Contributions to Discrete Mathematics

Volume 15, Number 3, Pages 1–24
ISSN 1715-0868

A COMPLETE SOLUTION TO THE SPECTRUM PROBLEM FOR GRAPHS WITH SIX VERTICES AND UP TO NINE EDGES

EMRE KOLOTOĞLU

ABSTRACT. Let G be a graph. A G-design of order n is a decomposition of the complete graph K_n into disjoint copies of G. The existence problem of graph designs has been completely solved for all graphs with up to five vertices, and all graphs with six vertices and up to seven edges; and almost completely solved for all graphs with six vertices and eight edges leaving two cases of order 32 unsettled. Up to isomorphism there are 20 graphs with six vertices and nine edges (and no isolated vertex). The spectrum problem has been solved completely for 11 of these graphs, and partially for 2 of these graphs. In this article, the two missing graph designs for the six-vertex eight-edge graphs are constructed, and a complete solution to the spectrum problem for the six-vertex nine-edge graphs is given; completing the spectrum problem for all graphs with six vertices and up to nine edges.

1. INTRODUCTION

Let $\mathcal{G} = \{G_1, G_2, \ldots, G_t\}$ be a set of (finite, simple, undirected) graphs, and K_n denote a complete graph with n vertices (or points). A $\mathcal{G}$-design of order n is a pair $(X, \mathcal{B})$, where X is the vertex set of K_n, and $\mathcal{B}$ is a set of subgraphs of K_n, called *blocks*, such that each block is isomorphic to some $G_i \in \mathcal{G}$ and the edges of the blocks partition the edge set of K_n. When $\mathcal{G} = \{G\}$, then a $\mathcal{G}$-design is simply denoted as a G-design. When $\mathcal{G} = \{K_{k_1}, K_{k_2}, \ldots, K_{k_t}\}$, then a $\mathcal{G}$-design of order n is a *pairwise balanced design* with block sizes in $\{k_1, k_2, \ldots, k_t\}$, and is denoted as a PBD$(n, \{k_1, k_2, \ldots, k_t\})$.

Let $K = K_{n_1,n_2,\ldots,n_r}$ denote the complete multipartite graph with the vertex set $X = \bigcup_{i=1}^{r} X_i$, where X_i are the parts of the multipartition, and $|X_i| = n_i$. Let $\mathcal{G} = \{G_1, G_2, \ldots, G_t\}$, $Y = \{X_i : 1 \le i \le r\}$, $T = [n_1, n_2, \ldots, n_r]$ (a multiset), and $\mathcal{B}$ be a set of subgraphs of K, called blocks, each isomorphic to some graph in $\mathcal{G}$, whose edges partition the edge

Received by the editors July 21, 2015, and in revised form October 31, 2019.
2000 *Mathematics Subject Classification.* 05B30, 05C51.
Key words and phrases. graph design.

set of K. Then the triple $(X, Y, \mathcal{B})$ is called a *$\mathcal{G}$-group divisible design* (or $\mathcal{G}$-GDD for short) *of type* T. Usually the type is denoted by exponential form, for example, the type $g_1^{u_1} g_2^{u_2} \ldots g_s^{u_s}$ denotes u_i occurences of g_i in T for $1 \leq i \leq s$. The parts of size greater than one in the multipartition of K are called *holes* of the GDD. Obviously, a $\mathcal{G}$-design of order n is a $\mathcal{G}$-GDD of type 1^n (with no holes). When $\mathcal{G} = \{K_{k_1}, K_{k_2}, \ldots, K_{k_t}\}$, then a $\mathcal{G}$-GDD is simply denoted as a $\{k_1, k_2, \ldots, k_t\}$-GDD. Moreover, if $t = 1$, it is simply denoted as a k_1-GDD. A k-GDD of type n^k is called a *transversal design*, and is denoted by $\mathrm{TD}(k, n)$.

There are three obvious necessary conditions for the existence of a G-design. If a G-design of order n exists, then $n = 1$ or $n \geq |V(G)|$, $n(n-1) \equiv 0 \pmod{2|E(G)|}$, and $n - 1 \equiv 0 \pmod{d}$, where $V(G)$ and $E(G)$ denote the set of vertices and edges of G respectively, and d is the g.c.d. of the degrees of all vertices in G.

The *spectrum* for a graph G is the set of positive integers n such that there exists a G-design of order n. Numerous articles have been written on the existence of G-designs. The results known by 2008 on the spectrum of graphs may be found in [4, 7]. For the latest results, see [6]. The results given in [4] show that the spectrum problem has been completely solved for all graphs with up to four vertices; and almost completely solved for all graphs with five vertices, and graphs with six vertices and up to eight edges. For graphs with five vertices, and graphs with six vertices and up to seven edges, the results in [4] have left some possible exceptions. These exceptions have since been dealt with in [24, 19, 13].

For graphs with six vertices and eight edges, Kang et al. [17] have given an almost complete solution leaving the case of order 32 for two of these graphs unsettled. These two graphs (H_{12} and H_{13} with the notation of [4]) are shown in Figure 1. These two missing graph designs of order 32 are constructed in the Appendix (see Examples A.1 and A.2), completing the spectrum problem for the six-vertex eight-edge graphs.

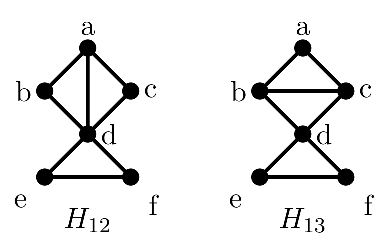

Figure 1. Two graphs with six vertices and eight edges

Up to isomorphism there are 20 graphs with six vertices and nine edges, excluding those with isolated vertices (see [16]). These graphs are shown in Figure 2.

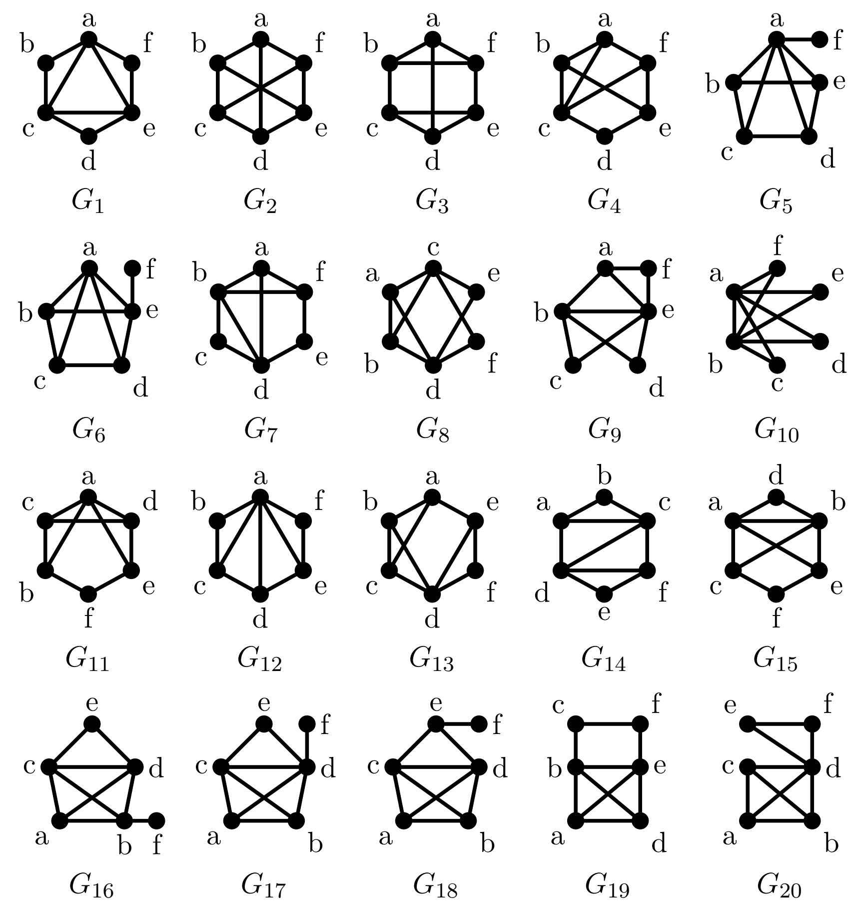

FIGURE 2. All graphs with six vertices and nine edges (and no isolated vertex)

The spectrum problem has been completely solved for G_1 in [22], for G_2 in [15], for G_3 in [8], and for G_{10} in [3]. The graphs $G_1, G_2, \ldots, G_9$ has been considered in [18], and it has been claimed that the spectrum problem for these graphs has been completely solved, obtaining the following result.

Theorem 1.1 ([22, 15, 8, 3, 18])**.** *Let* $1 \leq i \leq 10$. *There exists a* G_i*-design of order* n *if and only if* $n \equiv 1, 9 \pmod{18}$ *when* $i = 1$, $n \equiv 1 \pmod 9$ *and* $n \neq 10$ *when* $i = 2$, $n \equiv 1 \pmod 9$ *when* $i = 3$, $n \equiv 0, 1 \pmod 9$ *and* $n \neq 9$ *when* $i \in \{4, 5, 6, 8, 9, 10\}$, *and* $n \equiv 0, 1 \pmod 9$ *when* $i = 7$.

Although this result is correct, G_i-designs of order 18 have not been constructed for $i \in \{8, 9\}$ in [18]. These two designs, which are crucial for the recursive constructions given there, are constructed in this article to complete the proof of Theorem 1.1. For $i = 9$, the required design is constructed directly in Example A.5. For $i = 8$, we can first construct a G_8-GDD of type $1^8 10^1$ (Example A.35), and then fill in the hole with a G_8-design of order 10 which has been constructed in [18]. These two constructions complete the proof of Theorem 1.1.

In addition to the results in Theorem 1.1, the spectrum problem has been completely solved for the graph G_{16} in [20], where references have been given to [21] for some constructions. Also, some partial results have been obtained on the spectrums of the graphs G_{11} and G_{18}, in [23] and [11], respectively. Since [21] is not easy to obtain, and the results for the graphs G_{11} and G_{18} are incomplete, these three graphs, G_{11}, G_{16}, and G_{18} are included in this article for completeness. A complete solution to the spectrum problem for all graphs G_i, $11 \leq i \leq 20$ is given. We can see that for $11 \leq i \leq 20$, the necessary conditions for the existence of a G_i-design of order n is that $n \equiv 0, 1 \pmod 9$. The main result of this article is the following theorem, which completes the spectrum problem for all graphs with six vertices and up to nine edges.

Theorem 1.2. *Let $11 \leq i \leq 20$. There exists a G_i-design of order n if and only if $n \equiv 0, 1 \pmod 9$, $n \neq 9$, and $(i, n) \notin \{(18, 10), (20, 10)\}$.*

The spectrum problems solved in this article have also been solved independently in [9, 10]. Although these two articles have been published first, our work has actually been done earlier with an exception of a G_{20}-design of order 18. In the original version of our article, a G_{20}-design of order 18 was missing which has later been constructed in [9]. For completeness, we take this design from [9] and include here.

In what follows, as a block in a design, all graphs with six vertices are denoted by $[a, b, c, d, e, f]$ according to the vertex labels in Figures 1 and 2. Also, the complete graph on vertices $x_1, x_2, \ldots, x_n$ is denoted by $\{x_1, x_2, \ldots, x_n\}$. In the constructions, an ordered pair (x, y) is denoted by x_y.

2. Nonexistence results

In this section, we prove the nonexistence results given in Theorem 1.2 for the orders that satisfy the necessary conditions. The constructions for the remaining orders will be given in the following sections by using direct and recursive construction techniques.

Lemma 2.1. *There does not exist a G_i-design of order 9 for $11 \leq i \leq 20$.*

Proof. A G_i-design of order 9 would consist of 4 blocks. For $16 \leq i \leq 20$, the graph G_i contains K_4 as a subgraph, and 4 K_4's cannot be packed in K_9. Therefore, a G_i-design of order 9 cannot exist. The proofs for the cases $11 \leq i \leq 15$ are also straightforward but more tedious. We omit those proofs here. □

Lemma 2.2. *There does not exist a G_i-design of order 10 for $i \in \{18, 20\}$.*

Proof. A G_i-design of order 10 would consist of 5 blocks. Both G_{18} and G_{20} contain K_4 as a subgraph. Up to isomorphism, there is a unique way of packing 5 K_4's in K_{10}, namely by taking the blocks $\{0,1,2,3\}$, $\{0,4,5,6\}$, $\{1,4,7,8\}$, $\{2,5,7,9\}$, and $\{3,6,8,9\}$. After removing the edges in these 5 K_4's from K_{10}, we are left with a 3-regular graph with 10 vertices. One can check that this graph cannot be decomposed into 5 $K_{1,3}$'s (for G_{18}), or into 5 K_3's (for G_{20}). □

3. Fundamental Tools for Recursive Constructions

The following theorems on TDs, PBDs, and GDDs will be useful in the recursive constructions, and may be found in [1, 2, 12, 14].

Theorem 3.1 ([2]). *There exists a* $\mathrm{TD}(k,q)$ *for any prime power* q*, and* $k \le q+1$.

Theorem 3.2 ([2]). *There exists a* $\mathrm{TD}(3,n)$ *if and only if* $n \ge 2$.

Theorem 3.3 ([2]). *There exists a* $\mathrm{TD}(4,n)$ *if and only if* $n \ge 3$*, and* $n \ne 6$.

Theorem 3.4 ([1]). *There exists a* $\mathrm{PBD}(n,\{3,4,5\})$ *if and only if* $n \ge 3$ *and* $n \notin \{6,8\}$.

Theorem 3.5 ([1]). *There exists a* $\mathrm{PBD}(n,\{4,5,6\})$ *if and only if* $n \ge 4$ *and* $n \notin \{7,8,9,10,11,12,14,15,18,19,23\}$.

Theorem 3.6 ([12]). *There exists a* 4-GDD *of type* 2^7.

Theorem 3.7 ([12]). *There exists a* 4-GDD *of type* 3^k *if and only if* $k \equiv 0,1 \pmod 4$.

Theorem 3.8 ([14]). *There exists a* $\{4,7\}$-GDD *of type* 3^k *if and only if* $k \ge 4$*, and* $k \ne 6$.

The following fundamental recursive constructions may be found in [5].

Theorem 3.9 (Wilson's Fundamental Construction [5]). *Let* G *be a graph,* $(X,Y,\mathcal{B})$ *be a* $\{k_1,k_2,\dots,k_t\}$-GDD*, and* $\omega : X \to \mathbb{Z}^+ \cup \{0\}$ *be a weight function. Suppose that for each block* $B \in \mathcal{B}$*, there exists a* G-GDD *of type* $[\omega(x) : x \in B]$*. Then there exists a* G-GDD *of type* $[\sum_{x\in X_i} \omega(x) : X_i \in Y]$.

Theorem 3.10 (Inflation [5]). *Let* G *be a* k*-colorable graph, i.e. a subgraph of some complete* k*-partite graph. Suppose that there exists a* G-GDD *of type* T*, and a* $\mathrm{TD}(k,m)$*. Then there exists a* G-GDD *of type* mT.

Note that the graph G_i is 3-colorable if $11 \le i \le 15$, and 4-colorable if $16 \le i \le 20$. Therefore, we get the following corollaries of Theorems 3.2, 3.3, and 3.10.

Proposition 3.11 (Inflation I). *Let* $11 \le i \le 15$ *and suppose that there exists a* G_i-GDD *of type* T*. Then there exists a* G_i-GDD *of type* mT *for all* $m \ge 2$.

Proposition 3.12 (Inflation II). *Let* $16 \le i \le 20$ *and suppose that there exists a* G_i-GDD *of type* T*. Then there exists a* G_i-GDD *of type* mT *for all* $m \ge 3$ *and* $m \ne 6$.

4. Spectrum of the Graphs G_i for $11 \leq i \leq 15$

In this section, we consider the graphs G_i for $11 \leq i \leq 15$, and determine their spectrums by giving the necessary constructions. Note that these five graphs are 3-colorable graphs, but the last five graphs are not. Our main strategy is based on the following corollaries of Theorems 3.4 and 3.9.

Proposition 4.1. *Let $a \in \{0,1\}$ and suppose that there exists a G-GDD of type g^k for all $k \in \{3,4,5\}$, and a G-design of order $g+a$. Then there exists a G-design of order $gu+a$ for all $u \geq 3$ and $u \notin \{6,8\}$.*

Proof. Take a $\{3,4,5\}$-GDD of type 1^u from Theorem 3.4, and apply Wilson's construction by giving weight g to each point, and using G-GDDs of types g^3, g^4, and g^5, to obtain a G-GDD of type g^u. Finally add a points and fill in the holes with G-designs of order $g+a$. □

Proposition 4.2. *Let $a \in \{0,1\}$ and suppose that there exists a G-GDD of type g^k and a G-design of order $g(k-1)+a$ for all $k \in \{3,4,5\}$. Then there exists a G-design of order $gu+a$ for all $u \geq 2$ and $u \notin \{5,7\}$.*

Proof. Take a PBD$(u+1,\{3,4,5\})$ from Theorem 3.4 and remove one point to obtain a $\{3,4,5\}$-GDD of type $2^a3^b4^c$ for some a,b,c with $2a+3b+4c=u$. Then use Wilson's construction by assigning weight g to each point to obtain a G-GDD of type $(2g)^a(3g)^b(4g)^c$. Add a points and fill in the holes with G-designs of orders $2g+a$, $3g+a$, and $4g+a$ to obtain a G-design of order $gu+a$. □

Our goal is to use Propositions 4.1 and 4.2 with $g=9$. So, we first construct G_i-GDDs of types 9^u for $u \in \{3,4,5\}$ (Examples A.31−A.33).

Consider the case $n \equiv 1 \pmod 9$. Let $n = 9u+1$. For $u \in \{1,2\}$, we construct G_i-designs of order n directly in Examples A.3, A.4, and A.10. For $u \geq 3$ and $u \notin \{6,8\}$, apply Proposition 4.1 with $(g,a)=(9,1)$. Finally for $u \in \{6,8\}$, apply Proposition 4.2 with $(g,a)=(9,1)$ to settle the case $n \equiv 1 \pmod 9$.

Now let $n=9u$. For $u=1$, a G_i-design of order n does not exist by Lemma 2.1. Therefore we cannot apply Proposition 4.1 in this case. However, we can still apply Proposition 4.2. For $u \in \{2,3,4\}$, we make direct constructions in Examples A.5−A.7, A.11, and A.13. Now Proposition 4.2 can be applied for $u \geq 6$ and $u \neq 7$ with $(g,a)=(9,0)$. This leaves only the cases $u=5$ and $u=7$ unsettled. For these cases, we first construct G_i-GDDs of types $1^{35}10^1$ and $1^{35}28^1$ (Examples A.37 and A.39), and then fill in the holes with G_i-designs of orders 10 and 28 to obtain G_i-designs of orders 45 and 63. This settles the case $n \equiv 0 \pmod 9$ and we obtain the following theorem.

Theorem 4.3. *Let $11 \leq i \leq 15$. There exists a G_i-design of order n if and only if $n \equiv 0,1 \pmod 9$ and $n \neq 9$.*

5. Spectrum of the Graphs G_i for $16 \le i \le 19$

Since the graphs G_i for $16 \le i \le 20$ contain K_4 as a subgraph, a G_i-GDD of type 9^3 cannot exist, and hence the strategy used in the previous section cannot be applied for these graphs. However, we may use the following construction whose proof is analogous to the proof of Proposition 4.2, where Theorem 3.5 is used instead of Theorem 3.4.

Proposition 5.1. *Let $a \in \{0, 1\}$ and suppose that there exists a G-GDD of type g^k and a G-design of order $g(k-1)+a$ for all $k \in \{4, 5, 6\}$. Then there exists a G-design of order $gu+a$ for all $u \ge 3$ and $u \notin \{6, 7, 8, 9, 10, 11, 13, 14, 17, 18, 22\}$.*

We could also state a construction analogous to Proposition 4.1 here, but such a construction is not going to be needed. Our goal is to use Proposition 5.1 with $g = 9$. So, we need to construct G_i-GDDs of types 9^u for $u \in \{4, 5, 6\}$. All graphs G_i for $16 \le i \le 20$ are 4-colorable graphs, and we may expect that there exist G_i-GDDs of types 9^4, 9^5, and 9^6. However, one can show using counting arguments that a G_{20}-GDD of type g^4 cannot exist when g is odd. Therefore, we consider only the graphs G_i for $16 \le i \le 19$ in this section. We deal with the graph G_{20} in the next section.

Our first goal is to construct G_i-GDDs of types 9^u for $u \in \{4, 5, 6\}$ and $16 \le i \le 19$. Since there does not exist a G_{18}-design of order 10 (see Lemma 2.2), our strategy will be slightly different for the graph G_{18}. We construct G_{18}-GDDs of types 9^4 and 9^5 directly (Examples A.32 and A.33). For a G_{18}-GDD of type 9^6, inflate (i.e. use Proposition 3.12) a G_{18}-GDD of type 3^6 (Example A.25) by a factor of 3. For the graphs G_{16}, G_{17}, and G_{19}, we do much better and prove the following lemma.

Lemma 5.2. *There exists a G_i-GDD of type 9^u for all $i \in \{16, 17, 19\}$ and $u \ge 4$.*

Proof. We first construct G_i-GDDs of types 3^4, 3^6, and 3^7 (Examples A.24–A.27). Now, for $u \ge 4$ and $u \ne 6$, take a $\{4, 7\}$-GDD of type 3^u (Theorem 3.8) and apply Wilson's construction by assigning weight 3 to each point and using G_i-GDDs of types 3^4 and 3^7, to obtain a G_i-GDD of type 9^u. For $u = 6$, inflate a G_i-GDD of type 3^6 by a factor of 3 to obtain a G_i-GDD of type 9^6. □

Using Lemma 5.2, we get the following result.

Lemma 5.3. *There exists a G_i-design of order $9u + 1$ for $u \ge 1$ and $i \in \{16, 17, 19\}$.*

Proof. For $u \in \{1, 2, 3\}$, see Examples A.3, A.10, and A.12. For $u \ge 4$, take a G_i-GDD of type 9^u (Lemma 5.2), add one point, and fill in the holes with G_i-designs of order 10. □

For the remaining cases, our goal is to apply Proposition 5.1. We first make some direct constructions for small orders.

Lemma 5.4. *Let $16 \leq i \leq 19$, $1 \leq u \leq 7$, and $a \in \{0,1\}$. Then, there exists a G_i-design of order $9u+a$ except when $(u,a)=(1,0)$ or $(u,a,i)=(1,1,18)$.*

Proof. For the nonexistence results in the cases $(u,a)=(1,0)$ or $(u,a,i)=(1,1,18)$, see Lemmas 2.1 and 2.2. For $i \in \{16,17,19\}$, $1 \leq u \leq 7$, and $a=1$, see Lemma 5.3. For $2 \leq u \leq 7$ (where $a=0$ if $i \in \{16,17,19\}$, and $a \in \{0,1\}$ if $i=18$) a G_i-design of order $9u+a$ is either constructed directly (Examples A.8−A.17 and A.19−A.21) or by constructing G_i-GDDs of types $1^{35}m^1$ with $m \in \{10,19,28\}$ (Examples A.37−A.39) and then filling in the holes with G_i-designs of orders 10, 19, or 28. □

Applying Proposition 5.1, we get the following result.

Lemma 5.5. *Let $16 \leq i \leq 19$ and $a \in \{0,1\}$. There exists a G_i-design of order $9u+a$ for all $u \geq 1$ and $u \notin \{8,9,10,11,13,14,17,18,22\}$, except when $(u,a)=(1,0)$ or $(u,a,i)=(1,1,18)$.*

To deal with the remaining orders, we make the following constructions.

Lemma 5.6. *There exist G_i-GDDs of types 18^u and $18^{u-1}27^1$ for $u \equiv 0,1 \pmod 4$ and $16 \leq i \leq 19$.*

Proof. Take a 4-GDD of type 3^u (Theorem 3.7), and apply Wilson's construction by assigning weight 6 or 9 to all points in one group, and weight 6 to the remaining points. The input G_i-GDDs of types 6^4 and $6^3 9^1$ come from Examples A.28 and A.44. □

Lemma 5.7. *There exist G_i-GDDs of types $18^5 27^1$, 18^7, and $45^4 18^1$ for $16 \leq i \leq 19$.*

Proof. For $18^5 27^1$, inflate a G_i-GDD of type $6^5 9^1$ (Example A.46) by a factor of 3. For 18^7, take a 4-GDD of type 2^7 (Theorem 3.6), and apply Wilson's construction by assigning weight 9 to each point and using G_i-GDDs of type 9^4 (Lemma 5.2 and Example A.32). For $45^4 18^1$, take a TD$(5,5)$ (Theorem 3.1) and apply Wilson's construction by assigning weight 0 to three points in the same group, weight 9 to the remaining points, and using G_i-GDDs of types 9^4 and 9^5 (Lemma 5.2 and Examples A.32, A.33). □

To settle the unresolved cases in Lemma 5.5 (i.e. $u \in \{8,9,10,11,13,14,17,18,22\}$), take G_i-GDDs of types 18^4, $18^3 27^1$, 18^5, $18^4 27^1$, $18^5 27^1$, 18^7, $18^7 27^1$, 18^9, and $45^4 18^1$ constructed in Lemmas 5.6 and 5.7, add 0 or 1 points, and fill in the holes with G_i-designs of orders 18, 19, 27, 28, 45, or 46.

We obtain the final result of this section.

Theorem 5.8. *For $i \in \{16,17,19\}$, there exists a G_i-design of order n if and only if $n \equiv 0,1 \pmod 9$ and $n \neq 9$. There exists a G_{18}-design of order n if and only if $n \equiv 0,1 \pmod 9$ and $n \notin \{9,10\}$.*

6. Spectrum of the Graph G_{20}

In Lemmas 2.1 and 2.2, we have shown that G_{20}-designs of orders 9 or 10 do not exist. Also, as noted in the previous section, one can show using counting arguments that a G_{20}-GDD of type g^4 cannot exist when g is odd, and hence the strategy used in the previous section cannot be applied for the graph G_{20}. However, a G_{20}-GDD of type 18^4 may exist. Our strategy in this section is based on considering the cases $n \equiv 0, 1, 9, 10 \pmod{18}$ separately. The constructions made here do not make use of the existence of a G_{20}-design of order 18. This design has been constructed in [9] and we give an isomorphic copy of it here in Example A.5.

Lemma 6.1. *There exists a G_{20}-GDD of type 18^u for all $u \geq 4$.*

Proof. For $u \geq 4$ and $u \neq 6$, take a $\{4, 7\}$-GDD of type 3^u (Theorem 3.8) and apply Wilson's construction by assigning weight 6 to each point and using G_{20}-GDDs of types 6^4 and 6^7 (Examples A.28 and A.30). For $u = 6$, inflate a G_{20}-GDD of type 6^6 (Example A.29) by a factor of 3. □

Lemma 6.2. *There exists a G_{20}-design of order $18u + 1$ for all $u \geq 1$.*

Proof. For $u \in \{1, 2, 3\}$, see Examples A.10, A.15, and A.19. For $u \geq 4$, take a G_{20}-GDD of type 18^u from Lemma 6.1, add one point, and fill in the holes with G_{20}-designs of order 19. □

Lemma 6.3. *There exists a G_{20}-GDD of type $(2k)^3k^1$ for all $k \geq 2$, $k \neq 3$.*

Proof. Take a TD$(4, 2k)$ (Theorem 3.3) and label the points with the elements of $\mathbb{Z}_{2k} \times \{1, 2, 3, 4\}$, where the holes are on $\mathbb{Z}_{2k} \times \{b\}$ for $b \in \{1, 2, 3, 4\}$. For any $m, n \in \mathbb{Z}_{2k}$ where $0 \leq n < k$, there exist unique blocks containing the edges $\{m_1, n_4\}$ and $\{m_1, (k + n)_4\}$, say the blocks $\{m_1, p_2, q_3, n_4\}$ and $\{m_1, r_2, s_3, (k + n)_4\}$, where we necessarily have $p \neq r$ and $q \neq s$. For all m, n with $0 \leq m < 2k$ and $0 \leq n < k$, replace these two blocks with the block $[n_4, p_2, q_3, m_1, r_2, s_3]$. Finally, remove the points $(k+n)_4$ for $0 \leq n < k$ to obtain a G_{20}-GDD of type $(2k)^3k^1$. □

Lemma 6.4. *There exists a G_{20}-GDD of type $(36k)^3(18k + 3m)^1$ for all $k \geq 1$ and $0 \leq m \leq 9k$.*

Proof. Take a TD$(4, 9k)$ (Theorem 3.3), assign weight 5 to m points, and weight 2 to the remaining points in one of the groups, and weight 4 to all points in the remaining three groups. Apply Wilson's construction by using G_{20}-GDDs of types $4^3 2^1$ (Lemma 6.3) and $4^3 5^1$ (Example A.43). □

Lemma 6.5. *There exists a G_{20}-design of order n for all $n \equiv 0, 1 \pmod{9}$ and $19 \leq n \leq 91$.*

Proof. For $n \in \{19, 37, 55, 73, 91\}$, see Lemma 6.2. For $n \in \{27, 28, 36, 45, 46, 54, 63, 64, 81\}$ see Examples A.11–A.13, A.16–A.18, A.20, and A.22–A.23. For $n \in \{72, 82, 90\}$, take a G_{20}-GDD of type $1^{(n-27)}27^1$ (Examples A.40–A.42) and fill in the hole with a G_{20}-design of order 27. □

Proposition 6.6. *Let $u \geq 11$ and suppose that there exists a* $\mathrm{PBD}(\frac{u+m}{2}+1,\{4,5,6\})$ *for some $m \in \{0,1,2,3\}$. Then there exists a G_{20}-design of order $9u+a$ for all $a \in \{0,1\}$.*

Proof. Take a $\mathrm{PBD}(\frac{u+m}{2}+1,\{4,5,6\})$ and remove one point to obtain a $\{4,5,6\}$-GDD of type $3^a4^b5^c$ for some a,b,c with $3a+4b+5c=\frac{u+m}{2}$. Assign weight 9 to m points in the same group and weight 18 to the remaining points. Apply Wilson's construction by using G_{20}-GDDs of types 18^4, 18^5, 18^6, 18^39^1, 18^49^1, and 18^59^1 (Lemmas 6.1, 6.3 and Examples A.53, A.54) to obtain a G_{20}-GDD of type $(54)^d(72)^e(90)^f(9t)^1$ for some d,e,f,t with $3 \leq t \leq 10$ and $54d+72e+90f+9t=18(\frac{u+m}{2}-m)+9m=9u$. Add a points and fill in the holes with G_{20}-designs of orders $54+a$, $72+a$, $90+a$, and $9t+a$ (Lemma 6.5) to obtain a G_{20}-design of order $9u+a$. □

Theorem 6.7. *There exists a G_{20}-design of order n if and only if $n \equiv 0,1 \pmod 9$ and $n \notin \{9,10\}$.*

Proof. For $n \in \{9,10\}$, see Lemmas 2.1 and 2.2. For $n=18$, see Example A.5. For $19 \leq n \leq 91$, see Lemma 6.5. For $n \geq 99$, write $n=9u+a$ where $a \in \{0,1\}$ and $u \geq 11$. For $a=1$ and u even, see Lemma 6.2. For $a=0$, u even, and $u \notin \{12,14,16,18,20,22,26,28,34,36,44\}$, use Theorem 3.5 and apply Proposition 6.6 with $m=0$. For $a=0$ and $u \in \{22,28,36,44\}$, apply Proposition 6.6 with $m=2$. For $a \in \{0,1\}$, u odd, and $u \notin \{11,13,15,17,19,21,25,27,33,35,43\}$, apply Proposition 6.6 with $m=1$. For $a \in \{0,1\}$ and $u \in \{21,27,35,43\}$, apply Proposition 6.6 with $m=3$. These constructions leave the cases $u \in \{12,14,16,18,20,26,34\}$, $a=0$; and $u \in \{11,13,15,17,19,25,33\}$, $a \in \{0,1\}$.

For $u \in \{15,16,17,33,34\}$, take G_{20}-GDDs of types 36^327^1, 36^4, 36^345^1, 72^381^1, and 72^390^1 from Lemma 6.4, add a points, and fill in the holes.

For $u \in \{11,13\}$ and $a=1$, or $u \in \{18,19,26\}$, inflate G_{20}-GDDs of types 6^49^1, 6^59^1, 9^6, 12^49^1 and 15^418^1 (Examples A.34 and A.45–A.48) by a factor of 3 to obtain G_{20}-GDDs of types 18^427^1, 18^527^1, 27^6, 36^427^1 and 45^454^1. Then add a points and fill in the holes.

For $u \in \{20,25\}$, inflate a G_{20}-GDD of type 9^5 (Example A.33) by a factor of 4 or 5 to obtain G_{20}-GDDs of types 36^5 and 45^5. Then add a points and fill in the holes.

Finally, for $u \in \{11,12,13,14\}$ and $a=0$, take a G_{20}-GDD of type $16^5(9u-86)^1$ (Examples A.49–A.52), add 6 points, and fill in the holes with G_{20}-GDDs of type $1^{16}6^1$ (Example A.36) and a G_{20}-design of order $9u-80$.
□

Appendix A.

In what follows, a G-design of order n is denoted by $G-D(n)$, and a G-GDD of type T is denoted by $G-\mathrm{GDD}(T)$. In a few constructions we take $X=\{1,2,...,n\}$ and list all blocks explicitly. In all of the other constructions, we take $X=\left(\bigcup_{k\in S}(\mathbb{Z}_{m_k}\times U_k)\right)\cup(\{\infty\}\times U_\infty)$ for some S,

m_k, U_k and U_∞, where U_∞ is possibly empty. We denote (x, y) as x_y, and when $|U_\infty| = 1$ we omit the subscript of ∞. Then the construction is made by developing the given base blocks with the permutation $x_y \to (x+1)_y$, where addition is modulo m_k when $y \in U_k$ for all $k \in S$. The infinite points are fixed.

Example A.1. *$H_{12} - D(32)$ on $X = (\mathbb{Z}_4 \times \{1,2,3,4,5,6\}) \cup (\mathbb{Z}_2 \times \{7,8\}) \cup (\{\infty\} \times \{a,b,c,d\})$. There are 12 orbits of length 4, 6 short orbits of length 2, and 2 fixed blocks.*

$[0_1, 1_1, 3_2, 2_2, 3_3, \infty_c]$	$[0_1, 0_2, 2_3, 3_3, 1_2, \infty_d]$	$[0_1, 0_3, 1_3, 2_4, 3_1, \infty_d]$
$[0_1, 0_4, 1_4, 1_5, 2_1, \infty_a]$	$[0_1, 0_5, 2_5, 1_6, 0_2, \infty_a]$	$[0_2, 0_3, 3_4, 0_6, 2_1, \infty_b]$
$[0_6, 0_1, 1_1, 1_7, 0_2, 0_4]$	$[0_2, 1_4, 2_4, 0_5, 0_3, 1_7]$	$[0_5, 1_2, 1_3, 3_5, 2_2, \infty_b]$
$[0_4, 0_6, 0_8, 1_4, 2_3, \infty_b]$	$[0_5, 3_3, 0_8, 0_6, 1_2, 3_6]$	$[0_8, 0_3, 1_5, 3_6, 1_3, 1_4]$
$[0_7, 0_2, 2_2, 0_8, 0_1, 2_1]$	$[0_7, 0_3, 2_3, 1_8, 0_2, 2_2]$	$[0_7, 0_4, 2_4, \infty_a, 0_3, 2_3]$
$[0_7, 0_5, 2_5, \infty_c, 0_4, 2_4]$	$[0_7, 0_6, 2_6, \infty_d, 0_5, 2_5]$	$[0_8, 1_1, 3_1, \infty_c, 0_6, 2_6]$
$[\infty_a, 0_8, 1_8, \infty_b, 0_7, 1_7]$	$[\infty_c, \infty_a, \infty_b, \infty_d, 0_8, 1_8]$	

Example A.2. *$H_{13} - D(32)$ on $X = (\mathbb{Z}_4 \times \{1,2,3,4\}) \cup (\mathbb{Z}_2 \times \{5,6,7,8,9\}) \cup (\{\infty\} \times \{a,b,c,d,e,f\})$. There are 11 orbits of length 4, 7 short orbits of length 2, and 4 fixed blocks.*

$[0_5, 0_1, 0_2, \infty_a, 0_3, 1_4]$	$[0_5, 1_1, 3_2, \infty_b, 0_3, 2_4]$	$[0_5, 0_3, 1_3, 0_6, 0_1, 1_1]$
$[0_5, 0_4, 1_4, 0_6, 0_2, 1_2]$	$[0_7, 1_1, 0_3, \infty_c, 0_2, 0_4]$	$[\infty_d, 0_2, 1_4, 0_7, 1_2, 0_4]$
$[\infty_d, 0_1, 2_3, 1_8, 1_1, 3_4]$	$[0_8, 1_2, 0_3, 0_9, 1_1, 0_2]$	$[\infty_e, 0_1, 1_4, 0_9, 1_3, 0_4]$
$[\infty_e, 0_2, 1_3, \infty_f, 0_1, 3_4]$	$[0_1, 0_3, 0_4, 2_2, 1_1, 2_3]$	
$[0_5, 0_6, \infty_b, 0_7, 0_1, 2_1]$	$[0_5, 1_6, 0_7, 0_8, 0_2, 2_2]$	$[0_5, 0_8, 0_9, 1_7, 0_3, 2_3]$
$[0_6, 0_9, \infty_e, 1_8, 0_4, 2_4]$	$[\infty_e, 0_5, 1_7, \infty_a, 0_6, 1_9]$	
$[\infty_d, 0_6, 0_8, \infty_c, 0_5, 1_9]$	$[\infty_d, 0_7, 0_9, \infty_f, 0_5, 1_8]$	
$[\infty_a, \infty_b, \infty_c, \infty_d, 0_5, 1_5]$	$[\infty_e, \infty_a, \infty_d, \infty_f, 0_6, 1_6]$	
$[\infty_b, \infty_e, \infty_f, \infty_c, 0_7, 1_7]$	$[\infty_a, 0_8, 1_8, \infty_b, 0_9, 1_9]$	

Example A.3. *$G_i - D(10)$ for $i \in \{11,12,14,15,16,17,19\}$ on $X = \mathbb{Z}_5 \times \{1,2\}$.*

i	*Base Block*	i	*Base Block*
11	$[0_1, 1_1, 0_2, 2_2, 1_2, 3_1]$	12	$[0_1, 1_1, 4_2, 1_2, 0_2, 3_1]$
14	$[0_2, 0_1, 1_1, 2_2, 1_2, 4_1]$	15	$[1_1, 0_2, 0_1, 3_1, 2_2, 3_2]$
16	$[0_1, 2_2, 1_1, 4_2, 4_1, 1_2]$	17	$[0_1, 2_2, 1_1, 4_2, 4_1, 0_2]$
19	$[0_1, 1_1, 3_1, 2_2, 4_2, 3_2]$		

Example A.4. *$G_{13} - D(10)$ on $X = \{1,2,3,\ldots,10\}$.*

$[1,2,3,4,5,6]$	$[1,4,7,8,9,2]$	$[3,5,8,10,9,4]$
$[10,3,7,6,1,8]$	$[9,5,7,2,6,10]$	

Example A.5. *G_i–$D(18)$ for $i \in \{9, 11, 12, 13, 20\}$ on $X = \{1, 2, 3, \ldots, 18\}$.*

i	*Blocks*		
9	$[2,3,4,5,1,6]$	$[7,8,9,10,1,11]$	$[12,13,14,15,1,16]$
	$[17,2,7,12,18,1]$	$[3,7,10,13,6,16]$	$[4,8,11,14,6,17]$
	$[5,9,12,15,6,18]$	$[2,8,3,18,13,4]$	$[3,12,4,8,15,10]$
	$[3,9,2,4,11,18]$	$[5,14,3,12,17,15]$	$[7,12,10,11,5,4]$
	$[8,5,2,13,16,17]$	$[9,13,10,11,17,7]$	$[2,15,11,18,14,10]$
	$[7,14,4,9,16,15]$	$[16,18,4,9,10,11]$	
11	$[1,2,3,4,5,7]$	$[1,6,7,8,9,12]$	$[1,10,11,12,13,15]$
	$[14,15,1,16,2,8]$	$[3,17,6,13,18,1]$	$[2,13,9,15,17,5]$
	$[2,6,10,12,18,8]$	$[4,2,11,6,9,5]$	$[3,5,8,11,15,6]$
	$[14,3,7,18,6,16]$	$[10,3,9,14,4,12]$	$[5,10,18,11,14,17]$
	$[12,7,15,5,16,9]$	$[8,4,17,12,14,13]$	$[4,7,16,15,18,17]$
	$[17,9,11,13,16,18]$	$[10,7,13,8,16,11]$	
12	$[1,2,3,4,5,6]$	$[1,7,8,9,10,11]$	$[1,12,13,14,15,16]$
	$[17,1,18,3,5,8]$	$[2,17,4,6,7,9]$	$[2,5,10,12,14,16]$
	$[2,8,11,13,15,18]$	$[5,7,11,9,14,18]$	$[12,3,6,8,15,5]$
	$[13,3,7,4,16,5]$	$[15,3,9,17,11,4]$	$[10,7,15,6,13,17]$
	$[17,12,7,14,6,16]$	$[14,8,4,10,3,11]$	$[16,3,8,10,18,7]$
	$[18,8,13,9,6,11]$	$[12,11,16,9,4,18]$	
13	$[1,2,3,5,4,8]$	$[1,5,6,9,7,11]$	$[1,8,9,12,10,14]$
	$[1,11,12,15,13,17]$	$[1,14,15,18,16,2]$	$[4,6,10,17,18,1]$
	$[2,4,7,14,11,3]$	$[2,9,13,4,17,11]$	$[6,3,12,4,15,16]$
	$[2,6,14,16,12,17]$	$[5,10,15,9,17,14]$	$[10,2,8,15,3,7]$
	$[17,3,8,13,7,12]$	$[14,5,13,11,8,6]$	$[8,7,16,5,18,12]$
	$[7,6,18,13,10,16]$	$[16,3,9,18,11,10]$	
20	$[1,2,3,4,5,6]$	$[1,7,8,9,10,4]$	$[1,10,14,6,12,15]$
	$[1,11,12,13,5,9]$	$[1,16,17,5,2,18]$	$[4,8,17,15,1,18]$
	$[2,7,16,6,3,9]$	$[2,8,14,12,5,10]$	$[2,10,17,13,15,16]$
	$[2,11,15,9,12,16]$	$[3,5,11,8,10,16]$	$[3,7,17,12,4,18]$
	$[4,11,16,14,3,13]$	$[5,7,14,15,3,10]$	$[7,10,11,18,3,16]$
	$[6,8,18,13,4,7]$	$[9,14,18,17,6,11]$	

Example A.6. *G_{14}–$D(18)$ on $X = \mathbb{Z}_2 \times \{1, 2, 3, \ldots, 9\}$. There are 4 orbits of length 2, and 9 fixed blocks.*

$[0_4,0_2,1_6,0_3,0_5,1_9]$	$[0_4,0_8,0_1,0_6,0_7,1_9]$	$[0_4,1_5,1_7,0_5,0_9,0_8]$
$[0_5,1_2,1_1,0_2,0_7,0_8]$		
$[0_3,0_9,0_1,1_1,1_9,1_3]$	$[0_3,1_4,0_2,1_2,0_4,1_3]$	$[0_7,1_6,1_3,0_3,0_6,1_7]$
$[0_9,1_8,0_4,1_4,0_8,1_9]$	$[0_6,1_1,1_5,0_5,0_1,1_6]$	$[0_8,1_2,1_6,0_6,0_2,1_8]$
$[0_1,1_4,1_7,0_7,0_4,1_1]$	$[0_3,1_5,0_8,1_8,0_5,1_3]$	$[0_2,1_7,1_9,0_9,0_7,1_2]$

Example A.7. *$G_{15} - D(18)$ on $X = (\mathbb{Z}_2 \times \{1, 2, 3, \ldots, 7\}) \cup (\{\infty\} \times \{a, b, c, d\})$. There are 5 orbits of length 2, and 7 fixed blocks.*

$[0_4, 0_7, 0_6, \infty_a, \infty_b, 0_3]$	$[0_7, \infty_c, 0_5, 1_4, 1_6, \infty_a]$
$[0_4, 1_6, 0_2, 1_2, 0_3, 0_1]$	$[0_1, 1_3, 0_4, 0_7, 1_5, 1_1]$
$[0_3, 1_5, 0_2, 1_2, 0_7, 1_1]$	
$[0_1, 1_1, \infty_b, \infty_a, \infty_d, \infty_c]$	$[0_2, 1_2, \infty_c, \infty_b, \infty_d, \infty_a]$
$[0_3, 1_3, \infty_a, \infty_c, \infty_d, \infty_b]$	$[0_4, 1_4, 0_5, \infty_d, 1_5, \infty_b]$
$[0_5, 1_5, 0_6, \infty_d, 1_6, \infty_b]$	$[0_6, 1_6, 0_1, \infty_d, 1_1, \infty_c]$
$[0_7, 1_7, 0_2, \infty_d, 1_2, \infty_a]$	

Example A.8. *$G_i - D(18)$ for $i \in \{16, 17, 18\}$ on $X = \mathbb{Z}_{17} \cup \{\infty\}$.*

$[4, 6, 0, 1, 8, \infty]$

Example A.9. *$G_{19} - D(18)$ on $X = \mathbb{Z}_2 \times \{1, 2, 3, \ldots, 9\}$. There are 7 orbits of length 2, and 3 fixed blocks.*

$[0_1, 1_3, 1_1, 0_4, 1_8, 0_5]$	$[0_1, 1_4, 1_8, 0_6, 1_9, 1_5]$	$[0_3, 1_2, 0_8, 0_5, 1_6, 0_1]$
$[0_4, 1_2, 1_5, 0_7, 1_9, 1_1]$	$[0_3, 1_5, 1_7, 0_9, 0_7, 0_1]$	$[0_2, 1_6, 1_8, 0_8, 1_9, 0_3]$
$[0_3, 0_4, 0_2, 1_7, 0_6, 0_7]$		
$[0_1, 0_2, 0_3, 1_1, 1_2, 1_3]$	$[0_4, 0_5, 0_6, 1_4, 1_5, 1_6]$	$[0_7, 0_8, 0_9, 1_7, 1_8, 1_9]$

Example A.10. *$G_i - D(19)$ for $11 \le i \le 20$ on $X = \mathbb{Z}_{19}$.*

i	*Base Block*	i	*Base Block*
11	$[0, 1, 3, 9, 5, 12]$	12	$[0, 1, 3, 8, 15, 6]$
13	$[0, 1, 3, 10, 4, 15]$	14	$[0, 1, 3, 7, 2, 13]$
15	$[0, 1, 3, 5, 9, 15]$	16	$[0, 1, 3, 7, 12, 9]$
17	$[0, 1, 3, 7, 12, 15]$	18	$[0, 1, 3, 7, 12, 4]$
19	$[0, 1, 3, 4, 9, 15]$	20	$[0, 1, 3, 8, 2, 12]$

Example A.11. *$G_i - D(27)$ for $11 \le i \le 20$ on $X = (\mathbb{Z}_{13} \times \{1, 2\}) \cup \{\infty\}$.*

i	*Base Blocks*		
11	$[0_1, 1_1, 3_1, 7_1, 0_2, \infty]$	$[0_1, 5_1, 1_2, 2_2, 4_2, 0_2]$	$[0_1, 5_2, 12_2, 7_2, 10_2, 7_1]$
12	$[0_1, 1_1, 3_1, 7_1, 0_2, \infty]$	$[0_1, 1_2, 2_2, 5_1, 3_2, 8_2]$	$[0_1, 4_2, 7_2, 9_2, 5_2, 12_2]$
13	$[0_1, 1_1, 3_1, 7_1, 0_2, \infty]$	$[0_1, 5_1, 1_2, 0_2, 2_2, 3_1]$	$[0_1, 4_2, 7_2, 0_2, 5_2, 2_1]$
14	$[0_1, 1_1, 3_1, 7_1, \infty, 0_2]$	$[0_1, 5_1, 0_2, 1_2, 2_1, 4_2]$	$[0_1, 4_2, 9_2, 3_2, 5_2, 11_1]$
15	$[0_1, 2_1, 5_1, 6_1, 12_2, \infty]$	$[0_1, 1_1, 3_2, 5_2, 8_2, 10_1]$	$[0_2, 1_2, 3_2, 5_2, 0_1, 9_2]$
16	$[0_1, 1_1, 3_1, 9_1, 0_2, \infty]$	$[0_1, 8_2, 5_2, 9_2, 6_1, \infty]$	$[0_1, 0_2, 2_2, 7_2, 1_1, 2_1]$
17	$[0_1, 1_1, 3_1, 9_1, 0_2, \infty]$	$[0_1, 5_2, 8_2, 9_2, 10_1, \infty]$	$[0_1, 0_2, 2_2, 7_2, 1_1, 4_1]$
18	$[0_1, 1_1, 3_1, 0_2, 7_1, \infty]$	$[0_1, 6_1, 1_2, 11_2, 2_2, \infty]$	$[0_1, 5_1, 7_2, 9_2, 1_2, 11_1]$
19	$[0_1, 3_1, 1_1, 9_1, 4_2, \infty]$	$[0_1, 1_1, 9_1, 7_2, 11_2, 5_2]$	$[0_1, 0_2, 1_2, 2_2, 5_2, 2_1]$
20	$[0_1, 2_1, 7_2, 10_2, 4_1, \infty]$	$[0_1, 1_1, 0_2, 2_2, 1_2, 6_2]$	$[0_1, 3_2, 9_2, 5_1, 1_1, 8_1]$

Example A.12. *$G_i - D(28)$ for $16 \le i \le 20$ on $X = \mathbb{Z}_7 \times \{1,2,3,4\}$.*

i	*Base Blocks*		
16	$[0_1,1_1,3_1,0_2,1_2,3_2]$	$[0_1,1_2,3_2,0_3,0_2,2_3]$	$[0_2,2_3,0_4,2_4,3_1,0_1]$
	$[0_3,4_2,1_4,2_4,0_1,5_4]$	$[0_3,1_3,2_1,3_3,0_4,3_2]$	$[0_1,4_3,0_4,3_4,4_2,1_1]$
17	$[0_1,1_1,3_1,0_2,1_2,5_1]$	$[0_1,1_2,3_2,0_3,0_2,1_1]$	$[0_1,1_3,2_3,4_3,6_1,3_2]$
	$[0_2,2_3,0_4,1_4,0_1,3_1]$	$[0_2,5_3,2_4,5_4,6_1,2_2]$	$[0_2,3_3,4_4,6_4,2_1,4_3]$
18	$[0_1,1_1,3_1,0_2,1_2,6_1]$	$[0_1,1_2,3_2,0_3,0_2,0_4]$	$[0_1,1_3,2_3,4_3,1_2,0_4]$
	$[0_2,5_4,5_3,2_4,6_1,6_4]$	$[0_4,1_4,4_2,6_3,5_4,0_1]$	$[0_1,3_3,1_4,6_4,4_1,2_3]$
19	$[0_1,1_1,3_1,0_2,2_2,6_1]$	$[0_1,4_2,0_2,0_3,5_2,4_3]$	$[0_1,1_3,0_2,2_3,4_3,0_4]$
	$[0_3,0_4,3_1,6_4,2_2,2_3]$	$[0_3,2_4,0_2,4_4,4_1,6_4]$	$[0_4,3_4,2_2,2_3,4_1,5_4]$
20	$[0_1,1_1,3_1,0_2,2_1,3_2]$	$[0_1,3_2,0_3,2_3,0_2,1_2]$	$[0_2,2_2,0_3,3_4,5_1,4_3]$
	$[0_1,2_2,0_4,1_4,2_1,4_4]$	$[0_1,1_3,5_3,3_4,3_2,6_3]$	$[0_1,3_3,4_3,4_4,0_2,2_4]$

Example A.13. *$G_i - D(36)$ for $i \in \{11,12,13,14,15,19,20\}$ on $X = (\mathbb{Z}_7 \times \{1,2,3,4,5\}) \cup \{\infty\}$.*

i	*Base Blocks*		
11	$[0_1,1_1,3_1,0_2,1_2,\infty]$	$[0_3,2_3,4_1,0_4,\infty,0_5]$	$[0_1,2_2,5_2,3_2,0_3,3_1]$
	$[0_1,1_3,2_3,6_3,0_4,0_2]$	$[0_2,0_3,2_4,3_3,1_4,2_2]$	$[0_2,2_3,5_4,3_4,4_4,1_5]$
	$[0_2,2_5,6_3,3_5,5_5,4_1]$	$[0_3,0_5,4_4,2_5,1_5,0_2]$	$[0_1,0_5,5_4,1_4,2_5,3_2]$
	$[0_5,4_5,1_1,0_4,3_1,5_4]$		
12	$[\infty,0_1,6_2,3_3,4_4,6_5]$	$[0_1,1_1,3_1,0_2,1_2,3_2]$	$[0_1,2_2,5_2,0_3,1_3,3_3]$
	$[0_1,2_3,5_3,0_4,4_3,1_4]$	$[0_1,6_3,4_4,2_4,3_4,6_4]$	$[0_4,2_1,0_5,0_2,0_3,1_2]$
	$[0_5,0_1,1_5,5_2,3_3,2_4]$	$[0_5,1_1,4_5,1_3,2_5,5_1]$	$[0_2,1_3,1_5,3_3,5_5,4_4]$
	$[0_5,1_4,3_2,4_4,1_2,3_4]$		
13	$[0_1,1_1,0_2,3_1,1_2,\infty]$	$[\infty,0_3,3_4,5_1,0_5,1_1]$	$[0_1,2_2,3_2,0_2,0_3,1_3]$
	$[0_1,1_3,3_3,4_1,0_4,2_3]$	$[0_1,1_4,2_4,0_3,4_4,4_3]$	$[0_1,6_4,0_5,3_2,1_5,0_3]$
	$[0_1,3_5,4_5,2_2,5_5,0_3]$	$[0_2,2_3,1_4,3_2,0_4,2_4]$	$[0_3,0_5,2_5,5_4,4_5,1_4]$
	$[0_5,1_3,4_5,5_2,0_4,5_5]$		
14	$[0_1,1_1,3_1,0_2,0_3,\infty]$	$[0_1,3_2,5_2,0_4,\infty,0_5]$	$[0_1,1_2,2_2,6_2,0_3,1_3]$
	$[0_1,1_3,4_3,6_3,4_1,3_4]$	$[0_1,0_3,3_4,5_3,2_1,0_4]$	$[0_1,4_4,2_4,4_5,2_1,3_5]$
	$[0_1,1_4,5_5,0_5,6_2,2_3]$	$[0_2,4_4,4_3,3_5,0_1,6_5]$	$[0_2,0_4,1_4,0_5,2_4,3_2]$
	$[0_3,0_5,2_2,1_5,4_3,5_4]$		
15	$[0_1,1_1,3_1,0_2,2_2,\infty]$	$[0_3,0_4,1_1,2_1,\infty,0_5]$	$[0_1,3_2,4_2,5_2,0_3,0_2]$
	$[0_1,1_3,2_3,3_3,4_3,1_2]$	$[0_2,2_3,0_4,3_4,6_4,3_1]$	$[0_2,6_3,1_4,2_4,5_4,0_4]$
	$[0_2,5_3,1_5,4_5,3_5,5_1]$	$[0_1,0_5,0_4,1_4,1_5,3_2]$	$[0_5,2_5,5_1,0_3,4_4,0_4]$
	$[0_5,3_5,1_2,6_3,6_4,1_5]$		
19	$[0_1,1_1,3_1,0_2,2_2,\infty]$	$[0_3,1_4,5_1,\infty,2_5,1_1]$	$[0_1,3_2,5_1,4_2,0_3,1_3]$
	$[0_1,1_3,0_2,5_3,0_4,3_2]$	$[0_1,2_3,3_1,4_3,2_4,0_4]$	$[0_1,1_4,2_1,5_4,0_5,1_5]$
	$[0_2,0_3,4_5,3_4,3_5,0_1]$	$[0_2,6_3,4_5,5_5,0_5,2_1]$	$[0_2,2_3,4_2,6_4,2_5,5_5]$
	$[0_4,5_2,5_4,6_4,4_5,0_2]$		
20	$[0_1,1_1,3_1,0_2,2_1,\infty]$	$[0_3,3_4,\infty,6_5,2_1,5_2]$	$[0_1,1_2,2_2,0_3,1_1,2_3]$
	$[0_1,2_3,5_3,0_4,0_2,2_2]$	$[0_1,3_3,2_4,3_4,0_2,4_2]$	$[0_1,4_3,1_4,0_5,2_1,3_5]$
	$[0_2,3_3,4_3,4_5,5_2,0_3]$	$[0_1,4_4,6_4,6_5,1_2,1_3]$	$[0_2,1_3,2_4,3_5,1_2,1_5]$
	$[0_1,2_5,3_5,5_4,1_2,2_4]$		

Example A.14. *$G_i - D(36)$ for $i \in \{16, 17, 18\}$ on $X = \mathbb{Z}_{35} \cup \{\infty\}$.*

i	*Base Blocks*	
16	$[0, 4, 10, 17, 2, 20]$	$[0, 1, 3, 12, 17, \infty]$
17	$[0, 4, 10, 17, 2, 1]$	$[0, 1, 3, 12, 17, \infty]$
18	$[0, 4, 10, 17, 2, 18]$	$[0, 1, 3, 12, 17, \infty]$

Example A.15. *$G_i - D(37)$ for $i \in \{18, 20\}$ on $X = \mathbb{Z}_{37}$.*

i	*Base Blocks*	
18	$[0, 1, 4, 9, 15, 2]$	$[0, 2, 12, 19, 33, 11]$
20	$[0, 1, 4, 9, 2, 19]$	$[0, 2, 15, 21, 7, 32]$

Example A.16. *$G_i - D(45)$ for $i \in \{18, 20\}$ on $X = (\mathbb{Z}_{11} \times \{1, 2, 3, 4\}) \cup \{\infty\}$.*

i	*Base Blocks*		
18	$[0_1, 3_2, 0_3, \infty, 0_4, 1_1]$	$[0_1, 1_1, 3_1, 0_2, 7_1, 2_1]$	$[0_1, 1_2, 2_2, 5_2, 7_1, 3_2]$
	$[0_1, 2_3, 4_3, 8_3, 1_2, 6_2]$	$[0_2, 2_2, 0_3, 1_3, 2_1, 3_3]$	$[0_1, 3_3, 6_3, 0_4, 10_1, 1_4]$
	$[0_3, 1_4, 6_1, 2_4, 0_4, 2_1]$	$[0_2, 2_3, 0_4, 5_4, 8_1, 1_4]$	$[0_2, 4_3, 3_4, 10_4, 8_2, 3_3]$
	$[0_2, 5_3, 1_4, 9_4, 5_2, 2_4]$		
20	$[0_1, 1_1, 3_1, 0_2, 4_1, \infty]$	$[0_1, 5_1, 3_2, 0_3, 0_4, \infty]$	$[0_1, 4_1, 5_2, 6_2, 2_1, 0_3]$
	$[0_1, 1_3, 2_3, 4_3, 5_1, 0_4]$	$[0_1, 3_3, 7_3, 0_4, 1_1, 9_3]$	$[0_1, 5_3, 3_4, 4_4, 2_1, 0_4]$
	$[0_2, 2_2, 2_3, 7_4, 0_1, 1_4]$	$[0_2, 3_2, 1_4, 3_4, 6_1, 0_4]$	$[0_2, 4_2, 6_4, 3_3, 2_2, 8_3]$
	$[0_2, 5_2, 9_3, 4_4, 7_2, 3_3]$		

Example A.17. *$G_i - D(46)$ for $i \in \{18, 20\}$ on $X = \mathbb{Z}_{23} \times \{1, 2\}$.*

i	*Base Blocks*	
18	$[0_1, 1_1, 6_1, 16_1, 0_2, 1_2]$	$[0_1, 2_1, 11_1, 0_2, 2_2, 5_2]$
	$[0_1, 3_1, 2_2, 6_2, 5_1, 0_2]$	$[0_1, 4_1, 8_2, 15_2, 3_2, 13_2]$
	$[0_1, 5_2, 13_2, 19_2, 3_1, 12_2]$	
20	$[0_1, 1_1, 3_1, 10_1, 2_1, 0_2]$	$[0_1, 5_1, 11_1, 0_2, 1_1, 2_2]$
	$[0_1, 4_1, 6_2, 7_2, 13_1, 1_2]$	$[0_1, 5_2, 14_2, 9_2, 1_1, 16_2]$
	$[0_2, 3_2, 11_2, 7_1, 4_2, 17_2]$	

Example A.18. *$G_{20} - D(54)$ on $X = (\mathbb{Z}_6 \times \{1, 2, 3, 4, 5, 6\}) \cup (\mathbb{Z}_3 \times \{7, 8, 9, a, b, c\})$. There are 22 orbits of length 6, and 9 short orbits of length 3.*

$[0_1, 1_1, 0_2, 2_7, 1_2, 0_4]$	$[0_1, 2_1, 1_8, 3_2, 0_3, 0_7]$	$[0_1, 2_2, 1_9, 0_3, 0_5, 2_7]$
$[0_1, 4_2, 2_9, 0_5, 0_6, 0_7]$	$[0_5, 2_6, 1_7, 3_6, 0_1, 2_3]$	$[0_1, 1_3, 0_4, 0_9, 1_4, 2_4]$
$[0_5, 1_5, 1_9, 5_6, 0_1, 3_3]$	$[0_1, 4_3, 0_a, 5_3, 0_5, 2_8]$	$[0_1, 1_4, 1_a, 5_4, 1_2, 0_8]$
$[0_1, 4_4, 2_a, 1_6, 0_4, 0_8]$	$[0_1, 2_4, 2_b, 4_6, 5_4, 1_8]$	$[0_3, 2_3, 1_8, 0_6, 0_1, 0_b]$
$[0_1, 3_4, 2_c, 4_5, 2_1, 0_b]$	$[0_2, 1_2, 1_4, 5_6, 3_1, 4_5]$	$[0_2, 5_3, 3_4, 1_c, 1_1, 4_5]$
$[0_2, 1_3, 0_6, 2_c, 1_1, 0_5]$	$[0_2, 2_2, 2_b, 3_6, 1_6, 0_a]$	$[0_3, 3_4, 3_6, 0_c, 0_2, 2_6]$
$[0_2, 2_4, 0_5, 4_5, 0_3, 1_4]$	$[0_2, 2_3, 5_5, 2_a, 1_2, 4_5]$	$[0_3, 2_4, 0_b, 2_5, 1_2, 1_a]$
$[0_3, 0_4, 5_5, 2_b, 1_2, 1_3]$		
$[0_1, 3_1, 0_8, 0_7, 0_9, 0_a]$	$[0_2, 3_2, 0_9, 0_8, 1_7, 0_b]$	$[0_3, 3_3, 1_7, 0_9, 1_8, 0_c]$
$[0_4, 3_4, 1_7, 0_7, 1_8, 0_b]$	$[0_5, 3_5, 1_8, 0_8, 1_9, 0_c]$	$[0_6, 3_6, 1_9, 0_9, 2_7, 1_a]$
$[0_a, 0_c, 2_c, 2_7, 0_b, 1_c]$	$[0_b, 0_a, 1_a, 2_8, 0_c, 2_a]$	$[0_c, 0_b, 1_b, 2_9, 1_a, 2_b]$

Example A.19. *$G_i - D(55)$ for $i \in \{18, 20\}$ on $X = \mathbb{Z}_{55}$.*

i	*Base Blocks*		
18	$[0, 1, 5, 11, 18, 38]$	$[0, 2, 19, 27, 1, 23]$	$[0, 3, 15, 24, 1, 17]$
20	$[0, 1, 5, 11, 3, 34]$	$[0, 2, 16, 29, 9, 46]$	$[0, 3, 15, 22, 1, 31]$

Example A.20. *$G_i - D(63)$ for $i \in \{18, 19, 20\}$ on $X = (\mathbb{Z}_{31} \times \{1, 2\}) \cup \{\infty\}$.*

i	*Base Blocks*	
18	$[0_1, 1_1, 6_1, 13_1, 22_1, \infty]$	$[0_1, 2_1, 10_1, 0_2, 1_2, \infty]$
	$[0_1, 3_1, 14_1, 2_2, 0_2, 4_1]$	$[0_1, 4_1, 1_2, 7_2, 20_1, 15_2]$
	$[0_1, 5_2, 8_2, 16_2, 24_1, 2_2]$	$[0_1, 6_2, 10_2, 24_2, 5_2, 11_1]$
	$[0_2, 7_2, 18_1, 22_2, 1_2, 21_1]$	
19	$[0_1, 1_1, 11_1, 12_1, 0_2, \infty]$	$[0_1, 2_1, 7_1, 8_1, 15_1, 24_1]$
	$[0_1, 3_1, 0_2, 1_2, 7_2, 5_1]$	$[0_1, 4_1, 15_2, 14_2, 13_2, 4_2]$
	$[0_1, 5_2, 13_1, 21_2, 17_2, 0_2]$	$[0_1, 6_2, 3_2, 24_2, 16_2, 22_1]$
	$[0_2, 7_2, 4_1, 2_2, 11_1, 19_2]$	
20	$[0_1, 1_1, 7_1, 0_2, 2_1, \infty]$	$[0_1, 2_1, 11_1, 14_2, 4_1, 19_2]$
	$[0_1, 3_1, 18_1, 23_2, 9_2, 0_2]$	$[0_1, 4_1, 8_2, 14_1, 2_2, 0_2]$
	$[0_1, 5_1, 1_2, 2_2, 15_1, 7_1]$	$[0_1, 6_2, 13_2, 16_2, 26_1, 7_1]$
	$[0_1, 7_2, 11_2, 22_2, 9_2, 3_2]$	

Example A.21. *$G_{18} - D(64)$ on $X = (\mathbb{Z}_{14} \times \{1, 2, 3\}) \cup (\mathbb{Z}_7 \times \{4, 5, 6\}) \cup \{\infty\}$. There are 13 orbits of length 14, and 6 short orbits of length 7.*

$[0_1, 6_1, 11_1, 8_2, 1_1, 4_6]$	$[0_1, 2_1, 3_2, 12_2, 8_1, 3_6]$	$[0_1, 1_1, 0_2, 6_2, 2_6, 2_1]$
$[0_1, 2_3, 5_3, 4_4, 1_1, 0_6]$	$[0_1, 6_3, 8_3, 0_5, 10_1, 0_6]$	$[0_1, 3_3, 7_3, 1_6, 10_1, 5_3]$
$[0_1, 0_3, 1_3, 1_4, 2_1, 0_4]$	$[0_3, 4_4, 2_2, 5_2, 3_4, 1_1]$	$[0_2, 1_2, 0_3, 3_5, 4_1, 0_5]$
$[0_2, 0_4, 4_2, 11_3, 1_5, 6_1]$	$[0_2, 0_6, 1_3, 10_3, 1_5, 0_1]$	$[0_2, 2_2, 6_3, 1_5, 3_2, 0_4]$
$[0_2, 2_3, 8_3, 4_6, 3_2, 2_6]$		
$[0_4, 1_4, 3_4, 0_5, 1_6, 4_4]$	$[0_5, 1_5, 3_5, 0_6, 1_4, 4_5]$	$[0_6, 1_6, 3_6, 0_4, 1_5, 4_6]$
$[0_4, 5_5, 0_1, 7_1, \infty, 0_6]$	$[0_5, 5_6, 0_2, 7_2, \infty, 0_4]$	$[0_6, 5_4, 0_3, 7_3, \infty, 0_5]$

Example A.22. *$G_{20} - D(64)$ on $X = (\mathbb{Z}_{21} \times \{1, 2\}) \cup (\mathbb{Z}_7 \times \{3, 4, 5\}) \cup \{\infty\}$. There are 9 orbits of length 21, and 5 short orbits of length 7.*

$[0_1, 1_1, 3_1, 0_2, 2_1, \infty]$	$[0_1, 8_1, 3_2, 0_4, 3_1, 18_1]$	$[0_1, 10_1, 4_5, 4_2, 11_1, 6_4]$
$[0_1, 9_2, 5_3, 6_2, 15_1, 6_4]$	$[0_2, 9_2, 1_4, 5_2, 16_1, 3_4]$	$[0_1, 9_1, 5_2, 0_5, 1_1, 5_1]$
$[0_1, 1_2, 2_2, 3_3, 1_1, 6_1]$	$[0_2, 6_2, 3_5, 14_1, 1_2, 6_3]$	$[0_2, 10_2, 1_5, 2_2, 12_1, 6_3]$
$[0_1, 7_1, 14_1, 0_3, 1_3, 3_3]$	$[0_2, 7_2, 14_2, 0_3, 1_4, 3_4]$	$[0_3, 0_5, \infty, 0_4, 1_5, 3_5]$
$[2_4, 6_4, 1_5, 0_3, 3_5, 6_5]$	$[4_4, 5_4, 2_5, 0_3, 4_5, 5_5]$	

Example A.23. *$G_{20}-D(81)$ on $X=(\mathbb{Z}_{18}\times\{1,2,3\})\cup(\mathbb{Z}_9\times\{4,5,6\})$. There are 17 orbits of length 18, and 6 short orbits of length 9.*

$[0_1,1_1,3_1,0_6,4_1,15_1]$	$[0_1,4_1,10_1,2_6,7_1,0_2]$	$[0_2,1_2,3_2,6_6,5_2,1_3]$
$[0_2,4_2,10_2,8_6,0_3,2_3]$	$[0_3,1_3,4_3,2_6,7_3,17_3]$	$[0_1,5_1,10_3,7_4,0_2,9_3]$
$[0_1,0_2,13_2,6_4,3_2,15_3]$	$[0_1,1_2,2_3,1_4,5_1,15_2]$	$[0_1,2_2,9_3,3_4,8_1,2_3]$
$[0_1,3_2,13_3,8_4,0_2,3_3]$	$[0_1,4_2,15_2,1_5,2_1,0_3]$	$[0_1,5_2,3_3,3_5,1_1,0_2]$
$[0_1,6_2,11_3,7_5,2_1,1_3]$	$[0_1,7_2,4_3,6_5,4_2,8_3]$	$[0_1,8_2,1_3,4_5,0_3,5_3]$
$[0_1,9_2,8_3,15_3,1_1,15_2]$	$[0_1,0_3,6_3,16_2,4_1,11_3]$	
$[0_1,9_1,0_4,0_5,2_5,5_5]$	$[0_2,9_2,0_5,0_6,2_6,5_6]$	$[0_3,9_3,0_6,2_4,4_4,7_4]$
$[1_4,8_5,2_6,0_4,1_5,3_6]$	$[4_4,1_5,8_6,0_5,5_4,1_6]$	$[1_4,4_5,1_6,0_6,3_4,5_5]$

Example A.24. *$G_i-\mathrm{GDD}(3^4)$ for $i\in\{16,17,19\}$ on $X=\mathbb{Z}_3\times\{1,2,3,4\}$, where the holes are on $\{j\}\times\{1,2,3\}$ for $j\in\{0,1,2\}$, and $\mathbb{Z}_3\times\{4\}$.*

i	*Base Blocks*	
16	$[0_1,1_1,2_2,1_4,0_2,0_3]$	$[0_2,1_3,2_3,0_4,1_1,1_4]$
17	$[0_1,1_1,2_2,0_4,1_2,2_1]$	$[0_2,0_4,1_3,2_3,0_1,2_4]$
19	$[0_1,1_1,0_2,2_3,0_4,2_1]$	$[0_2,1_2,0_4,1_4,2_3,0_3]$

Example A.25. *$G_i-\mathrm{GDD}(3^6)$ for $16\le i\le 18$ on $X=(\mathbb{Z}_{15}\times\{1\})\cup(\mathbb{Z}_3\times\{2\})$, where the holes are on $\{(a+5j)_1:0\le j\le 2\}$ for $a\in\{0,1,2,3,4\}$, and $\mathbb{Z}_3\times\{2\}$.*

i	*Base Block*	i	*Base Block*
16	$[0_1,1_1,3_1,7_1,1_2,0_2]$	17	$[0_1,1_1,3_1,7_1,1_2,0_2]$
18	$[0_1,1_1,3_1,7_1,1_2,2_1]$		

Example A.26. *$G_{19}-\mathrm{GDD}(3^6)$ on $X=\mathbb{Z}_6\times\{1,2,3\}$, where the holes are on $\{(a+2j)_b:0\le j\le 2\}$ for $a\in\{0,1\}$ and $b\in\{1,2,3\}$. There are 2 orbits of length 6, and one short orbit of length 3.*

$[0_1,1_1,2_2,5_2,4_3,3_2]$	$[0_1,2_2,4_3,0_3,5_3,3_1]$	$[0_1,0_2,0_3,3_1,3_2,3_3]$

Example A.27. *$G_i-\mathrm{GDD}(3^7)$ for $i\in\{16,17,19,20\}$ on $X=\mathbb{Z}_{21}$, where the holes are on $\{a+7j:0\le j\le 2\}$ for $a\in\{0,1,2,3,4,5,6\}$.*

i	*Base Block*	i	*Base Block*
16	$[0,1,3,9,13,6]$	17	$[0,1,3,9,13,4]$
19	$[0,1,3,6,9,13]$	20	$[0,1,4,16,3,5]$

Example A.28. *$G_i-\mathrm{GDD}(6^4)$ for $16\le i\le 20$ on $X=\mathbb{Z}_{24}$, where the holes are on $\{a+4j:0\le j\le 5\}$ for $a\in\{0,1,2,3\}$.*

i	*Base Block*	i	*Base Block*
16	$[0,1,3,10,16,6]$	17	$[0,1,3,10,16,5]$
18	$[0,1,3,10,16,11]$	19	$[0,1,6,18,3,16]$
20	$[0,1,3,10,4,15]$		

Example A.29. *$G_{20} - \mathrm{GDD}(6^6)$ on $X = (\mathbb{Z}_{30} \times \{1\}) \cup (\mathbb{Z}_6 \times \{2\})$, where the holes are on $\{(a+5j)_1 : 0 \le j \le 5\}$ for $a \in \{0,1,2,3,4\}$ and $\mathbb{Z}_6 \times \{2\}$.*

$[0_1, 1_1, 0_2, 8_1, 2_1, 11_1]$	$[0_1, 2_1, 3_2, 13_1, 1_1, 17_1]$

Example A.30. *$G_{20} - \mathrm{GDD}(6^7)$ on $X = \mathbb{Z}_{42}$, where the holes are on $\{a + 7j : 0 \le j \le 5\}$ for $a \in \{0,1,2,3,4,5,6\}$.*

$[0, 1, 4, 9, 19, 31]$	$[0, 2, 15, 26, 1, 7]$

Example A.31. *$G_i - \mathrm{GDD}(9^3)$ for $11 \le i \le 15$ on $X = \mathbb{Z}_{27}$, where the holes are on $\{a+3j : 0 \le j \le 8\}$ for $a \in \{0,1,2\}$.*

i	*Base Block*	i	*Base Block*
11	$[0, 1, 5, 13, 11, 18]$	12	$[0, 1, 11, 7, 5, 13]$
13	$[0, 1, 5, 12, 14, 4]$	14	$[0, 1, 11, 4, 17, 9]$
15	$[0, 1, 5, 8, 17, 3]$		

Example A.32. *$G_i - \mathrm{GDD}(9^4)$ for $i \in \{11,12,13,14,15,18\}$ on $X = (\mathbb{Z}_{27} \times \{1\}) \cup (\mathbb{Z}_9 \times \{2\})$, where the holes are on $\{(a+3j)_1 : 0 \le j \le 8\}$ for $a \in \{0,1,2\}$, and $\mathbb{Z}_9 \times \{2\}$.*

i	*Base Blocks*	
11	$[0_2, 3_1, 14_1, 4_1, 17_1, 1_2]$	$[0_1, 0_2, 1_1, 5_1, 7_1, 15_1]$
12	$[0_2, 2_1, 12_1, 25_1, 14_1, 22_1]$	$[0_1, 0_2, 1_1, 5_1, 7_1, 1_2]$
13	$[0_1, 1_1, 5_1, 0_2, 7_1, 15_1]$	$[0_2, 0_1, 2_1, 16_1, 26_1, 4_2]$
14	$[0_1, 1_1, 0_2, 2_1, 3_2, 7_1]$	$[0_2, 6_1, 14_1, 3_1, 17_1, 7_1]$
15	$[0_2, 2_1, 13_1, 12_1, 16_1, 8_2]$	$[0_1, 1_1, 0_2, 5_1, 8_1, 6_1]$
18	$[0_1, 1_1, 0_2, 5_1, 12_1, 1_2]$	$[0_1, 2_1, 5_2, 13_1, 3_1, 11_1]$

Example A.33. *$G_i - \mathrm{GDD}(9^5)$ for $i \in \{11,12,13,14,15,18,20\}$ on $X = \mathbb{Z}_{45}$, where the holes are on $\{a+5j : 0 \le j \le 8\}$ for $a \in \{0,1,2,3,4\}$.*

i	*Base Blocks*	
11	$[0, 1, 4, 11, 23, 7]$	$[0, 2, 19, 27, 14, 23]$
12	$[0, 1, 4, 11, 23, 9]$	$[0, 2, 19, 6, 24, 8]$
13	$[0, 1, 4, 10, 18, 32]$	$[0, 2, 19, 31, 7, 18]$
14	$[0, 1, 4, 11, 3, 25]$	$[0, 2, 18, 12, 3, 31]$
15	$[0, 1, 4, 7, 9, 20]$	$[0, 2, 14, 23, 19, 1]$
18	$[0, 1, 4, 12, 21, 2]$	$[0, 2, 16, 23, 10, 28]$
20	$[0, 1, 4, 12, 6, 28]$	$[0, 2, 9, 26, 8, 39]$

Example A.34. *$G_{20} - \mathrm{GDD}(9^6)$ on $X = \mathbb{Z}_{27} \times \{1,2\}$, where the holes are on $\{(a+3j)_b : 0 \le j \le 8\}$ for $a \in \{0,1,2\}$ and $b \in \{1,2\}$.*

$[0_1, 1_1, 8_1, 0_2, 2_1, 7_1]$	$[0_1, 2_1, 13_1, 15_2, 4_1, 7_2]$	$[0_1, 4_1, 14_2, 16_2, 7_1, 11_2]$
$[0_1, 5_2, 1_2, 10_1, 0_2, 7_2]$	$[0_1, 7_2, 6_2, 23_2, 2_1, 10_2]$	

Example A.35. *$G_8 - \mathrm{GDD}(1^8 10^1)$ on $X = (\mathbb{Z}_8 \times \{1,2\}) \cup (\mathbb{Z}_2 \times \{3\})$, where the hole is on $(\mathbb{Z}_8 \times \{2\}) \cup (\mathbb{Z}_2 \times \{3\})$. There is one orbit of length 8, and one short orbit of length 4.*

$[0_1, 2_1, 0_2, 4_2, 1_1, 3_1]$	$[0_1, 4_1, 1_1, 5_1, 0_3, 1_3]$

Example A.36. *$G_{20}-\mathrm{GDD}(1^{16}6^1)$ on $X=(\mathbb{Z}_3\times\{1,2,3,4,5,6,7\})\cup\{\infty\}$, where the hole is on $\mathbb{Z}_3\times\{6,7\}$.*

$[0_1,1_1,0_2,0_3,\infty,0_6]$	$[0_1,1_2,2_4,0_7,1_1,\infty]$	$[0_2,2_4,\infty,1_5,0_3,0_4]$
$[0_1,0_4,1_4,0_6,1_1,2_3]$	$[0_1,0_5,1_7,1_5,2_1,0_6]$	$[0_3,1_3,2_4,1_7,1_2,1_4]$
$[0_2,1_3,0_5,0_6,2_4,2_5]$	$[0_3,0_5,2_7,1_2,0_2,2_6]$	

Example A.37. *$G_i-\mathrm{GDD}(1^{35}10^1)$ for $i\in\{11,12,13,14,15,16,17,19\}$ on $X=(\mathbb{Z}_{35}\times\{1\})\cup(\mathbb{Z}_5\times\{2,3\})$, where the hole is on $\mathbb{Z}_5\times\{2,3\}$.*

i	*Base Blocks*	
11	$[0_1,1_1,7_1,23_1,10_1,0_2]$	$[0_1,2_1,20_1,9_1,1_2,23_1]$
	$[0_1,3_1,8_1,0_3,4_1,2_3]$	
12	$[0_1,1_1,9_1,21_1,11_1,0_2]$	$[0_1,2_1,17_1,22_1,3_2,16_1]$
	$[0_1,3_1,7_1,0_3,6_1,2_3]$	
13	$[0_1,1_1,6_1,17_1,9_1,0_2]$	$[0_1,2_1,15_1,27_1,4_2,9_1]$
	$[0_1,3_1,0_3,7_1,14_1,3_3]$	
14	$[0_1,1_1,6_1,15_1,33_1,0_2]$	$[0_1,2_1,21_1,11_1,2_2,34_1]$
	$[0_1,3_3,3_1,7_1,15_1,4_3]$	
15	$[0_1,1_1,9_1,11_1,13_1,0_2]$	$[0_1,2_1,17_1,16_1,0_2,11_1]$
	$[0_1,3_1,7_1,0_3,1_3,2_1]$	
16	$[0_1,1_1,6_1,17_1,26_1,0_2]$	$[0_1,2_1,10_1,23_1,1_2,2_2]$
	$[0_3,3_1,0_1,7_1,1_3,0_2]$	
17	$[0_1,1_1,5_1,16_1,23_1,0_2]$	$[0_1,2_1,10_1,23_1,0_2,1_2]$
	$[0_1,3_3,3_1,9_1,0_3,0_2]$	
19	$[0_1,1_1,5_1,11_1,28_1,0_2]$	$[0_1,2_1,15_1,3_2,14_1,30_1]$
	$[0_1,3_1,8_1,0_3,9_1,2_3]$	

Example A.38. *$G_i-\mathrm{GDD}(1^{35}19^1)$ for $i\in\{16,17,18,19\}$ on $X=(\mathbb{Z}_{35}\times\{1\})\cup(\mathbb{Z}_7\times\{2,3\})\cup(\mathbb{Z}_5\times\{4\})$, where the hole is on $(\mathbb{Z}_7\times\{2,3\})\cup(\mathbb{Z}_5\times\{4\})$.*

i	*Base Blocks*	
16	$[0_1,1_1,10_1,22_1,15_1,3_4]$	$[0_1,2_2,11_1,19_1,0_4,3_1]$
	$[0_1,3_3,15_1,18_1,3_4,5_1]$	$[0_2,4_1,0_1,6_1,0_3,1_3]$
17	$[0_1,1_1,7_1,17_1,28_1,3_4]$	$[0_1,2_2,8_1,23_1,5_2,0_4]$
	$[0_1,3_3,9_1,31_1,1_2,0_4]$	$[0_1,4_3,2_1,5_1,0_4,3_3]$
18	$[0_1,1_1,7_1,15_1,3_4,6_1]$	$[0_1,2_1,11_1,0_2,6_1,0_4]$
	$[0_1,3_1,13_1,3_3,30_1,0_4]$	$[0_1,4_2,12_1,16_1,0_3,1_1]$
19	$[0_1,1_1,8_1,0_2,6_1,0_4]$	$[0_1,2_1,10_1,0_4,19_1,31_1]$
	$[0_1,3_1,12_1,0_3,13_1,1_2]$	$[0_1,4_1,1_2,15_1,6_3,3_1]$

Example A.39. *G_i − GDD($1^{35}28^1$) for $i \in \{11, 12, 13, 14, 15, 16, 17\}$ on $X = (\mathbb{Z}_{35} \times \{1\}) \cup (\mathbb{Z}_7 \times \{2, 3, 4, 5\})$, where the hole is on $\mathbb{Z}_7 \times \{2, 3, 4, 5\}$.*

i	*Base Blocks*	
11	$[0_1, 1_1, 8_1, 0_2, 3_1, 4_2]$	$[0_1, 2_1, 11_1, 0_3, 5_1, 0_2]$
	$[0_3, 3_1, 9_1, 22_1, 6_1, 0_4]$	$[0_1, 4_1, 0_4, 15_1, 0_5, 18_1]$
	$[0_5, 5_1, 17_1, 27_1, 9_1, 0_4]$	
12	$[0_1, 1_1, 11_1, 28_1, 13_1, 0_2]$	$[0_1, 2_1, 16_1, 4_2, 12_1, 3_2]$
	$[0_1, 3_3, 3_1, 1_3, 4_1, 6_3]$	$[0_1, 4_4, 5_1, 0_4, 9_1, 3_4]$
	$[0_1, 5_5, 6_1, 1_5, 8_1, 4_5]$	
13	$[0_1, 1_1, 11_1, 27_1, 14_1, 0_2]$	$[0_1, 2_1, 4_2, 19_1, 7_1, 3_2]$
	$[0_3, 3_1, 0_1, 1_3, 5_1, 9_1]$	$[0_4, 4_1, 9_1, 1_4, 0_1, 6_1]$
	$[0_5, 5_1, 13_1, 2_5, 2_1, 17_1]$	
14	$[0_1, 1_5, 1_1, 14_1, 2_1, 4_5]$	$[0_3, 2_1, 0_1, 8_1, 6_5, 28_1]$
	$[0_1, 3_1, 4_3, 9_1, 0_4, 15_1]$	$[0_1, 4_1, 0_2, 10_1, 5_4, 15_1]$
	$[0_2, 5_1, 16_1, 34_1, 15_1, 2_4]$	
15	$[0_1, 1_1, 6_1, 13_1, 21_1, 3_2]$	$[0_1, 2_1, 9_1, 0_2, 18_1, 0_3]$
	$[0_1, 3_1, 2_2, 0_3, 2_3, 1_1]$	$[0_1, 4_1, 0_4, 1_4, 0_5, 1_1]$
	$[0_1, 5_5, 8_1, 10_1, 11_1, 6_4]$	
16	$[0_1, 1_2, 5_1, 13_1, 0_3, 2_1]$	$[0_2, 2_1, 0_1, 3_1, 0_5, 1_4]$
	$[0_3, 3_1, 9_1, 21_1, 2_4, 10_1]$	$[0_5, 4_1, 13_1, 23_1, 5_3, 18_1]$
	$[0_1, 5_4, 4_1, 15_1, 3_5, 2_1]$	
17	$[0_1, 1_2, 10_1, 19_1, 5_2, 3_3]$	$[0_1, 2_3, 17_1, 29_1, 0_2, 5_5]$
	$[0_3, 3_1, 7_1, 18_1, 6_4, 0_5]$	$[0_1, 4_4, 1_1, 3_1, 1_4, 10_1]$
	$[0_1, 5_5, 5_1, 13_1, 0_5, 27_1]$	

Example A.40. *G_{20} − GDD($1^{45}27^1$) on $X = (\mathbb{Z}_{10} \times \{1, 2, 3, 4, 6, 7\}) \cup (\mathbb{Z}_5 \times \{5, 8\}) \cup (\{\infty\} \times \{a, b\})$, where the hole is on $(\mathbb{Z}_{10} \times \{6, 7\}) \cup (\mathbb{Z}_5 \times \{8\}) \cup (\{\infty\} \times \{a, b\})$. There are 23 orbits of length 10, and 3 short orbits of length 5.*

$[0_1, 0_3, \infty_b, 0_4, 1_1, 0_6]$	$[0_2, 0_3, \infty_a, 1_4, 0_1, 0_6]$	$[0_1, 1_1, 3_1, 4_6, 6_1, 0_5]$
$[0_1, 4_1, 0_2, 6_6, 1_1, 3_5]$	$[0_1, 5_2, 7_2, 0_7, 1_1, 2_5]$	$[0_1, 8_2, 1_3, 2_7, 4_1, 2_5]$
$[0_1, 2_2, 3_2, 7_6, 0_2, 4_2]$	$[0_1, 1_2, 4_2, 0_8, 1_1, 0_2]$	$[0_1, 2_3, 3_3, 1_8, 3_1, 1_3]$
$[0_1, 6_3, 2_4, 2_8, 0_2, 0_4]$	$[0_1, 4_3, 1_7, 3_4, 1_2, 4_8]$	$[0_1, 5_3, 4_7, 7_4, 1_4, 0_8]$
$[0_1, 7_3, 9_3, 5_7, 2_1, 0_4]$	$[0_1, 4_4, 5_4, 7_7, 1_1, 7_4]$	$[0_2, 1_3, 3_5, 9_6, 7_2, 4_3]$
$[0_2, 2_3, 1_5, 7_7, 5_2, 4_3]$	$[0_2, 6_3, 4_5, 6_7, 8_2, 2_4]$	$[0_2, 5_4, 2_5, 1_7, 1_2, 9_3]$
$[0_2, 5_3, 1_6, 9_4, 1_2, 0_7]$	$[0_2, 4_3, 8_6, 7_4, 2_3, 6_7]$	$[0_2, 3_4, 6_4, 0_6, 0_3, 3_3]$
$[0_3, 8_4, 1_5, 1_6, 2_3, 8_3]$	$[0_4, 2_4, 1_5, 8_6, 6_3, 3_4]$	
$[0_1, 5_1, \infty_a, 0_5, 1_5, 0_8]$	$[0_2, 5_2, \infty_b, 0_5, 2_5, 3_8]$	$[0_3, 5_3, 2_8, 0_5, 0_4, 5_4]$

Example A.41. *G_{20} − GDD($1^{55}27^1$) on $X = (\mathbb{Z}_{55} \times \{1\}) \cup (\mathbb{Z}_{11} \times \{2, 3\}) \cup (\mathbb{Z}_5 \times \{4\})$, where the hole is on $(\mathbb{Z}_{11} \times \{2, 3\}) \cup (\mathbb{Z}_5 \times \{4\})$.*

$[0_1, 1_1, 8_1, 19_1, 2_1, 0_2]$	$[0_1, 2_1, 12_1, 6_2, 4_1, 17_1]$	$[0_1, 3_1, 2_3, 23_1, 8_1, 9_2]$
$[0_1, 4_1, 0_4, 26_1, 1_1, 0_2]$	$[0_1, 5_1, 0_3, 14_1, 30_1, 2_4]$	$[0_1, 6_1, 27_1, 9_3, 2_1, 26_1]$

Example A.42. *G_{20} − GDD($1^{63}27^1$) on $X = (\mathbb{Z}_{21} \times \{1,2,6\}) \cup (\mathbb{Z}_7 \times \{3,4,5\}) \cup (\mathbb{Z}_3 \times \{7,8\})$, where the hole is on $(\mathbb{Z}_{21} \times \{6\}) \cup (\mathbb{Z}_3 \times \{7,8\})$. There are 18 orbits of length 21, and one short orbit of length 7. After developing the blocks, insert a G_{20}-GDD of type 3^7 (Example A.27) on $\mathbb{Z}_7 \times \{3,4,5\}$, where the holes are on $\{j_3, j_4, j_5\}$ for $j \in \mathbb{Z}_7$.*

$[0_1, 5_1, 1_7, 6_3, 1_1, 7_6]$	$[0_1, 4_2, 0_7, 0_4, 1_1, 0_6]$	$[0_2, 2_2, 0_7, 0_5, 3_1, 12_6]$
$[0_1, 4_1, 2_8, 4_3, 2_1, 17_6]$	$[0_1, 7_2, 0_8, 1_4, 3_1, 0_6]$	$[0_2, 4_2, 1_8, 1_5, 2_1, 10_6]$
$[0_1, 2_1, 8_1, 11_1, 0_3, 18_6]$	$[0_1, 1_1, 1_2, 9_2, 0_3, 0_6]$	$[0_1, 7_1, 13_2, 3_6, 8_2, 0_3]$
$[0_2, 6_2, 3_3, 5_6, 7_1, 18_2]$	$[0_2, 1_2, 2_3, 7_6, 3_1, 6_4]$	$[0_1, 14_2, 2_2, 2_4, 5_1, 0_6]$
$[0_1, 15_2, 12_2, 13_6, 1_1, 3_5]$	$[0_1, 16_2, 1_5, 5_6, 12_1, 17_2]$	$[0_1, 17_2, 5_5, 11_6, 1_1, 4_5]$
$[0_1, 18_2, 0_5, 1_6, 5_2, 4_4]$	$[0_1, 19_2, 3_2, 0_6, 0_2, 4_4]$	$[0_1, 20_2, 2_6, 10_2, 1_4, 3_6]$
$[0_2, 7_2, 14_2, 0_3, 0_4, 0_5]$		

Example A.43. *G_{20} − GDD($4^3 5^1$) on $X = (\mathbb{Z}_6 \times \{1,2\}) \cup (\mathbb{Z}_3 \times \{3\}) \cup (\mathbb{Z}_2 \times \{4\})$, where the holes are on $\{(a+3j)_b : 0 \le j \le 1$ and $1 \le b \le 2\}$ for $a \in \{0,1,2\}$, and $(\mathbb{Z}_3 \times \{3\}) \cup (\mathbb{Z}_2 \times \{4\})$.*

$[0_3, 4_2, 2_1, 0_1, 1_1, 0_4]$	$[0_3, 1_1, 2_2, 0_2, 1_2, 0_4]$

Example A.44. *G_i−GDD($6^3 9^1$) for $16 \le i \le 19$ on $X = (\mathbb{Z}_6 \times \{1,2,3,4\}) \cup (\mathbb{Z}_3 \times \{5\})$, where the holes are on $\mathbb{Z}_6 \times \{b\}$ for $b \in \{1,2,3\}$, and $(\mathbb{Z}_6 \times \{4\}) \cup (\mathbb{Z}_3 \times \{5\})$.*

i	*Base Blocks*		
16	$[0_1, 1_2, 2_3, 3_4, 4_1, 2_1]$	$[0_1, 4_2, 3_3, 2_4, 1_2, 1_4]$	$[0_1, 1_4, 2_2, 5_3, 2_4, 3_1]$
	$[0_2, 0_5, 0_1, 0_3, 0_4, 1_3]$	$[0_5, 2_3, 1_1, 4_2, 2_5, 0_4]$	
17	$[0_1, 1_2, 5_3, 3_4, 4_1, 3_1]$	$[0_1, 4_2, 3_3, 2_4, 2_2, 4_1]$	$[0_1, 1_4, 2_2, 4_3, 5_4, 0_4]$
	$[0_1, 2_3, 1_5, 5_2, 2_1, 0_4]$	$[0_1, 0_5, 0_2, 0_3, 1_5, 0_4]$	
18	$[0_1, 1_2, 3_3, 1_4, 5_1, 2_2]$	$[0_1, 4_2, 5_3, 0_4, 1_2, 4_4]$	$[0_1, 3_4, 2_2, 1_3, 0_4, 2_1]$
	$[0_2, 2_5, 1_1, 3_3, 0_4, 0_3]$	$[0_1, 0_5, 0_2, 0_3, 1_5, 2_1]$	
19	$[0_1, 0_2, 1_1, 0_3, 0_5, 2_3]$	$[0_1, 1_2, 0_3, 2_5, 3_3, 0_4]$	$[0_1, 2_2, 1_5, 5_3, 1_4, 3_1]$
	$[0_1, 3_2, 0_4, 4_3, 3_4, 1_1]$	$[0_1, 2_3, 3_4, 4_2, 0_4, 2_2]$	

Example A.45. *G_{20} − GDD($6^4 9^1$) on $X = (\mathbb{Z}_{12} \times \{1,2\}) \cup (\mathbb{Z}_6 \times \{3\}) \cup (\mathbb{Z}_3 \times \{4\})$, where the holes are on $\{(a+2j)_b : 0 \le j \le 5\}$ for $a \in \{0,1\}$ and $b \in \{1,2\}$, and $(\mathbb{Z}_6 \times \{3\}) \cup (\mathbb{Z}_3 \times \{4\})$.*

$[0_1, 1_1, 0_4, 0_2, 3_1, 6_1]$	$[0_1, 2_2, 1_4, 3_2, 2_1, 0_3]$	$[0_1, 5_2, 10_2, 5_3, 3_1, 10_1]$
$[0_1, 4_2, 7_2, 3_3, 3_1, 11_2]$		

Example A.46. *G_i−GDD($6^5 9^1$) for $16 \le i \le 20$ on $X = (\mathbb{Z}_{10} \times \{1,2,3\}) \cup (\mathbb{Z}_5 \times \{4\}) \cup (\mathbb{Z}_2 \times \{5,6\})$, where the holes are on $\{(a+5j)_b : 0 \le j \le 1$ and $1 \le$*

$b \leq 3\}$ *for* $a \in \{0,1,2,3,4\}$ *and* $(\mathbb{Z}_5 \times \{4\}) \cup (\mathbb{Z}_2 \times \{5,6\})$.

i	*Base Blocks*		
16	$[0_1, 1_1, 3_1, 2_2, 9_1, 0_5]$	$[0_1, 4_2, 7_2, 0_5, 0_3, 6_1]$	$[0_1, 2_3, 6_2, 3_3, 2_2, 1_5]$
	$[0_1, 1_3, 4_3, 0_6, 5_1, 5_3]$	$[0_3, 3_4, 1_2, 2_2, 0_6, 4_1]$	$[0_2, 2_2, 4_3, 0_4, 7_1, 1_4]$
	$[0_3, 2_4, 2_1, 8_3, 3_4, 0_1]$		
17	$[0_1, 1_1, 3_1, 2_2, 0_5, 6_1]$	$[0_1, 4_2, 6_1, 3_2, 0_5, 0_2]$	$[0_1, 3_3, 2_3, 0_6, 3_1, 0_2]$
	$[0_3, 2_3, 3_2, 9_2, 0_4, 0_6]$	$[0_1, 4_3, 2_4, 7_3, 6_1, 0_5]$	$[0_2, 2_2, 0_4, 6_3, 0_1, 0_5]$
	$[0_2, 2_3, 8_3, 4_4, 0_1, 1_1]$		
18	$[0_1, 1_1, 3_1, 2_2, 9_1, 0_5]$	$[0_1, 4_2, 6_2, 2_3, 8_1, 0_5]$	$[0_1, 7_2, 1_3, 0_4, 3_1, 0_6]$
	$[0_1, 6_3, 7_3, 0_6, 0_2, 1_4]$	$[0_2, 3_3, 4_2, 4_4, 7_2, 0_6]$	$[0_2, 1_2, 2_3, 0_5, 5_3, 3_3]$
	$[0_3, 2_4, 1_1, 4_3, 4_4, 0_1]$		
19	$[0_1, 1_1, 3_1, 4_1, 2_2, 0_5]$	$[0_1, 3_2, 4_1, 4_2, 7_2, 0_5]$	$[0_2, 4_1, 0_3, 1_3, 3_3, 0_5]$
	$[0_2, 2_2, 1_6, 9_3, 0_4, 0_1]$	$[0_3, 2_1, 4_3, 3_3, 0_6, 0_2]$	$[0_3, 6_1, 0_4, 9_3, 4_4, 3_1]$
	$[0_3, 4_2, 0_4, 3_4, 6_3, 8_2]$		
20	$[0_1, 1_1, 3_1, 2_2, 5_1, 0_5]$	$[0_1, 4_1, 3_4, 6_3, 8_1, 0_5]$	$[0_1, 4_2, 0_4, 8_2, 0_3, 1_5]$
	$[0_1, 3_2, 0_6, 6_2, 4_2, 4_4]$	$[0_1, 3_3, 1_6, 4_3, 5_1, 2_4]$	$[0_1, 1_3, 7_3, 1_4, 2_2, 0_3]$
	$[0_2, 1_2, 4_3, 7_3, 6_2, 5_3]$		

Example A.47. $G_{20} - \mathrm{GDD}(12^4 9^1)$ *on* $X = (\mathbb{Z}_{48} \times \{1\}) \cup (\mathbb{Z}_3 \times \{2,3,4\})$, *where the holes are on* $\{(a+4j)_1 : 0 \leq j \leq 11\}$ *for* $a \in \{0,1,2,3\}$, *and* $\mathbb{Z}_3 \times \{2,3,4\}$.

$[0_2, 0_1, 1_1, 23_1, 8_1, 14_1]$	$[0_3, 0_1, 2_1, 13_1, 10_1, 31_1]$	$[0_4, 0_1, 5_1, 19_1, 2_1, 9_1]$

Example A.48. $G_{20} - \mathrm{GDD}(15^4 18^1)$ *on* $X = (\mathbb{Z}_{45} \times \{1\}) \cup (\mathbb{Z}_{15} \times \{2\}) \cup (\mathbb{Z}_9 \times \{3,4\})$, *where the holes are on* $\{(a+3j)_1 : 0 \leq j \leq 14\}$ *for* $a \in \{0,1,2\}$, $\mathbb{Z}_{15} \times \{2\}$, *and* $\mathbb{Z}_9 \times \{3,4\}$.

$[0_1, 1_1, 26_1, 6_2, 4_1, 0_3]$	$[0_1, 2_1, 13_1, 8_3, 8_1, 7_2]$	$[0_1, 4_1, 12_2, 2_3, 17_1, 1_1]$
$[0_1, 5_1, 22_1, 3_2, 14_1, 2_4]$	$[0_1, 7_1, 1_2, 5_4, 15_1, 1_1]$	$[0_1, 0_2, 0_4, 8_1, 18_1, 2_4]$

Example A.49. $G_{20} - \mathrm{GDD}(16^5 13^1)$ *on* $X = (\mathbb{Z}_{80} \times \{1\}) \cup (\mathbb{Z}_8 \times \{2\}) \cup (\mathbb{Z}_5 \times \{3\})$, *where the holes are on* $\{(a+5j)_1 : 0 \leq j \leq 15\}$ *for* $a \in \{0,1,2,3,4\}$, *and* $(\mathbb{Z}_8 \times \{2\}) \cup (\mathbb{Z}_5 \times \{3\})$.

$[0_1, 1_1, 23_1, 37_1, 66_1, 4_2]$	$[0_1, 2_1, 3_2, 21_1, 60_1, 32_1]$	$[0_1, 3_1, 0_2, 12_1, 50_1, 0_3]$
$[0_1, 4_1, 1_3, 17_1, 50_1, 1_1]$	$[0_1, 6_1, 24_1, 32_1, 59_1, 25_1]$	

Example A.50. $G_{20} - \mathrm{GDD}(16^5 22^1)$ *on* $X = (\mathbb{Z}_{80} \times \{1\}) \cup (\mathbb{Z}_{16} \times \{2\}) \cup (\mathbb{Z}_2 \times \{3,4,5\})$, *where the holes are on* $\{(a+5j)_1 : 0 \leq j \leq 15\}$ *for* $a \in \{0,1,2,3,4\}$, *and* $(\mathbb{Z}_{16} \times \{2\}) \cup (\mathbb{Z}_2 \times \{3,4,5\})$.

$[0_1, 1_1, 13_1, 29_1, 50_1, 0_3]$	$[0_1, 2_1, 11_1, 19_1, 50_1, 0_4]$	$[0_1, 3_1, 27_1, 6_2, 58_1, 36_1]$
$[0_1, 4_1, 1_2, 41_1, 55_1, 7_1]$	$[0_1, 6_1, 4_2, 42_1, 60_1, 1_2]$	$[0_1, 7_1, 0_2, 33_1, 56_1, 0_5]$

Example A.51. $G_{20} - \mathrm{GDD}(16^5 31^1)$ *on* $X = (\mathbb{Z}_{80} \times \{1\}) \cup (\mathbb{Z}_{16} \times \{2\}) \cup (\mathbb{Z}_5 \times \{3,4,5\})$, *where the holes are on* $\{(a+5j)_1 : 0 \leq j \leq 15\}$ *for* $a \in$

$\{0,1,2,3,4\}$, *and* $(\mathbb{Z}_{16}\times\{2\})\cup(\mathbb{Z}_5\times\{3,4,5\})$.

$[0_1,1_1,29_1,9_2,54_1,15_1]$	$[0_1,2_1,0_2,14_1,51_1,30_1]$	$[0_1,3_1,7_2,34_1,70_1,1_2]$
$[0_1,4_1,1_2,27_1,51_1,4_3]$	$[0_1,6_1,0_3,19_1,67_1,41_1]$	$[0_1,7_1,18_1,0_4,51_1,9_1]$
$[0_1,8_1,17_1,0_5,51_1,4_1]$		

Example A.52. *$G_{20}-\mathrm{GDD}(16^5 40^1)$ on $X=(\mathbb{Z}_{80}\times\{1\})\cup(\mathbb{Z}_{40}\times\{2\})$, where the holes are on $\{(a+5j)_1 : 0\le j\le 15\}$ for $a\in\{0,1,2,3,4\}$, and $(\mathbb{Z}_{40}\times\{2\})$.*

$[0_1,1_1,12_1,0_2,10_1,46_1]$	$[0_1,2_1,26_1,18_2,16_1,48_1]$
$[0_1,3_1,16_1,1_2,10_1,48_1]$	$[0_1,4_1,18_1,7_2,10_1,51_1]$
$[0_1,6_1,23_1,27_2,15_1,48_1]$	$[0_1,7_1,29_1,13_2,27_1,48_1]$
$[0_1,8_1,27_1,22_2,14_1,45_1]$	$[0_1,9_1,37_1,20_2,11_1,45_1]$

Example A.53. *$G_{20}-\mathrm{GDD}(18^4 9^1)$ on $X=(\mathbb{Z}_{72}\times\{1\})\cup(\mathbb{Z}_9\times\{2\})$, where the holes are on $\{(a+4j)_1 : 0\le j\le 17\}$ for $a\in\{0,1,2,3\}$, and $(\mathbb{Z}_9\times\{2\})$.*

$[0_1,1_1,35_1,2_2,7_1,21_1]$	$[0_1,2_1,9_1,27_1,4_1,0_2]$	$[0_1,3_1,13_1,42_1,25_1,4_2]$
$[0_1,5_1,11_1,26_1,4_1,45_1]$		

Example A.54. *$G_{20}-\mathrm{GDD}(18^5 9^1)$ on $X=(\mathbb{Z}_{90}\times\{1\})\cup(\mathbb{Z}_9\times\{2\})$, where the holes are on $\{(a+5j)_1 : 0\le j\le 17\}$ for $a\in\{0,1,2,3,4\}$, and $(\mathbb{Z}_9\times\{2\})$.*

$[0_1,1_1,17_1,1_2,3_1,22_1]$	$[0_1,2_1,14_1,43_1,15_1,2_2]$	$[0_1,3_1,11_1,24_1,1_1,0_2]$
$[0_1,4_1,31_1,38_1,1_1,47_1]$	$[0_1,6_1,32_1,54_1,3_1,21_1]$	

References

1. R. J. R. Abel, F. E. Bennett, M. Greig, *PBD-Closure*, in Handbook of Combinatorial Designs, 2nd ed, C. J. Colbourn and J. H. Dinitz (Editors), Chapman & Hall/CRC, Boca Raton, FL, 2007, 247–255.
2. R. J. R. Abel, C. J. Colbourn, and J. H. Dinitz, *Mutually orthogonal Latin squares*, in Handbook of Combinatorial Designs, 2nd ed, C. J. Colbourn and J. H. Dinitz (Editors), Chapman & Hall/CRC, Boca Raton, FL, 2007, 160–193.
3. P. Adams, E. J. Billington, and D. G. Hoffman, *On the spectrum for $K_{m+2}\setminus K_m$ designs*, Journal of Combinatorial Designs **5** (1997), 49–60.
4. P. Adams, D. Bryant, and M. Buchanan, *A survey on the existence of G-designs*, Journal of Combinatorial Designs **16** (2008), 373–410.
5. T. Beth, D. Jungnickel, and H. Lenz, *Design Theory*, 2nd ed., Cambridge University Press, Cambridge, UK, 1999.
6. D. Bryant and T. McCourt, *Existence results for G-designs*, 2011, `http://wiki.smp.uq.\edu.au/G-designs`.
7. D. Bryant and S. El-Zanati, *Graph decompositions*, in Handbook of Combinatorial Designs, 2nd ed, C. J. Colbourn and J. H. Dinitz (Editors), Chapman & Hall/CRC, Boca Raton, FL, 2007, 477–486.
8. J. E. Carter, *Designs on cubic multigraphs*, Ph.D. Thesis, Department of Mathematics and Statistics, McMaster University, Canada, 1989.
9. A. D. Forbes and T. S. Griggs, *Designs for graphs with six vertices and nine edges*, Australasian Journal of Combinatorics **70** (2018), 52–74.
10. A. D. Forbes, T. S. Griggs, and K. A. Forbes, *Completing the design spectra for graphs with six vertices and eight edges*, Australasian Journal of Combinatorics **70** (2018), 386–389.

11. Q. Gao, *Graph design of a graph with six vertices and nine edges*, High Perf Network Comput Commun Sys, Commun Comput Info Sci **163** (2011), 188–192.
12. G. Ge, *Group divisible designs*, in Handbook of Combinatorial Designs, 2nd ed, C. J. Colbourn and J. H. Dinitz (Editors), Chapman & Hall/CRC, Boca Raton, FL, 2007, 255–261.
13. G. Ge, S. Hu, E. Kolotoğlu, and H. Wei, *A complete solution to spectrum problem for five-vertex graphs with application to traffic grooming in optical networks*, Journal of Combinatorial Designs, **23(6)** (2015), 233–273.
14. G. Ge and A. C. H. Ling, *Constructions of quad-rooted double covers*, Graphs and Combinatorics **21** (2005), 231–238.
15. R. K. Guy and L. W. Beineke, *The coarseness of the complete graph*, Canadian Journal of Mathematics **20** (1968), 888–894.
16. F. Harary, *Graph Theory*, Addison-Wesley Publishing Co., Reading, Mass.-Menlo Park, Calif.-London, 1969.
17. Q. Kang, L. Yuan, S. Liu, *Graph designs for all graphs with six vertices and eight edges*, Acta Mathematicae Applicatae Sinica, English Series **21** (2005), 469–484.
18. Q. Kang, H. Zhao, and C. P. Ma, *Graph designs for nine graphs with six vertices and nine edges*, Ars Combinatoria **88** (2008), 379–395.
19. E. Kolotoğlu, *The existence and construction of* $(K_5 \setminus e)$*-designs of orders 27, 135, 162 and 216*, Journal of Combinatorial Designs *21(7)* (2013), 280–302.
20. H. Liu, L. Wang, *Completing the spectrum for a class of graph designs*, High Performance Networking, Computing and Communication Systems, Communications in Computer and Information Science **163** (2011), 10–14.
21. C. P. Ma, *Graph designs for 16 graphs*, Master Thesis, Hebei Normal University, 2004.
22. R. C. Mullin, A. L. Poplove, and L. Zhu, *Decomposition of Steiner triple systems into triangles*, Proceedings of the First Carbondale Combinatorics Conference (Carbondale, III., 1986) (1987), 149–174.
23. C. Wang, *A graph design for nine graphs with six vertices and nine edges*, Advances in Computer Science and Engineering, Advances in Intelligent and Soft Computing **141** (2012), 129–134.
24. L. Wang, *The spectrum for a class of graph designs*, Utilitas Mathematica **87** (2012), 199–206.

Dept. of Mathematics, Yildiz Technical Univ., İstanbul, Turkey
E-mail address: `kolot@yildiz.edu.tr`