Contributions to Discrete Mathematics

Volume 12, Number 1, Pages 106–109
ISSN 1715-0868

COMMENTS ON THE GOLDEN PARTITION CONJECTURE

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Abstract. We generalize the result of Zaguia that the 1/3–2/3 Conjecture is satisfied by every N-free finite poset which is not a chain: we show a wider class of posets which satisfy the Golden Partition Conjecture. We generalize the result of Pouzet that the 1/3–2/3 Conjecture is satisfied by every finite poset with a non-trivial automorphism: we show that such posets satisfy the Golden Partition Conjecture.

1. Introduction

Throughout the whole paper, P denotes a finite poset $(V, \leq)$, where $\leq$ is a reflexive, antisymmetric, and transitive relation on a set V. By $<$ we denote the non-reflexive (asymmetric and transitive) counterpart of $\leq$. For $x, y \in V$ we say that x is a *lower cover* of y if $x < y$ and there is no element $z \in V$ such that $x < z < y$. We say that P contains an N-poset if there exist four distinct elements $a, b, c, d \in V$ such that the element a is a lower cover of the element b, the element c is a lower cover of the elements b and d, and these are the only relations between the elements a, b, c, d, see Figure 1. We say that P is N-free if it does not contain an N-poset.

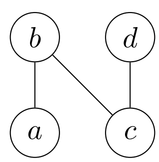

Figure 1. An N-poset

A bijection $\alpha\colon V \to V$ is an *automorphism* of P if for every $x, y \in V$, it holds that $x < y$ if and only if $\alpha(x) < \alpha(y)$. We say that an automorphism is *non-trivial* if it is not the identity map.

We define a *comparison* on P as a pair (x, y) of two distinct elements $x, y \in V$ for which we ask an oracle about a relation between x and y.

Received by the editors June 23, 2015 and in revised form August 10, 2016.
2010 *Mathematics Subject Classification.* 06A06, 06A07, 06A11.
Key words and phrases. Poset, Linear extension, 1/3–2/3 Conjecture, Golden Partition Conjecture.

If the chosen elements are incomparable in P, then there are two possible outcomes: either x precedes y or x succeeds y. In the former case we extend the relation of P by $x < y$ and in the latter case by $y < x$. In both cases we close the relation transitively. If the chosen elements are comparable in P, then the oracle answers according to the relation and the poset remains unchanged.

Related to our discussion is the 1/3–2/3 Conjecture which was formulated independently by Kislitsyn, Fredman, and Linial [1, 3, 4]. Here we use an equivalent formulation: if P is not a chain then we can point out a comparison such that regardless of the oracle's answer, the following inequality holds:

$$t_0 \geq \frac{3}{2}t_1,$$

where t_0 and t_1 denotes the number of linear extensions of the poset P and the poset obtained after the comparison, respectively. Zaguia proved that an N-free poset cannot be a counterexample to the 1/3–2/3 Conjecture, see Theorem 1 in [6]. In Section 2 of [2], authors quote the argument of Pouzet that proves that every poset with a non-trivial automorphism cannot be a counterexample to the 1/3–2/3 Conjecture.

We formulated the Golden Partition Conjecture (GPC) in [5]: if P is not a chain then we can point out two consecutive comparisons such that regardless of the oracle's answers the following inequality holds:

$$t_0 \geq t_1 + t_2,$$

where t_0, t_1, t_2 denotes the number of linear extensions of the poset P, the poset obtained after the first comparison, the poset obtained after both comparisons, respectively. The GPC generalizes the 1/3–2/3 Conjecture, see Proposition 1 in [5].

We generalize the result of Zaguia in Section 2. We show a class of finite posets containing all not totally ordered N-free posets, but also many other posets. Every member of this class cannot be a counterexample to the GPC and hence it cannot be a counterexample to the 1/3–2/3 Conjecture as well. We generalize the result of Pouzet in Section 3. We show that every poset with a non-trivial automorphism cannot be a counterexample to the GPC. We benefit from three facts, but first we introduce an additional notation.

For $x, y \in V$, if $x \neq y$ and $y \not< x$ then by $P + xy$ we denote the poset $(V, <')$, where $<'$ is the transitive closure of the relation $<$ extended by $x < y$. By $P + xy + uv$, we mean $(P + xy) + uv$. We denote by $e(P)$ the number of linear extensions of P. If $x < y$ then $e(P + xy) = e(P)$, and by convention we take $e(P + yx) = 0$, however there is no poset $P + yx$.

Fact 1. Let $x, y, z \in V$ be three distinct elements. A triple (x, y, z) is called a *balanced triple* in P if

$$e(P + xy + yz) \leq \max\{e(P + yx), e(P + zy)\} \leq \frac{1}{2}e(P).$$

We proved that if P contains a balanced triple, then it cannot be a counterexample to the GPC, see Lemma 1 in [5].

Fact 2. Let (x, y) be an incomparable pair in P. An element z is called a *slave* for the pair (x, y) if z is above x and incomparable with y or z is below y and incomparable with x. We proved that if an incomparable pair (x, y) has at most one slave in P and $e(P + xy) \geq e(P)/2$, then P cannot be a counterexample to the GPC, see Lemma 2 in [5].

Fact 3. Let $x, y, z \in V$ be three distinct elements. A triple (x, y, z) is called a *cyclic triple* in P if

$$e(P + xy) > \frac{1}{2}e(P), \quad e(P + yz) > \frac{1}{2}e(P), \quad e(P + zx) > \frac{1}{2}e(P).$$

We proved that if P contains a cyclic triple, then it cannot be a counterexample to the GPC, see Lemma 3 in [5].

2. Not only N-free posets

For $x \in V$, we denote by $U(x)$ the *upper set* of the element x, i.e. $U(x) = \{y \in V : x < y\}$. Note that there may exist elements u, v such that $u \in U(x)$, $v < u$ and the elements x and v are incomparable in P.

We define a class $\mathcal{P}$ of finite posets as follows. If $P \in \mathcal{P}$ then P is not a chain and P contains two distinct elements x, y with the same set of lower covers and such that elements in sets $U(x) \cup \{x\}$ and $U(y) \cup \{y\}$ form two chains. Note that the common set of lower covers may be empty and the sets $U(x)$ and $U(y)$ do not need to be disjoint. The proof of Theorem 1 in [6] shows that every finite not totally ordered N-free poset belongs to the class $\mathcal{P}$. Obviously $\mathcal{P}$ contains many other posets not necessarily N-free.

Now we prove that every poset in $\mathcal{P}$ cannot be a counterexample to the GPC. Without loss of generality we can label the two elements in the definition of the class $\mathcal{P}$ such that $e(P + xy) \geq e(P)/2$. Let $x = x_1 < x_2 < x_3 < \ldots$ be the chain of elements of the set $U(x) \cup \{x\}$. Let r be the largest index for which $e(P + x_r y) \geq e(P)/2$.

If there exists a successor x_{r+1} incomparable in P with y, then (x_r, y, x_{r+1}) is a balanced triple in P. Indeed, we have $\max\{e(P + yx_r), e(P + x_{r+1}y)\} \leq e(P)/2$. As elements x and y have the same set of lower covers, then for every z such that $z < y$, it holds that $z < x$ and thus also $z < x_r$ because $U(x) \cup \{x\}$ is a chain. Moreover, for every z such that $x_r < z$, it holds that $z = x_{r+1}$ or $x_{r+1} < z$. Therefore, in every linear extension of $P + x_r y + y x_{r+1}$, the elements between x_r and y are incomparable in P with x_r and y. Hence, if we exchange the elements x_r and y, we obtain a linear extension of $P + yx_r$. This means that $e(P + x_r y + y x_{r+1}) \leq e(P + yx_r)$. The proof is complete by Fact 1.

If there is no successor x_{r+1} incomparable in P with y then the pair (x_r, y) has no slave. The proof is complete by Fact 2.

Observe that we can define another class of posets satisfying the GPC replacing the assumption that $U(y) \cup \{y\}$ forms a chain by the assumption that $e(P + xy) \geq e(P)/2$.

3. Posets with a non-trivial automorphism

We assume now that P has a non-trivial automorphism α. This implies that P is not a chain.

If P contains a pair (x, y) such that $e(P + xy) = e(P)/2$, then we take this pair as the first comparison. We have $t_1 = t_0/2$ and $t_2 \leq t_1$. Therefore P satisfies the GPC.

If P does not contain a pair (x, y) such that $e(P + xy) = e(P)/2$, then V contains at least three elements. We define a relation $\ll$ on V such that $x \ll y$ if $e(P + xy) > e(P)/2$. Because $e(P + xy) = e(P + \alpha(x)\alpha(y))$, α respects $\ll$, i.e. $x \ll y$ if and only if $\alpha(x) \ll \alpha(y)$. If $\ll$ was transitive then it would be a linear order and α would be the identity, which contradicts the assumption. Hence the relation $\ll$ is not transitive and P contains a cyclic triple. The proof is complete by Fact 3.

Note that if P is additionally cycle-free, then it contains a pair (x, y) such that $e(P + xy) = e(P)/2$, see the main theorem in [2].

References

1. G. Brightwell, *Balanced pairs in partial orders*, Discrete Math. **201** (1999), 25–52.
2. B. Ganter, G. Häfner, and W. Poguntke, *On linear extensions of ordered sets with a symmetry*, Discrete Math. **63** (1987), 153–156.
3. S.S. Kislitsyn, *Finite partially ordered sets and their associated sets of permutations*, Mat. Zametki **4** (1968), 511–518.
4. N. Linial, *The information theoretic bound is good for merging*, SIAM J. Comput. **13** (1984), 795–801.
5. M. Peczarski, *The gold partition conjecture*, Order **23** (2006), 89–95.
6. I. Zaguia, *The* 1/3–2/3 *conjecture for N-free ordered sets*, Electr. J. Comb. **19** (2012), #P29.

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