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THE COPNUMBER FOR LEXICOGRAPHIC PRODUCTS AND SUMS OF GRAPHS

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ABSTRACT. For the lexicographic product $G \bullet H$ of two graphs G and H so that G is connected, we prove that if the copnumber $c(G)$ of G is greater than or equal to 2, then $c(G \bullet H) = c(G)$. Moreover, if $c(G) = c(H) = 1$, then $c(G \bullet H) = 1$. If $c(G) = 1$, G has more than one vertex, and $c(H) \geq 2$, then $c(G \bullet H) = 2$. We also provide the copnumber for general lexicographic sums.

The game of cops and robbers on a graph, conceived originally in [4] and recently described comprehensively in [2], is played according to the following rules: Vertices $v_1, \ldots, v_n$ in a graph are chosen as the initial positions for cops $C_1, \ldots, C_n$. A vertex w is then chosen for a robber R. At the start, as well as throughout the game, multiple cops can occupy the same vertex. The cops' objective is to catch the robber by placing a cop on the same vertex with the robber. The robber's objective is to prevent this from happening. Both sides know the position of all cops and of the robber at all times. Each side alternately takes turns, starting with the cop. A cop move consists of each of the cops either staying at their current vertex or moving to an adjacent vertex. In a robber move, the robber either stays at their current vertex or moves to an adjacent vertex. The smallest number of cops needed to capture the robber in a given graph G is called the graph's **copnumber** $c(G)$. A graph with $c(G) = 1$ is called **cop-win**.

It is customary to label the vertex at which the cop C_k is located and the vertex at which the robber R is located by C_k and R respectively which we will use throughout the paper. Moreover, since neither the cops nor the robber can leave the component in which each of them started the game, the game is typically assumed to be played on a connected graph. Trivially, because the robber is placed after the cops are placed, the copnumber of a disconnected graph is the sum of the copnumbers of its components. See [2, Section 4.2] for a survey of the copnumbers for several types of products for graphs. Aside from results on the cartesian product and the strong

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product in [2, p. 85], it is stated that the question when one cop has a winning strategy is "wide open" and little is known about the copnumber in general for other products. One such product is the lexicographic product, for which we state the copnumber in Theorem 2.7. Lexicographic products can be viewed as special cases of lexicographic sums (see Definition 1.1). Theorems 1.4 and 2.6 provide the copnumber for lexicographic sums.

1. Cops and Robbers in Lexicographic Sums

Lexicographic products can be viewed as special instances of lexicographic sums as follows.

Definition 1.1. *Let $H_t = (V_t, E_t)$ be a family of pairwise disjoint nonempty graphs indexed by the set T. Let $G = (T, F)$ be a graph whose vertices are the elements of T. Define the graph $\sum_{t\in T} H_t$ to be the graph with vertex set $V = \bigcup_{t\in T} V_t$ and edge set E, where, for $v_1 \in V_{t_1}$ and $v_2 \in V_{t_2}$, we have $\{v_1, v_2\} \in E$ if and only if*

1. *$t_1 \neq t_2$ and $\{t_1, t_2\} \in F$, or*
2. *$t_1 = t_2$ and $\{v_1, v_2\} \in E_{t_1}$.*

The graph $\sum_{t\in T} H_t$ is called the **lexicographic sum** *of the graphs $H_t = (V_t, E_t)$ over the* **index graph** *$G = (T, F)$. The* **index function** *$i : V \to T$ maps each $v \in V$ to the unique index $t =: i(v)$ so that $v \in V_t$.*

From the perspective of lexicographic sums, the **lexicographic product** $G \bullet H$ of the graphs G and H is the lexicographic sum of graphs H_t that are isomorphic to H over the index graph $G = (T, F)$. Throughout this paper, we consider lexicographic sums $\sum_{t\in T} H_t$ of the graphs $H_t = (V_t, E_t)$ over the index graph $G = (T, F)$.

Note that a lexicographic sum is connected if and only if the index graph is connected. Hence, we will ultimately need to also consider disconnected subgraphs H_t. On the positive side, the special structure of lexicographic sums allows for a cop in the same subgraph H_t as the robber to limit the robber's moves to H_t (see Lemma 1.2). This observation assures that most of the game can be played in a structure that is isomorphic to the index graph G (see Theorem 1.4).

Lemma 1.2. *If at any stage of the game there is a cop C and robber R so that we have $i(C) = i(R)$, then keeping the cop C in the vertex set $V_{i(C)} = V_{i(R)}$, with instructions to capture R if R moves to a vertex adjacent to C, then this forces the robber to also stay in $V_{i(C)} = V_{i(R)}$.*

The proof of Lemma 1.2 is immediate.

Definition 1.3. *Let $G = (V, E)$ be a graph. A function $r : V \to V$ is called a* **retraction** *if and only if r is idempotent and $\{v, w\} \in E$ implies $r(v) = r(w)$ or $\{r(v), r(w)\} \in E$. For a retraction r, the induced subgraph $G[r[V]]$ is also called a* **retract** *of G.*

By [1] or [2, Theorem 1.9], the copnumber of a retract is less than or equal to that of the original graph.

Theorem 1.4. *For $c(G) > 1$, we have that $c\left(\sum_{t\in T} H_t\right) = c(G)$. For $c(G) = 1$ and G having at least two vertices, we have that $c\left(\sum_{t\in T} H_t\right) \leq 2$.*

Proof. Let $S = (V, E) = \sum_{t\in T} H_t$. For every $t \in T$, pick a vertex $v_t \in V_t$. Then the function r from V to $U = \{v_t : t \in T\}$ that maps each $v \in V_t$ to v_t is a retraction from S to the induced subgraph $S[U]$, which is isomorphic to $G = (T, F)$ via $i|_U$. Hence $c(S) \geq c(G)$ (see [1] or [2, Theorem 1.9]).

To prove $c(S) \leq c(G)$ (or $c(S) \leq 2$ if $c(G) = 1$), use $\max\{c(G), 2\}$ cops to chase the robber's "shadow" $r(R)$ in G until there is a cop C so that $i(C) = i(R)$. By Lemma 1.2, the robber cannot leave $V_{i(R)}$. In either of the cases, one additional cop C' is available to move to some H_t with t adjacent to $i(R)$ to capture the robber in the next round. $\square$

2. Dismantlable Lexicographic Sums

By Theorem 1.4, the case of a cop-win index graph is the only remaining case in a complete characterization of the copnumber for lexicographic sums. For cop-win index graphs with at least two vertices, the copnumber of a lexicographic sum is either 1 or 2 by Theorem 1.4. From [4] we know that a graph is cop-win if and only if it is dismantlable (see Definition 2.2). We can then compute the copnumbers of all lexicographic sums if we can characterize dismantlable lexicographic sums. This is done in Theorem 2.6.

Definition 2.1 (See [2, p. 11]). *For a vertex v in a graph $G = (V, E)$, define $N[v] = \{x \in V : x = v \text{ or } x \sim v\}$. When we specifically need to indicate the graph in which we consider the vertices and the adjacency relation, we also write $N_G[v]$ for $N[v]$. The vertex v is called a* **corner** *if and only if there is a vertex u so that $N[v] \subseteq N[u]$. A corner is called* **isolated** *if and only if it is not adjacent to another corner.*

Definition 2.2. *Let $C = (V, E)$ be a graph and let $W \subseteq V$. Then we say that C is* **dismantlable to** *the induced subgraph $D = C[W]$ if and only if there is an enumeration $v_1, \ldots, v_m$ of the vertices in $V \setminus W$ so that every v_i is a corner in $C[V \setminus \{v_1, \ldots, v_{i-1}\}]$. A graph that is dismantlable to the graph K_1 is also called* **dismantlable**.

The enumeration of the vertices is also called a **dismantling sequence**. *Note that dismantling sequences from a graph C to a graph D, and then from D to another graph B, can be concatenated to obtain a dismantling sequence from C to B.*

Lemma 2.3 shows that dismantlable subgraphs H_t are not a barrier to the dismantlability of a lexicographic sum because these subgraphs can be dismantled first. Therefore, after Lemma 2.3 we concentrate on nondismantlable subgraphs H_t.

Lemma 2.3. *Let $v \in T$. If H_v is dismantlable to the graph D, then $\sum_{t\in T} H_t$ is dismantlable to a graph $\sum_{t\in T} H'_t$, where, for $t \neq v$, we have that $H_t = H'_t$ and, for $t = v$, we have that H'_v is the graph D.*

Proof. In a lexicographic sum $\sum_{t\in T} H_t$, a corner in a subgraph H_{t_0} is also a corner in $\sum_{t\in T} H_t$.

Hence, a dismantling sequence from H_v to D is also a dismantling sequence that starts with $\sum_{t\in T} H_t$ and that ends with $\sum_{t\in T} H'_t$. □

Lemma 2.4 (c.f. [3, Lemma 5])**.** *Every retract of a dismantlable finite graph is dismantlable.*

Proof. Let D be a dismantlable graph and let B be a retract of D. Then for the copnumber of B, we obtain $c(B) \leq c(D) = 1$, that is, $c(B) = 1$. However, this means B is dismantlable. □

Lemma 2.5. *Let $C = (V, E)$ be a graph and let $v \in V$ be a corner in C. Then C is dismantlable if and only if the induced subgraph $C[V \setminus \{v\}]$ is dismantlable.*

Proof. In the forward direction, the function that maps a corner v to a vertex u so that $N[v] \subseteq N[u]$ while leaving all other vertices fixed is a retraction. By Lemma 2.4, the induced subgraph $C[V \setminus \{v\}]$ is dismantlable. In the other direction, attach v at the beginning of a dismantling sequence for the induced subgraph $C[V \setminus \{v\}]$. □

Theorem 2.6. *Let $G = (T, F)$ be a dismantlable graph. Then $\sum_{t\in T} H_t$ is dismantlable if and only if $|T| = 1$ and H_1 is dismantlable or $|T| > 1$ and there is a dismantling sequence $t_1, \ldots, t_n$ of G so that, for each t_i, there is a t_{i+k} so that $H_{t_{i+k}}$ is dismantlable and $N_{G[T\setminus\{t_1,\ldots,t_{i-1}\}]}[t_i] \subseteq N_{G[T\setminus\{t_1,\ldots,t_{i-1}\}]}[t_{i+k}]$.*

Proof. First note that, by Lemmas 2.3 and 2.5, it is sufficient to prove the case in which all dismantlable H_{t_j} have exactly one vertex. By the same Lemmas we can assume without loss of generality that none of the non-dismantlable graphs H_t have a corner.

Beginning with the reverse direction, to dismantle $\sum_{t\in T} H_t$, we first remove the vertices in V_{t_1}, then those in V_{t_2}, and so on. At each step, the vertices $v \in V_{t_i}$ are so that $N_{\sum_{t\in T\setminus\{t_1,\ldots,t_{i-1}\}} H_t}[v] \subseteq N_{\sum_{t\in T\setminus\{t_1,\ldots,t_{i-1}\}} H_t}[w]$, where w is the only vertex in the vertex set $V_{t_{i+k}}$ from the hypothesis. Hence the order in which the vertices in the individual V_{t_i} are removed does not matter.

In the forward direction, the proof is by induction on $|T|$. Note that the base case $|T| = 1$ is trivial. For the induction step $|T| - 1 \to |T|$, let v_1 be the first vertex in a dismantling sequence of $\sum_{t\in T} H_t$ and let $t_1 = i(v_1)$. Recall that, by our initial assumption, no subgraph H_t has a corner.

In case $|H_{t_1}| = 1$, there is a $w \notin V_{t_1}$ with $N[v_1] \subseteq N[w]$. With $u = i(w) \neq t_1$, we have $N_G[t_1] \subseteq N_G[u]$. Moreover, if H_u was not dismantlable, then

$N[v_1]$ would contain vertices of H_u that are not adjacent to w, which cannot be. Hence, $|H_u| = 1$.

In case $|H_{t_1}| > 1$, the subgraph H_{t_1} is not dismantlable and has no corners. In particular, v_1 is not a corner in H_{t_1}. Therefore, there is a $w \notin V_{t_1}$ with $N[v_1] \subseteq N[w]$. With $u = i(w) \neq t_1$, we have $N_G[t_1] \subseteq N_G[u]$. Moreover, if H_u was not dismantlable, then $N[v_1]$ would contain vertices of H_u that are not adjacent to w, which cannot be. Hence, $|H_u| = 1$.

In either case, t_1 is a corner in G and there is a $u \in T \setminus \{t_1\}$ so that $|H_u| = 1$ and $N_G[t_1] \subseteq N_G[u]$. Therefore, all vertices v of H_{t_1} are corners so that, with w being the unique element of H_u, we have $N[v] \subseteq N[w]$.

Remove from $\sum_{t\in T} H_t$ all vertices of H_{t_1} in any order. Because these vertices are corners, by Lemma 2.5, the lexicographic sum $\sum_{t\in T\setminus\{t_1\}} H_t$ is dismantlable. The result now follows from the induction hypothesis and from the fact that $N_G[t_1] \subseteq N_G[u]$ and $|H_u| = 1$. $\square$

Theorem 2.7. *Let G and H be graphs and let G be connected. If $c(G) \geq 2$, then $c(G \bullet H) = c(G)$. If $c(G) = c(H) = 1$, then $c(G \bullet H) = 1$. If $c(G) = 1$, $|T| \geq 2$ and $c(H) \geq 2$, then $c(G \bullet H) = 2$. If $|T| = 1$, then $c(G \bullet H) = c(H)$.*

Proof. The claim for $c(G) \geq 2$ follows from Theorem 1.4. For $c(G) = c(H) = 1$, by Theorem 2.6 we have that $c(G \bullet H) = 1$. For $c(G) = 1$, $|T| \geq 2$ and $c(H) \geq 2$, Theorem 2.6 implies $c(G \bullet H) = 2$. Finally, the claim for $|T| = 1$ is trivial because, in this case, $G \bullet H$ is isomorphic to H. $\square$

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