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Q_4-FACTORIZATION OF λK_n AND $\lambda K_{x(m)}$

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ABSTRACT. In this study, we show that necessary conditions for Q_4-factorization of λK_n and $\lambda K_{x(m)}$ (complete x partite graph with parts of size m) are sufficient. We proved that there exists a Q_4-factorization of $\lambda K_{x(m)}$ if and only if $mx \equiv 0 \pmod{16}$ and $\lambda m(x-1) \equiv 0 \pmod 4$. This result immediately gives that λK_n has a Q_4-factorization if and only if $n \equiv 0 \pmod{16}$ and $\lambda \equiv 0 \pmod 4$.

1. INTRODUCTION

Given a graph H, an *H-decomposition of a graph G* is a collection of edge-disjoint subgraphs of G, isomorphic to H, such that each edge of G belongs to exactly one subgraph. Each subgraph H is called a *block*. Such a decomposition is called *resolvable* if it is possible to partition the blocks into classes (often referred to as *parallel classes*) such that each vertex of G appears in exactly one block of each parallel class.

A resolvable H-decomposition of G is generally referred to as an *H-factorization of G*, and each parallel class is called an *H-factor of G*. If $H = K_2$ (a single edge), then the H-factorization is known as a *1-factorization* of G. In general, if the factors are regular of degree k, then the factorization is called a *k-factorization*. A *near-one-factor* of G is a set of edges that cover all but one vertex. A set of near-one-factors which covers every edge precisely once is called a *near-one-factorization*.

A *complete graph K_n* is a simple graph on n vertices in which each pair of distinct vertices are connected by a unique edge. If the edges are taken λ times, then the graph is denoted by λK_n. A *complete equipartite graph* $K_{x(m)}$ has xm vertices, partitioned into x different parts of size m, so that any two vertices are adjacent if and only if they are in different parts. If there are λ copies of each edge, then the graph is denoted by $\lambda K_{x(m)}$.

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Let k be a positive integer. A *group divisible design of index* 1, denoted by k-GDD, is a triple (V, G, B) where:

(1) V is a finite set of points of size mn,
(2) G is a set of n subsets of V each with size m, called groups, which partition V,
(3) $\mathcal{B}$ is a collection of subsets of V with size k, called blocks, such that every pair of points from distinct groups occurs in exactly one block, and
(4) no pair of points belonging to a group occurs in any block.

A k-GDD is said to be *resolvable* and denoted by k-RGDD if its blocks can be partitioned into parallel classes, each of which partitions the set of points. A k-GDD or k-RGDD with n groups, each group is of size m will be shown by k-GDD of type m^n and k-RGDD of type m^n, respectively. Note that a K_k-decomposition of $K_{x(m)}$ is a k-GDD of type m^x.

The *k-dimensional cube* or *k-cube* is the simple graph whose vertices are the k-tuples with entries in $\{0, 1\}$ and edges are the pairs of k-tuples that differ in exactly one position. This graph is bipartite and k-regular. The k-cube is denoted by Q_k. The number of vertices in a k-cube is 2^k and the number of edges is $k2^{k-1}$. In particular, Q_4, shown in Figure 1, has 16 vertices and 32 edges.

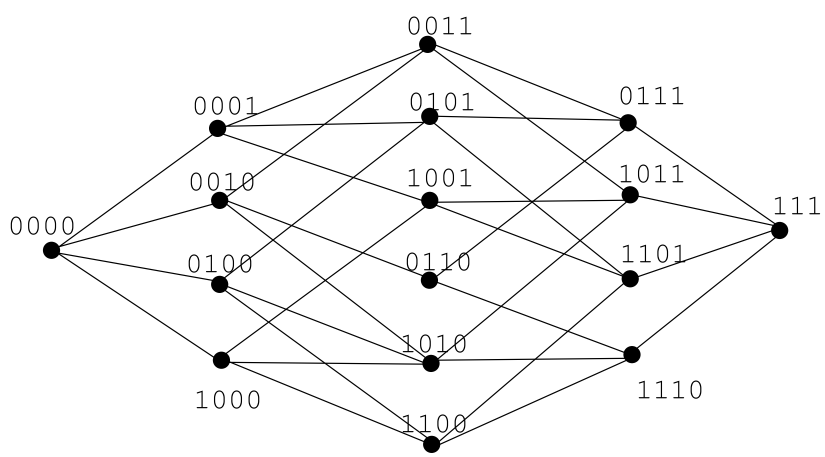

FIGURE 1. Q_4

In 1979, Kotzig posed two problems related to a Q_k-decomposition and a Q_k-factorization of K_n, which are Problems 15 and 16 in [8]. Those two open problems are:

Cube Decomposition Problem: For which values of n and k does there exist a Q_k-decomposition of K_n?

Cube Factorization Problem: For which values of n and k does there exist a Q_k-factorization of K_n?

Kotzig [8] established necessary conditions for a Q_k-decomposition of K_n: If there exists such a decomposition then

(a) if k is even, then $n \equiv 1 \pmod{k2^k}$ and
(b) if k is odd, then either
  (i) $n \equiv 1 \pmod{k2^k}$ or
  (ii) $n \equiv 0 \pmod{2^k}$ and $n \equiv 1 \pmod{k}$.

For even k, Kotzig [9] proved the sufficiency of necessary conditions. Moreover, for $k = 3$ [10] and $k = 5$ [3], the problems have been solved completely. In addition, a Q_3-decomposition of λK_n is solved in [1].

In 1976, Wilson [13] proved that for each k, there is a Q_k-decomposition of K_n for all sufficiently large n satisfying the necessary conditions. In addition in [7], it is proven that for each odd k, there is an infinite arithmetic progression of even integers n for which a Q_k-decomposition of K_n exists.

On the other hand, since these problems were introduced, progress on the cube factorization problem has been done for some special values of n, see [5] and [6]. Necessary conditions for the existence of a Q_k-factorization of K_n are

$$n \equiv 0 \left(\bmod\ 2^k\right) \text{ and } n \equiv 1 \pmod{k}.$$

The first condition implies that n must be even and the second condition implies that n must have opposite parity to k. Hence, if a Q_k-factorization of K_n exists, then k must be odd. For $k = 3$ [2], this problem is completely solved; the other cases are still open.

If we consider λK_n, then necessary conditions for a Q_4-factorization of λK_n are

$$n \equiv 0 \pmod{16} \text{ and } \lambda(n-1) \equiv 0 \pmod{4}.$$

Necessary conditions for a Q_4-factorization of $\lambda K_{x(m)}$ are

$$mx \equiv 0 \pmod{16} \text{ and } \lambda m(x-1) \equiv 0 \pmod{4}.$$

In [12] Q_3-factorizations of $\lambda K_{x(m)}$ are studied. The other cases are still open for $k > 3$.

In this study, we investigate the sufficiency of necessary conditions for Q_4-factorizations of λK_n and $\lambda K_{x(m)}$. Theorem 1.1 establishes sufficiency of necessary conditions.

Theorem 1.1. *There exists a Q_4-factorization of $\lambda K_{x(m)}$ if and only if $mx \equiv 0 \pmod{16}$ and $\lambda m(x-1) \equiv 0 \pmod{4}$.*

In Section 2, we establish required small examples and preliminary results. Theorem 1.1 is proven in Section 3. The result on complete graphs is also given in this section.

The following two results are used several times throughout this paper.

Theorem 1.2. *$\lambda K_{x(m)}$ has a 1-factorization if and only if xm is even* [4].

Theorem 1.3. *A 4-RGDD of type 4^m exists for every $m \in \mathbb{Z}^+$, except for $m \in \{2, 3, 6\}$* [11].

2. Preliminary Results

In this section, some important constructions and examples are given. These examples will be used in Section 3 to prove Theorem 1.1.

Example 2.1. *Q_4-factorization of $K_{4(4)}$.*

Let the parts of $K_{4(4)}$ be denoted by $X,\ Y,\ Z, W$ where $X = \{x_1, x_2, x_3, x_4\}$, $Y = \{y_1, y_2, y_3, y_4\}$, $Z = \{z_1, z_2, z_3, z_4\}$, and $W = \{w_1, w_2, w_3, w_4\}$. The labeling in Table 1 gives the blocks of the factorization: B_1, B_2, and B_3.

	0000	0001	0010	0100	1000	0011	0101	1001	0110	1010	1100	0111	1011	1101	1110	1111
B_1	x_1	z_1	z_2	z_3	z_4	x_2	x_3	x_4	y_1	y_2	y_3	w_1	w_2	w_3	w_4	y_4
B_2	x_1	w_1	w_2	w_3	w_4	z_2	z_3	x_4	z_4	x_3	x_2	y_4	y_3	y_2	y_1	z_1
B_3	x_1	y_1	y_2	y_3	y_4	w_3	w_2	x_4	w_1	x_3	x_2	z_1	z_2	z_3	z_4	w_4

Table 1. Q_4-factors of $K_{4(4)}$

Example 2.2. *Q_4-factorization of $K_{2(8)}$.*

Let X and Y be the parts of $K_{2(8)}$ where $X = \{x_1, x_2, ..., x_8\}$ and $Y = \{y_1, y_2, ..., y_8\}$. The labeling in Table 2 gives a Q_4-factorization of $K_{2(8)}$.

	0000	0001	0010	0100	1000	0011	0101	1001	0110	1010	1100	0111	1011	1101	1110	1111
B_1	x_1	y_1	y_2	y_3	y_4	x_2	x_3	x_4	x_5	x_6	x_7	y_5	y_6	y_7	y_8	x_8
B_2	x_1	y_5	y_6	y_7	y_8	x_7	x_6	x_4	x_5	x_3	x_2	y_1	y_2	y_3	y_4	x_8

Table 2. Q_4-factors of $K_{2(8)}$

Example 2.3. *Q_4-factorization of $K_{4(12)}$.*

Let the parts of $K_{4(12)}$ be denoted by $X,\ Y,\ Z,\ W$, where each of these parts are divided into 3 sets denoted by X_i, Y_i, Z_i, and W_i for $1 \leq i \leq 3$ containing 4 vertices each. Let $x_{i,j}$, $y_{i,j}$, $z_{i,j}$, $w_{i,j}$ denote the vertices of X_i, Y_i, Z_i, W_i, respectively for $1 \leq i \leq 3$ and $1 \leq j \leq 4$.

For each i, the parts X_i, Y_i, Z_i, and W_i form a copy of $K_{4(4)}$ which can be decomposed into 3 Q_4-factors by Example 2.1. Let these factors be denoted by $B_{i,1}$, $B_{i,2}$, and $B_{i,3}$.

Consider the blocks obtained by the labeling in Table 3.

	0000	0001	0010	0100	1000	0011	0101	1001	0110	1010	1100	0111	1011	1101	1110	1111
$B'_{1,1}$	$x_{2,1}$	$y_{3,1}$	$y_{3,2}$	$y_{3,3}$	$y_{3,4}$	$x_{2,2}$	$x_{2,3}$	$x_{2,4}$	$z_{2,1}$	$z_{2,2}$	$z_{2,3}$	$w_{3,1}$	$w_{3,2}$	$w_{3,3}$	$w_{3,4}$	$z_{2,4}$
$B''_{1,1}$	$x_{3,1}$	$y_{2,1}$	$y_{2,2}$	$y_{2,3}$	$y_{2,4}$	$x_{3,2}$	$x_{3,3}$	$x_{3,4}$	$z_{3,1}$	$z_{3,2}$	$z_{3,3}$	$w_{2,1}$	$w_{2,2}$	$w_{2,3}$	$w_{2,4}$	$z_{3,4}$
$B'_{1,2}$	$x_{2,1}$	$w_{3,4}$	$w_{3,3}$	$w_{3,2}$	$w_{3,1}$	$x_{2,2}$	$x_{2,3}$	$x_{2,4}$	$y_{2,4}$	$y_{2,3}$	$y_{2,2}$	$z_{3,1}$	$z_{3,2}$	$z_{3,3}$	$z_{3,4}$	$y_{2,1}$
$B''_{1,2}$	$x_{3,1}$	$w_{2,4}$	$w_{2,3}$	$w_{2,2}$	$w_{2,1}$	$x_{3,2}$	$x_{3,3}$	$x_{3,4}$	$y_{3,4}$	$y_{3,3}$	$y_{3,2}$	$z_{2,1}$	$z_{2,2}$	$z_{2,3}$	$z_{2,4}$	$y_{3,1}$
$B'_{1,3}$	$x_{2,1}$	$z_{3,4}$	$z_{3,3}$	$z_{3,2}$	$z_{3,1}$	$x_{2,2}$	$x_{2,3}$	$x_{2,4}$	$w_{2,1}$	$w_{2,2}$	$w_{2,3}$	$y_{3,4}$	$y_{3,3}$	$y_{3,2}$	$y_{3,1}$	$w_{2,4}$
$B''_{1,3}$	$x_{3,1}$	$z_{2,4}$	$z_{2,3}$	$z_{2,2}$	$z_{2,1}$	$x_{3,2}$	$x_{3,3}$	$x_{3,4}$	$w_{3,1}$	$w_{3,2}$	$w_{3,3}$	$y_{2,4}$	$y_{2,3}$	$y_{2,2}$	$y_{2,1}$	$w_{3,4}$

Table 3. Q_4-blocks of $K_{4(12)}$

Apply the permutation $P = (x_{2,j}, x_{1,j})(y_{2,j}, y_{1,j})(z_{2,j} z_{1,j})(w_{2,j}, w_{1,j})$ on the above blocks to obtain new blocks. These new blocks are named by the following permutation: $P(B'_{1,j}) = B'_{2,j}$ and $P(B''_{1,j}) = B''_{2,j}$ for $1 \leq j \leq 3$.

Independently, apply the permutation $R = (x_{3,j}, x_{1,j})(y_{3,j},\ y_{1,j})(z_{3,j} z_{1,j})$ $(w_{3,j}, w_{1,j})$ on the above blocks to obtain new blocks. These new blocks are named by the following permutation: $R(B'_{1,j}) = B'_{3,j}$ and $R(B''_{1,j}) = B''_{3,j}$ for $1 \leq j \leq 3$. Then, $\pi_{i,j} = \{B_{i,j}, B'_{i,j}, B''_{i,j},\ 1 \leq i \leq 3,\ 1 \leq j \leq 3\}$ form the 9 factors of the Q_4-factorization of $K_{4(12)}$.

Example 2.4. *Q_4-factorization of $2K_{8(2)}$.*

Let the parts be denoted by X, Y, Z, W, R, S, T, V, where each part has two vertices. Consider the following 4 factors in Table 4.

	0000	0001	0010	0100	1000	0011	0101	1001	0110	1010	1100	0111	1011	1101	1110	1111
B_1	x_1	y_1	y_2	w_2	v_2	x_2	z_2	t_2	z_1	t_1	r_1	w_1	v_1	s_1	s_2	r_2
B_2	x_1	r_1	z_1	s_2	t_2	t_1	y_2	z_2	v_2	r_2	w_1	w_2	x_2	v_1	y_1	s_1
B_3	x_1	w_1	z_2	v_1	r_2	y_2	r_1	v_2	s_2	t_1	w_2	t_2	s_1	x_2	y_1	z_1
B_4	x_1	y_1	y_2	t_1	s_1	x_2	v_1	r_1	v_2	r_2	w_1	t_2	s_2	z_1	z_2	w_2

TABLE 4. Q_4-factors of $2K_{8(2)}$

The remaining 3 factors are obtained by considering the $K_{4(4)}$ formed by the parts $X \cup Y$, $Z \cup W$, $R \cup S$, $T \cup V$ and taking the factors as in Example 2.1.

Lemma 2.5. *There exists a Q_4-factorization of $K_{4(16k+4)}$ for $k \geq 0$.*

Proof. There exists a 4-RGDD of type 4^{4k+1} for each $k \geq 0$ by Theorem 1.3. Let $b_{i,j}$ be the jth block of the ith parallel class. For each $1 \leq i \leq 4k+1$ and $1 \leq j \leq 4k+1$ blow-up vertices in each block by 4 to obtain a copy of $K_{4(4)}$ on $b_{i,j} \times \{1, 2, 3, 4\}$ for each i and j. Then place a Q_4-factorization of $K_{4(4)}$ on the blown-up blocks. There are 3 factors in each Q_4-factorization of $K_{4(4)}$ by Example 2.1; let the factors for each blown-up b_{ij} be $B_{i,j,k}$ for $1 \leq k \leq 3$. Then the followings are the factors of the Q_4-factorization:

$$\pi_{i,1} = \{B_{i,j,1}, 1 \leq j \leq 4k+1\},$$
$$\pi_{i,2} = \{B_{i,j,2}, 1 \leq j \leq 4k+1\},$$
$$\pi_{i,3} = \{B_{i,j,3}, 1 \leq j \leq 4k+1\},$$

where $1 \leq i \leq 4k+1$. The number of parallel classes is $12k+3$ and the number of Q_4's in each parallel class is $4k+1$ as expected. □

Lemma 2.6. *There exists a Q_4-factorization of $K_{4(16k+12)}$ for $k \geq 0$.*

Proof. Since there exists a 4-RGDD of type 4^{4k+3} for each $k \geq 1$ by Theorem 1.3, a Q_4-factorization of $K_{4(16k+12)}$ can be obtained as in the proof of Lemma 2.5. The case $k = 0$ is obtained in Example 2.3. □

Lemma 2.7. *There exists a Q_4-factorization of $K_{t(16k)}$ and $K_{2t(8k)}$ for $k, t \geq 1$.*

Proof. Consider a 1-factorization of $K_{t(2k)}$ which is known to exist for $k, t \geq 1$ by Theorem 1.2. Let the factors be $F_1, F_2, \ldots, F_n$, where $n = (2k)(t-1)$. Let the edges of the factors F_i be $E(F_i) = \{e_{i,1}, e_{i,2}, \ldots, e_{i,s}\}$, where $s = kt$. When each vertex of $K_{t(2k)}$ is blown-up by 8, then each edge in the 1-factors correspond to a copy of $K_{2(8)}$. By Example 2.2, $K_{2(8)}$ has a Q_4-factorization into two Q_4's. Let $B_{i,j,1}$, $B_{i,j,2}$ be the Q_4 factors of each copy of $K_{2(8)}$ corresponding to the edge $e_{i,j}$. Hence, parallel classes of the factorization of $K_{t(16k)}$ are:

$$\pi_{i,1} = \{B_{i,j,1}, 1 \leq j \leq s\}, \pi_{i,2} = \{B_{i,j,2}, 1 \leq j \leq s\} \text{ for } 1 \leq i \leq 2k(t-1).$$

Similarly, consider a 1-factorization of $K_{2t(k)}$ which is known to exist by Theorem 1.2. As above, blow-up each vertex of $K_{2t(k)}$ by 8. Hence, parallel classes of the factorization of $K_{2t(8k)}$ are:

$$\pi_{i,1} = \{B_{i,j,1}, 1 \leq j \leq s\}, \pi_{i,2} = \{B_{i,j,2}, 1 \leq j \leq s\} \text{ for } 1 \leq i \leq k(2t-1).$$

□

3. Q_4-Factorization of $\lambda K_{x(m)}$ and λK_n

We study this problem depending on the value of λ modulo 4 and the values of x and m. Recall that necessary conditions for a Q_4-factorization of $\lambda K_{x(m)}$ are:

$$mx \equiv 0 \pmod{16} \text{ and } \lambda m(x-1) \equiv 0 \pmod{4}. \tag{3.1}$$

Case 1: $\lambda \equiv 1$ *or* $3 \pmod 4$. By (3.1), if $\lambda \equiv 1$ or $3 \pmod 4$, necessary conditions for Q_4-factorizations of $\lambda K_{x(m)}$ reduce to $mx \equiv 0 \pmod{16}$ and $m \equiv 0 \pmod 4$. These are equivalent to necessary conditions for $\lambda = 1$. We will construct a Q_4-factorization of $K_{x(m)}$ and will take λ copies of the factors.

Two subcases on m will be considered.

Subcase 1.1: $m \equiv 4,\ 12 \pmod{16}$.

The first necessary condition implies that $4|x$. We look for a Q_4-factorization of $K_{4t(16k+4)}$ and $K_{4t(16k+12)}$ for $k \geq 0$ and $t \geq 1$.

Let the vertices of $K_{4t(16k+4)}$ be partitioned into t vertex-disjoint subgraphs each isomorphic to $K_{4(16k+4)}$. By Lemma 2.5, these subgraphs have Q_4-factorizations. The remaining edges of $K_{4t(16k+4)}$ correspond to a copy of $K_{t(64k+16)}$. By Lemma 2.7, this graph has a Q_4-factorization. Combining these factors gives the Q_4-factorization of $K_{4t(16k+4)}$.

Similarly, if vertices of $K_{4t(16k+12)}$ are partitioned into t vertex-disjoint subgraphs each isomorphic to $K_{4(16k+12)}$, by Lemma 2.6, these subgraphs have Q_4-factorizations. The remaining edges correspond to a copy of $K_{t(64k+48)}$ which has a Q_4-factorization by Lemma 2.7.

Subcase 1.2: $m \equiv 0,\ 8 \pmod{16}$.

When $m \equiv 0 \pmod{16}$, both of the necessary conditions are satisfied. So, we look for a Q_4-factorization of $K_{t(16k)}$ which follows by Lemma 2.7.

For $m \equiv 8 \pmod{16}$, to satisfy the first necessary condition, x should be even. We look for a Q_4-factorization of $K_{2t(16k+8)}$ which follows by Lemma 2.7.

Case 2: $\lambda \equiv 2 \pmod 4$.

By (3.1), necessary conditions for Q_4-factorizations of $\lambda K_{x(m)}$ reduce to $m \equiv 0 \pmod 2$ and $mx \equiv 0 \pmod{16}$. When $m \equiv 0 \pmod 4$, this problem is solved in Case 3.1 above. So, we only need to consider $m \equiv 2 \pmod 4$. We will construct a Q_4-factorization of $2K_{8t(2k)}$ and take $\lambda/2$ copies of it.

Example 3.1. *There exists a Q_4 factorization of $2K_{2(16)} - 2F$ where $2F$ represents two copies of a 2-factor of $2K_{2(16)}$ with 4-cycles.*

Let the parts of $2K_{2(16)}$ be denoted by X and Y and the vertices be labeled by x_i and y_i, respectively for $1 \leq i \leq 16$. Let F be a 2-factor consisting of the 4-cycles: $F = (x_{2i-1}, y_{2i}, x_{2i}, y_{2i-1})$ for $1 \leq i \leq 8$.

Consider the blocks in Table 5.

	0000	0001	0010	0100	1000	0011	0101	1001	0110	1010	1100	0111	1011	1101	1110	1111
B_1	x_1	y_3	y_4	y_5	y_7	x_2	x_7	x_5	x_8	x_6	x_3	y_6	y_8	y_1	y_2	x_4
B_1'	x_9	y_{11}	y_{12}	y_{13}	y_{15}	x_{10}	x_{15}	x_{13}	x_{16}	x_{14}	x_{11}	y_{14}	y_{16}	y_9	y_{10}	x_{12}
B_2	x_3	y_6	y_5	y_8	y_2	x_4	x_1	x_7	x_2	x_8	x_5	y_7	y_1	y_4	y_3	x_6
B_2'	x_{11}	y_{14}	y_{13}	y_{16}	y_{10}	x_{12}	x_9	x_{15}	x_{10}	x_{16}	x_{13}	y_{15}	y_9	y_{12}	y_{11}	x_{14}
B_3	x_1	y_3	y_5	y_6	y_8	x_7	x_8	x_6	x_2	x_4	x_3	y_4	y_2	y_1	y_7	x_5
B_3'	x_9	y_{11}	y_{13}	y_{14}	y_{16}	x_{15}	x_{16}	x_{14}	x_{10}	x_{12}	x_{11}	y_{12}	y_{10}	y_9	y_{15}	x_{13}

TABLE 5. Q_4-blocks of $2K_{16} - 2F$

$\{B_i, B_i'\}$, $1 \leq i \leq 3$ gives the 3 factors of the Q_4-factorization of $2K_{2(16)} - 2F$.

Let $X_1 = \{x_1, x_2, ..., x_8\}$ and $X_2 = \{x_9, x_{10}, ..., x_{16}\}$, and define Y_1 and Y_2 similarly. The edges between X_1 and Y_2 and also between Y_1 and X_2 form a copy of $2K_{2(8)}$ which has a Q_4-factorization by Example 2.2.

Lemma 3.2. *There exists a Q_4-factorization of $2K_{8(2k)}$ for $k \geq 1$.*

Proof. When $k = 1$, Example 2.4 gives the required factorization for $2K_{8(2)}$. If k is even, Case 1 gives the result. Let k be odd and $k \geq 3$. Consider Figure 2 representing $2K_{8(2k)}$. The edges in each rectangle form $2K_{8(2)}$ and the edges between any two rectangles form a copy of $2K_{2(16)} - 2F$, where F represents a 2-factor of $2K_{2(16)}$ with 4-cycles as in Example 3.1.

There exists a near-one-factorization of K_k for odd k [4]. Consider a near-one-factor of K_k where $V(K_k) = \{1, 2..., k\}$. For each isolated vertex s of near-one-factor, consider edges of the corresponding rectangle in Figure 2 and for each edge $\{i, j\}$ of near-one-factor, consider the edges between rectangles i and j. By Examples 2.4 and 3.1, $2K_{8(2)}$ and $2K_{2(16)} - 2F$ have Q_4-factorizations, respectively. For each near-one-factor of K_k, the corresponding Q_4-factor of $2K_{8(2k)}$ is obtained. This procedure is repeated for each near-one-factor of K_k and a Q_4-factorization of $2K_{8(2k)}$ is obtained. □

Lemma 3.3. *There exists a Q_4 factorization of $2K_{8t(2k)}$ for $k \geq 1$ and $t \geq 1$.*

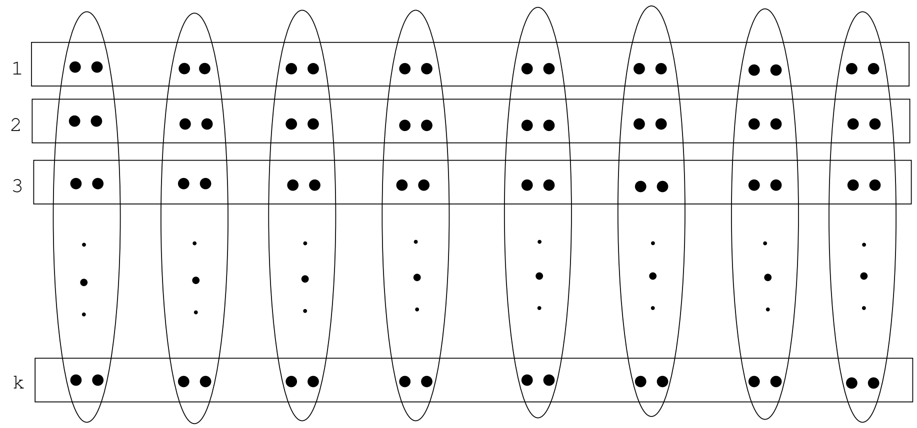

FIGURE 2. $2K_{8(2k)}$

Proof. Let $2K_{8t(2k)}$ be partitioned into t vertex-disjoint subgraphs each isomorphic to $2K_{8(2k)}$. By Lemma 3.2, these subgraphs have Q_4-factorizations. The remaining edges correspond to a copy of $2K_{t(16k)}$ which has a Q_4-factorization by Lemma 2.7. These factors together give the Q_4-factorization of $2K_{8t(2k)}$. □

Case 3: $\lambda \equiv 0 \pmod 4$ By (3.1), if $\lambda \equiv 0 \pmod 4$, necessary conditions reduce to $mx \equiv 0 \pmod{16}$. $4K_{4t(4k)}$, $4K_{2t(8k)}$ and $4K_{t(16k)}$ have Q_4-factorizations by Case 1. $\lambda/4$ copies of the factors of these factorizations give required Q_4-factorizations of $\lambda K_{2t(8k)}$ and $\lambda K_{t(16k)}$. So, two subcases on x will be considered.

Subcase 3.1: $x \equiv 8 \pmod{16}$. For this case, $m \equiv 0 \pmod 2$. $\lambda/2$ copies of the factors of a Q_4-factorization of $2K_{8t(2k)}$ given in Lemma 3.3 give the desired factorization of $\lambda K_{8t(2k)}$.

Subcase 3.2: $x \equiv 0 \pmod{16}$.

For this case, m is arbitrary; so, we look for a Q_4-factorization of $4K_{16t(k)}$. $\lambda/4$ copies of this factorization give a Q_4-factorization of $\lambda K_{16t(k)}$.

Consider the vertex disjoint subgraphs H_i of $4K_{16t(k)}$, where each H_i is isomorphic to $4K_{16(k)}$ for $1 \leq i \leq t$.

To get a Q_4-factorization of H_i, consider a resolvable (K_{16}, K_4)-design. Blow-up each vertex in each of 5 parallel classes by k and assume that each parallel class corresponds to a complete multipartite graph where the parts are the blocks of parallel classes. This graph corresponds to a $K_{4(4k)}$ which has a Q_4-factorization by Case 1. The number of factors in each $K_{4(4k)}$ is $3k$.

Let $\pi_{i,j,l}$ denote the Q_4 factors of H_i for the jth parallel class of $K_{4(4k)}$; $1 \leq i \leq t$, $1 \leq j \leq 5$, $1 \leq l \leq 3k$. Then the followings are the factors of the factorization of H_i's for each i: $\{\pi_{i,j,l},\ 1 \leq j \leq 5,\ 1 \leq l \leq 3k\}$.

The remaining edges of $4K_{16t(k)}$ correspond to a copy of $4K_{t(16k)}$, which has a Q_4-factorization by Lemma 2.7. Combining the factors gives a Q_4-factorization of $4K_{16t(k)}$.

Here, we restate Theorem 1.1 and it is proven by the above cases. Hence, the claim asserted in the introduction part will be completed.

Theorem 1.1. *There exists a Q_4-factorization of $\lambda K_{x(m)}$ if and only if $mx \equiv 0 \pmod{16}$ and $\lambda m(x-1) \equiv 0 \pmod{4}$.*

Proof. The Cases 1, 2, and 3 establish the proof of Theorem 1.1. □

By taking $m = 1$ and $n = x$, we immediately get the result on complete graphs: There exists a Q_4-factorization of λK_n if and only if $n \equiv 0 \pmod{16}$ and $\lambda \equiv 0 \pmod{4}$.

References

1. P. Adams, D. Bryant, and S. El-Zanati, *Lambda-fold cube decompositions.*, Australas. J. Combin. **11** (1995), 197–210.
2. P. Adams, D. Bryant, and B. Maenhaut, *Cube factorizations of complete graphs*, J. Combin. Des. **12** (2004), no. 5, 381–388.
3. D. Bryant, S. El-Zanati, B. Maenhaut, and C. Vanden Eynden, *Decomposition of complete graphs into 5-cubes*, J. Combin. Des. **14** (2006), no. 2, 159–166.
4. C. J. Colbourn, *Crc Handbook of Combinatorial Designs*, CRC Press, 2010.
5. S. El-Zanati, M. Plantholt, and C. Vanden Eynden, *Graph decompositions into generalized cubes*, Ars Combin. **49** (1998), 237–247.
6. S. El-Zanati and C. Vanden Eynden, *Factorizations of complete multipartite graphs into generalized cubes*, J. Graph Theory **33** (2000), no. 3, 144–150.
7. ———, *Decomposing complete graphs into cubes*, Discuss. Math. Graph Theory **26** (2006), no. 1, 141–147.
8. A. Kotzig, *Selected open problems in graph theory*, Graph Theory and Related Topics, Academic Press New York, 1979, pp. 258–267.
9. ———, *Decompositions of complete graphs into isomorphic cubes*, J. Combin. Theory Ser. B **31** (1981), no. 3, 292–296.
10. M. Maheo, *Strongly graceful graphs*, Discrete Math. **29** (1980), no. 1, 39–46.
11. X. Sun and G. Ge, *Resolvable group divisible designs with block size four and general index*, Discrete Math. **309** (2009), no. 10, 2982–2989.
12. J. Wang, *Cube factorizations of complete multipartite graphs*, Ars Combin. **99** (2011), 243–256.
13. R. M. Wilson, *Decompositions of complete graphs into subgraphs isomorphic to a given graph*, (1975).

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