12}$	$\{(v-1-3x, 2x), x = 0, 1, \ldots, (v-3)/3\}$
$4 \pmod{12}$	$\{(v-1-3x, 2x), x = 0, 1, \ldots, (v-1)/3\}$
$8 \pmod{12}$	$\{(v-1-3x, 2x), x = 0, 1, \ldots, (v-2)/3\}$

TABLE 2. The set $J_2(v)$.

In this paper, the main purpose is to investigate the existence problem of a (K_2, P_k)-URD$(v; r, s)$ of K_v for $k = 3, 4$. We completely solve the spectrum problem for such design; i.e., characterize the existence of uniformly resolvable $\{K_2, P_k\}$-designs of order v, by proving the following result:

Main Theorem.

(i) *A (K_2, P_3)-URD$(v; r, s)$ exists if and only if $v \equiv 0 \pmod 6$ and URD-$(v; K_2, P_3)$=$J_1(v)$.*

(ii) *A (K_2, P_4)-URD$(v; r, s)$ exists if and only if $v \equiv 0 \pmod 4$ and URD-$(v; K_2, P_3)$=$J_2(v)$.*

2. PRELIMINARIES AND NECESSARY CONDITIONS

In this section we will introduce some useful definitions, results, and give necessary conditions for the existence of a uniformly resolvable decomposition of K_v into r classes of 1-factors and s classes of paths P_k, $k = 3, 4$. For missing terms or results that are not explicitly explained in the paper, the reader is referred to [1] and its online updates. For some results below, we also cite this handbook instead of the original papers. A (resolvable) $\mathcal{H}$-decomposition of the complete multipartite graph with u parts each of size g is known as a resolvable group divisible design $\mathcal{H}$-RGDD of type g^u, the parts of size g are called the groups of the design. When $\mathcal{H} = K_n$ we will call it an n-(R)GDD. A (K_2, P_k)-URGDD (r, s) of type g^u is a uniformly resolvable decomposition of the complete multipartite graph with u parts each of size g into r classes containing only copies of 1-factors and s classes containing only copies of paths P_k.

If the blocks of an $\mathcal{H}$-GDD of type g^u can be partitioned into partial parallel classes, each of which contain all points except those of one group, we refer to the decomposition as a *frame*.

A incomplete resolvable (K_2, P_4)-decomposition of K_v with a hole of size h is an (K_2, P_4)-decomposition of $K_{v+h} - K_h$ in which there are two types of classes, full classes and partial classes which cover every point except those in the hole (the points of K_h are referred to as the hole). Specifically a (K_2, P_4)-IURD$(v + h, h; [r_1, s_1], [\bar{r}_1, \bar{s}_1])$ is a uniformly resolvable (K_2, P_4)-decomposition of $K_{v+h} - K_h$ with r_1 1-factors which cover only the points not in the hole, s_1 partial classes of paths P_4 which cover only the points not in the hole, $\bar{r}_1$ 1-factors and $\bar{s}_1$ full classes of paths P_4 which cover every point of K_{v+h}.

Lemma 2.1. *If there exists a (K_2, P_3)-URD$(v; r, s)$ of K_v, then $v \equiv 0 \pmod 6$ and $(r, s) \in J_1(v)$.*

Proof. The condition $v \equiv 0 \pmod 6$ is trivial. Let D be a (K_2, P_3)-URD$(v; r, s)$ of K_v. Counting the edges of K_v that appear in D we obtain

$$\frac{rv}{2} + \frac{2sv}{3} = \frac{v(v-1)}{2},$$

and hence

$$3r + 4s = 3(v - 1). \tag{2.1}$$

This equation implies that $3r \equiv 3(v - 1) \pmod 4$ and $4s \equiv 3(v - 1) \pmod 3$. Then we obtain

- $r \equiv 3 \pmod 4$ and $s \equiv 0 \pmod 3$ for $v \equiv 0 \pmod{12}$,
- $r \equiv 1 \pmod 4$ and $s \equiv 0 \pmod 3$ for $v \equiv 6 \pmod{12}$.

Letting now $s = 3x$, the equation (2) yields $r = (v - 1) - 4x$. Since r and s cannot be negative, and x is an integer, the value of x has to be in the range as given in the definition of $J_1(v)$. This completes the proof. □

Lemma 2.2. *If there exists a (K_2, P_4)-URD$(v; r, s)$ of K_v then $v \equiv 0 \pmod 4$ and $(r, s) \in J_2(v)$.*

Proof. The condition $v \equiv 0 \pmod 4$ is trivial. Let D be a (K_2, P_4)-URD$(v; r, s)$ of K_v. Counting the edges of K_v that appear in D we obtain

$$\frac{rv}{2} + \frac{3sv}{4} = \frac{v(v-1)}{2},$$

and hence

$$2r + 3s = 2(v - 1). \tag{2.2}$$

This equation implies that

$$2r \equiv 2(v - 1) \pmod 3 \quad \text{and} \quad 3s \equiv 2(v - 1) \pmod 2.$$

Then we obtain

- $r \equiv 2 \pmod 3$ and $s \equiv 0 \pmod 2$ for $v \equiv 0 \pmod{12}$,
- $r \equiv 0 \pmod 3$ and $s \equiv 0 \pmod 2$ for $v \equiv 4 \pmod{12}$,
- $r \equiv 1 \pmod 3$ and $s \equiv 0 \pmod 2$ for $v \equiv 8 \pmod{12}$.

Letting now $s = 2x$, the equation (2) yields $r = (v-1) - 3x$. Since r and s cannot be negative, and x is an integer, the value of x has to be in the range as given in the definition of $J_2(v)$. This completes the proof. □

We now recall some results that can be used to produce the main result.

Theorem 2.3. [10] *There exists a* (K_2, K_3)*-URD*$(v; r, s)$, $r, s > 0$, *if and only if*

(1) $v \equiv 0 \pmod 6$,
(2) $(r, s) \in \{(v-1-2x, x), x = 1, 2, \ldots, \frac{v-2}{2}\}$,
(3) *with the two exceptions* $(v, s) = (6, 2), (12, 5)$.

Theorem 2.4. [9] *Let* $v \equiv 0 \pmod 3$, $v \geq 9$. *The union of any two edge-disjoint parallel classes of* 3*-cycles of* K_v *can be decomposed into three parallel classes of* P_3.

We also need the following definitions. Let (s_1, t_1) and (s_2, t_2) be two pairs of non-negative integers. Define $(s_1, t_1) + (s_2, t_2) = (s_1 + s_2, t_1 + t_2)$. If X and Y are two sets of pairs of non-negative integers, then $X + Y$ denotes the set $\{(s_1, t_1) + (s_2, t_2) : (s_1, t_1) \in X, (s_2, t_2) \in Y\}$. If X is a set of pairs of non-negative integers and h is a positive integer, then $h * X$ denotes the set of all pairs of non-negative integers which can be obtained by adding any h elements of X together (repetitions of elements of X are allowed).

3. Small cases

Lemma 3.1. $URD(6; K_2, P_3) = \{(5, 0), (1, 3)\}$.

Proof. The case $(5, 0)$ corresponds to a 1-factorization of the complete bipartite graph K_6 which is known to exist [1]. For the case $(1, 3)$, let $V(K_{12}) = \mathbb{Z}_6$, and the classes as listed below:

$$\{\{0,1\},\{2,3\},\{4,5\}\},\{[1,4,5],[2,3,6]\},\{[3,1,5],[4,2,6]\},\{[1,6,4],[2,5,3]\}.$$

□

Lemma 3.2. *There exists a* (K_2, P_4)*-URGDD*(r, s) *of type* 6^2 *with* $(r, s) \in \{(0, 4), (3, 2), (6, 0)\}$.

Proof. The case $(6, 0)$ corresponds to a 1-factorization of the complete bipartite graph $K_{6,6}$ which is known to exist [1]. The case $(0, 4)$ corresponds to a (K_2, P_4)-URGDD$(0, 4)$ which is known to exist [15]. For the case $(3, 2)$ take the groups to be $\{1, 2, 3, 4, 5, 6, 7, 8\}, \{a, b, c, d, e, f\}$ and the classes listed below:

$$\{\{1,c\},\{2,d\},\{3,e\},\{4,f\},\{5,a\},\{6,b\}\},$$
$$\{\{1,d\},\{2,c\},\{3,f\},\{4,e\},\{5,b\},\{6,a\}\},$$
$$\{\{1,b\},\{2,e\},\{3,c\},\{4,a\},\{5,f\},\{6,d\}\},$$
$$\{[1,a,2,b],[3,d,4,c],[5,e,6,f]\},\{[4,b,3,a],[6,c,5,d],[e,1,f,2]\}.$$

□

Lemma 3.3. $URD(12; K_2, P_4) = \{(11,0), (8,2), (5,4), (2,6)\}$.

Proof. The case $(11,0)$ corresponds to a 1-factorization of the complete graph K_{12} which is known to exist [1]. The rest of the cases are given explicitly below.

- $(8,2), (5,4)$.
  Take a (K_2, P_4)-URGDD(r,s) of type 6^2 with $(r,s) \in \{(0,4),(3,2)\}$, which come from Lemma 3.2. Fill in each of the groups of size 6 with the same 1-factorization of K_6. This gives a (K_2, P_4)-URD$(12; r, s)$ for each $(r,s) \in \{(5,0) + 4 * \{(0,4),(3,2),(6,0)\}\}$.
- $(2,6)$.
  Let $V(K_{12}) = \{0,1,\dots,11\}$ be the vertex set and the classes listed below:
  $\{[0,1,2,3],[4,5,6,7],[8,9,10,11]\}$,$\{[1,3,0,2],[5,7,4,6],[9,11,8,10]\}$,
  $\{[0,4,1,5],[8,6,9,7],[10,2,11,3]\}$, $\{[1,7,0,6],[2,8,3,9],[11,5,10,4]\}$,
  $\{[9,4,8,5],[11,0,10,1],[3,6,2,7]\}$, $\{[2,5,3,4],[8,1,9,0],[10,7,11,6]\}$,
  $\{\{0,8\},\{1,11\},\{2,4\},\{3,7\},\{6,10\},\{5,9\}\}$,
  $\{\{0,5\},\{1,6\},\{2,9\},\{3,10\},\{4,11\},\{7,8\}\}$.

□

Lemma 3.4. *There exists a* (K_2, P_4)-*IURD*$(8,2;[1,0],[r,s])$ *with* $(r,s) \in \{(6,0),(3,2),(0,4)\}$.

Proof. Let the point set be $V = \{a,b,0,1,2,3,4,5\}$ and let $\{a,b\}$ be the hole. Let $\mathcal{F}$=$\{F_1, F_2, \dots, F_7\}$ be a 1-factorization of K_8 such that $\{a,b\} \in F_1$.

- A (K_2, P_4)-IURD$(8,2;[1,0],[6,0])$
  $F_1 - \{a,b\}$, $\{F_2,\dots,F_7\}$.
- A (K_2, P_4)-IURD$(8,2;[1,0],[3,2])$
  $F_1 - \{a,b\}$, $\{\{0,b\},\{1,5\},\{2,a\},\{3,4\}\}$,
  $\{\{4,b\},\{a,5\},\{2,3\},\{0,1\}\}$, $\{\{0,3\},\{b,5\},\{2,1\},\{3,0\}\}$,
  $\{[0,a,1,b],[3,5,2,4]\}$, $\{[2,b,3,a],[5,0,4,1]\}$.
- A (K_2, P_4)-IURD$(8,2;[1,0],[0,4])$
  $F_1 - \{a,b\}$, $\{[0,a,1,b],[3,5,2,4]\}$, $\{[2,b,3,a],[5,0,4,1]\}$,
  $\{[2,a,5,b],[1,0,3,4]\}$, $\{[0,b,4,a],[5,1,2,3]\}$.

□

Lemma 3.5. $URD(8; K_2, P_4) = \{(7,0),(4,2),(1,4)\}$.

Proof. The assertion follows from Lemma 3.4. □

4. Main results

Lemma 4.1. *For every* $v \equiv 0 \pmod 6$ $J_1(v) \subseteq URD(v; K_2, P_3)$.

Proof. For $v = 6$ the conclusion follows from Lemma 3.1. For $v \geq 12$, take a (K_2, K_3)-URD$(v; v-1-4t, 2t)$ with $t \in \{0,1,\dots,(v-4)/4\}$ for $v \equiv 0 \pmod{12}$ and $t \in \{0,1,\dots,(v-2)/4\}$ for $v \equiv 6 \pmod{12}$, which exists by Theorem 2.3. Applying Theorem 2.4 we obtain a (K_2, P_3)-URD$(v; v - 1 - 4t, 3t)$.

□

Lemma 4.2. *For every* $v \equiv 4 \pmod{12}$, $J_2(v) \subseteq URD(v; K_2, P_4)$.

Proof. Let $R_1, R_2, \dots, R_{\frac{v-1}{3}}$ be the parallel classes of a resolvable $\{K_4\}$-design R of order v. Place on each block of a given resolution class of R the same (K_2, P_4)-URD$(4; r, s)$ with $(r, s) \in \{(3, 0), (0, 2)\}$. Since R contains $(v-1)/3$ parallel classes the result is a (K_2, P_4)-URD$(v; r, s)$ of K_v for each $(r, s) \in (v-1)/3 * \{(3, 0), (0, 2)\}$. This implies

$$URD(v; K_2, P_4) \supseteq \left\{\frac{v-1}{3} * \{(3,0),(0,2)\}\right\}.$$

Since

$$\frac{v-1}{3} * \{(3,0),(0,2)\} = \left\{(v-1-3x, 2x), x = 0, \dots, \frac{v-1}{3}\right\} = J_2(v),$$

we obtain the proof. □

Lemma 4.3. *For every* $v \equiv 0 \pmod{12}$ $J_2(v) \subseteq URD(v; K_2, P_4)$.

Proof. For $v = 12$ the conclusion follows from Lemma 3.3. For $v \geq 24$ start with a 2-RGDD G of type $2^{\frac{v}{12}}$ [1]. Give weight 6 to each point of this 2-GDD and place on each edge of a given resolution class the same (K_2, P_4)-URGDD(r, s) of type 6^2, with $(r, s) \in \{(6, 0), (3, 2), (0, 4)\}$, which exists by Lemma 3.2. Fill the groups of sizes 12 with the same (K_2, P_4)-URD$(12; r, s)$, with $(r, s) \in \{(11, 0), (8, 2), (5, 4), (2, 6)\}$, which exists by Lemma 3.3. Since G contains $(v-12)/6$ resolution classes the result is a (K_2, P_4)-URD$(v; r, s)$ of K_v for each $(r, s) \in \{\{(11, 0), (8, 2), (5, 4), (2, 6)\} + (v-12)/6 * \{(6, 0), (3, 2), (0, 4)\}\}$. This implies

$$URD(v; K_2, P_4) \supseteq \left\{\{(11,0),(8,2),(5,4),(2,6)\} + \frac{(v-12)}{6} * \{(6,0),(3,2),(0,4)\}\right\}.$$

Since

$$\frac{v-12}{6} * \{(6,0),(3,2),(0,4)\} = \left\{(v-12-3x, 2x), x = 0, \dots, \frac{v-12}{3}\right\},$$

it easy to see that

$$\left\{\{(11,0),(8,2),(5,4),(2,6)\} + \frac{(v-12)}{6} * \{(6,0),(3,2),(0,4)\}\right\} = J_2(v).$$

This completes the proof. □

Lemma 4.4. *For every* $v \equiv 8 \pmod{12}$ $J_2(v) \subseteq URD(v; K_2, P_4)$.

Proof. For $v = 8$ the conclusion follows from Lemma 3.5. For $v > 8$ start with a $2-$frame F of type $1^{\frac{v-2}{6}}$ [14] with groups G_i, $i = 1, \ldots, (v-2)/6$. Let p_i be the partial parallel class which miss the group G_i. Expand each point 6 times and add a set H of 2 ideal points a_1, a_2. For each $i = 1, \ldots, (v-2)/6$, place on $G_i \times \{1, \ldots, 6\} \cup H$ the same (K_2, P_4)-IURD$(8, 2; [1, 0], [x, y])$ D_i of K_8-K_2 with $(x, y) \in \{(6, 0), (3, 2), (0, 4)\}$, which exists by Lemma 3.4, in such a way the hole covers the point of H. For each $i = 1, \ldots, (v-2)/6$, place on each block of the p_i partial parallel class the same (K_2, P_4)-URGDD(r_2, s_2) of type 6^2 with $(r_2, s_2) \in \{(6, 0), (3, 2), (0, 4)\}$, which exists by Lemma 3.2. Add the edge $\{a_1, a_2\}$ of H to the partial classes of D_i and form, on $\cup_{i=1}^{\frac{v-2}{6}} G_i \times \{1, \ldots, 6\} \cup H$, 1 class of 1-factors. For each $i = 1, \ldots, (v-2)/6$, add the full classes of D_i to the classes of p_i and form r_3 classes of 1-factors and s_3 classes of P_4-factors with $(r_3, s_3) \in \{(6, 0), (3, 2), (0, 4)\}$. Since each group G_i is missed by 1 partial parallel class of F we obtain a (K_2, P_4)-URD $(v; r, s)$ for each $(r, s) \in \{(1, 0) + (v-2)/6 * \{(6, 0), (3, 2), (0, 4)\}\}$. This implies

$$URD(v; K_2, P_4) \supseteq \left\{(1, 0) + \frac{v-2}{6} * \{(0, 4), (3, 2), (6, 0)\}\right\}.$$

Since

$$\frac{v-2}{6} * \{(0, 4), (3, 2), (6, 0)\} = \left\{(v - 1 - 3x, 2x), x = 0, \ldots, \frac{v-2}{3}\right\},$$

it easy to see that $\{(1, 0) + (v-2)/6 * \{(6, 0), (3, 2), (0, 4)\}\} = J_2(v)$. This completes the proof.

□

5. Conclusion

We are now in a position to prove the main result of the paper.

Theorem 5.1. *For every* $v \equiv 0 \pmod 6$*, we have* $URD(v; K_2, P_3) = J_1(v)$ *and, for every* $v \equiv 0 \pmod 4$*, we have* $URD(v; K_2, P_4) = J_2(v)$*.*

Proof. Necessity follows from Lemmas 2.1 and 2.2. Sufficiency follows from Lemmas 4.1, 4.2, 4.3 and 4.4. This completes the proof. □

Remark: Note that the existence of uniformly resolvable $\{K_2, P_k\}$-designs with $k > 4$ is very difficult to study and it is currently under investigation.

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