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DIAGONAL RECURRENCE RELATIONS FOR THE STIRLING NUMBERS OF THE FIRST KIND

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Abstract. The paper presents diagonal recurrence relations for the Stirling numbers of the first kind and recovers three explicit formulas for special values of the Bell polynomials of the second kind.

1. Introduction

In combinatorics, the Bell polynomials of the second kind, also known as the partial Bell polynomials, denoted by $\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ for $n \ge k \ge 0$, are defined by

$$\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{1\le i\le n, \ell_i\in\{0\}\cup\mathbb{N} \\ \sum_{i=1}^{n} i\,\ell_i = n \\ \sum_{i=1}^{n} \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{i!}\right)^{\ell_i}; \tag{1.1}$$

see [1, p. 134, Theorem A]. For more information on the Bell polynomials in general and Dyck paths in particular, please see the papers [7, 8] and the references therein.

In mathematics, the Stirling numbers arise in a variety of combinatorics problems and were introduced by James Stirling in the eighteenth century. There are two different kinds of the Stirling numbers. The Stirling numbers of the first kind, $s(n,k)$, which are also called the signed Stirling numbers of the first kind, can be generated by

$$\frac{[\ln(1+x)]^k}{k!} = \sum_{n=k}^{\infty} s(n,k)\frac{x^n}{n!}, \quad |x| < 1. \tag{1.2}$$

The unsigned Stirling numbers of the first kind, $(-1)^{n-k}s(n,k)$, can be interpreted as the number of permutations of $\{1, 2, \dots, n\}$ with k cycles.

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Several "triangular," "horizontal," and "vertical" recurrence relations for the Stirling numbers of the first kind $s(n,k)$ are listed in [1, pp. 214–215, Theorems A, B, and C] as

$$\begin{aligned}
s(n,k) &= s(n-1,k-1)-(n-1)s(n-1,k),\\
(n-k)s(n,k) &= \sum_{k+1\le \ell\le n}(-1)^{\ell-k}\binom{\ell}{k-1}s(n,\ell),\\
s(n,k) &= \sum_{k\le \ell\le n} s(n+1,\ell+1)n^{\ell-k},\\
ks(n,k) &= \sum_{k-1\le \ell\le n-1}(-1)^{n-\ell-1}\binom{n}{\ell}s(\ell,k-1),\\
s(n+1,k+1) &= \sum_{k\le \ell\le n}(-1)^{\ell-1}\prod_{q=1}^{n-\ell}(\ell+q)s(\ell,k),
\end{aligned}$$

where, by convention, the empty product equals 1. Observe that the term $(-1)^{\ell-1}$ in the last recurrence relation was misprinted as $(-1)^{n-1}$ in [1, p. 215, Theorem B].

The aim of this paper is to present diagonal recurrence relations for the Stirling numbers of the first kind $s(n,k)$ based on an integral representation, Faà di Bruno's formula, and properties of the Bell polynomials of the second kind $\mathrm{B}_{n,k}$. Three explicit formulas for special values of the Bell polynomials of the second kind $\mathrm{B}_{n,k}$ are recovered as by-products.

The main results are formulated in the following theorem.

Theorem 1.1. *For $n\ge k\ge 1$, we have*

$$\begin{aligned}
&\mathrm{B}_{n,k}\left(\frac{1!}{2},\frac{2!}{3},\dots,\frac{(n-k+1)!}{n-k+2}\right)\\
&\qquad= (-1)^{n-k}\frac{1}{k!}\sum_{m=1}^{k}(-1)^m\frac{\binom{k}{m}}{\binom{n+m}{n}}s(n+m,m),
\end{aligned}\tag{1.3}$$

$$\begin{aligned}
&\mathrm{B}_{n,k}(0,1!,\dots,(n-k)!)\\
&\qquad= (-1)^{n-k}\binom{n}{k}\sum_{m=0}^{k}(-1)^m\frac{\binom{k}{m}}{\binom{n-m}{n-k}}s(n-m,k-m),
\end{aligned}\tag{1.4}$$

and

$$s(n,k) = (-1)^k\sum_{m=1}^{n}(-1)^m\sum_{\ell=k-m}^{k-1}(-1)^\ell\binom{n}{\ell}\binom{\ell}{k-m}s(n-\ell,k-\ell)\tag{1.5}$$

$$= (-1)^{n-k}\sum_{\ell=0}^{k-1}(-1)^\ell\binom{n}{\ell}\binom{\ell-1}{k-n-1}s(n-\ell,k-\ell),\tag{1.6}$$

where the conventions that $\binom{0}{0}=1$, $\binom{-1}{-1}=1$, *and* $\binom{p}{q}=0$ *for* $p\ge 0>q$ *are adopted in Equation* (1.6)*.*

2. Proof of Theorem 1.1

Recently, three integral representations for the unsigned Stirling numbers of the first kind $(-1)^{n-k}s(n,k)$ were discovered in [10]. The first among them, [10, Theorem 2.1], states that, for $1\le k\le n$,

$$s(n,k)=\binom{n}{k}\lim_{x\to0}\frac{\mathrm{d}^{n-k}}{\mathrm{d}x^{n-k}}\left\{\left[\int_0^\infty\left(\int_{1/e}^1 t^{xu-1}\,\mathrm{d}t\right)e^{-u}\,\mathrm{d}u\right]^k\right\}. \tag{2.1}$$

In combinatorial analysis, Faà di Bruno's formula plays an important role and may be described in terms of the Bell polynomials of the second kind $\mathrm{B}_{n,k}$ by

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}[f\circ h(t)]=\sum_{k=1}^n f^{(k)}(h(t))\mathrm{B}_{n,k}\big(h'(t),h''(t),\dots,h^{(n-k+1)}(t)\big); \tag{2.2}$$

see, for instance, [1, p. 139, Theorem C]. The Bell polynomials of the second kind, $\mathrm{B}_{n,k}$, satisfy

$$\sum_{n=k}^\infty \mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})\frac{t^n}{n!}=\frac{1}{k!}\left(\sum_{m=1}^\infty x_m\frac{t^m}{m!}\right)^k, \tag{2.3}$$

$$\begin{aligned}\mathrm{B}_{n,k}&\big(abx_1,ab^2x_2,\dots,ab^{n-k+1}x_{n-k+1}\big)\\&=a^kb^n\mathrm{B}_{n,k}(x_1,x_n,\dots,x_{n-k+1}),\end{aligned} \tag{2.4}$$

$$\mathrm{B}_{n,k}\left(\frac{x_2}{2},\frac{x_3}{3},\dots,\frac{x_{n-k+2}}{n-k+2}\right)=\frac{n!}{(n+k)!}\mathrm{B}_{n+k,k}(0,x_2,\dots,x_{n+1}), \tag{2.5}$$

where a and b are arbitrary complex numbers; see [1, pp. 133, 135–136] for details.

Let

$$h(x)=\int_0^\infty\left(\int_{1/e}^1 t^{xu-1}\,\mathrm{d}t\right)e^{-u}\,\mathrm{d}u. \tag{2.6}$$

It is clear that, for $\ell\in\mathbb{N}$,

$$\begin{aligned}h^{(\ell)}(x)&=\int_0^\infty\left[\int_{1/e}^1 t^{xu-1}(\ln t)^\ell\,\mathrm{d}t\right]u^\ell e^{-u}\,\mathrm{d}u\\&\to\int_0^\infty\left[\int_{1/e}^1\frac{(\ln t)^\ell}{t}\,\mathrm{d}t\right]u^\ell e^{-u}\,\mathrm{d}u=\frac{(-1)^\ell\ell!}{\ell+1}\end{aligned}$$

as $x\to0$. Setting $f(v)=v^k$ in Equation (2.2) and using the function described in (2.6) to compute (2.1), we obtain

$$s(n,k)=\binom{n}{k}\lim_{x\to0}\sum_{m=1}^{n-k} f^{(m)}(h(x))\mathrm{B}_{n-k,m}\big(h'(x),\dots,h^{(n-k-m+1)}(x)\big) \tag{2.7}$$

$$= \begin{cases} \dbinom{n}{k} \lim_{x\to0} \displaystyle\sum_{m=1}^{k} f^{(m)}(h(x))\mathrm{B}_{n-k,m}\big(h'(x), \dots, h^{(n-k-m+1)}(x)\big), \\ \hfill n > 2k; \\ \dbinom{n}{k} \lim_{x\to0} \displaystyle\sum_{m=1}^{n-k} f^{(m)}(h(x))\mathrm{B}_{n-k,m}\big(h'(x), \dots, h^{(n-k-m+1)}(x)\big), \\ \hfill k \le n \le 2k \end{cases}$$

$$= \begin{cases} \dbinom{n}{k} \lim_{x\to0} \displaystyle\sum_{m=1}^{k} f^{(m)}(h(0))\mathrm{B}_{n-k,m}\big(h'(0), \dots, h^{(n-k-m+1)}(0)\big), \\ \hfill n > 2k; \\ \dbinom{n}{k} \lim_{x\to0} \displaystyle\sum_{m=1}^{n-k} f^{(m)}(h(0))\mathrm{B}_{n-k,m}\big(h'(0), \dots, h^{(n-k-m+1)}(0)\big), \\ \hfill k \le n \le 2k \end{cases}$$

$$= \begin{cases} \dbinom{n}{k} \displaystyle\sum_{m=1}^{k} \frac{k!}{(k-m)!}\mathrm{B}_{n-k,m}\bigg(-\frac{1!}{2}, \dots \\ \qquad \dots, \dfrac{(-1)^{n-k-m+1}(n-k-m+1)!}{n-k-m+2}\bigg), \qquad n > 2k; \\ \dbinom{n}{k} \displaystyle\sum_{m=1}^{n-k} \frac{k!}{(k-m)!}\mathrm{B}_{n-k,m}\bigg(-\frac{1!}{2}, \dots \\ \qquad \dots, \dfrac{(-1)^{n-k-m+1}(n-k-m+1)!}{n-k-m+2}\bigg), \quad k \le n \le 2k \end{cases}$$

Taking $x_m = m!/(m+1)$ in Equation (2.3) and using (1.2) give

$$\begin{aligned} \sum_{n=k}^{\infty} \mathrm{B}_{n,k}\bigg(\frac{1!}{2}, \frac{2!}{3}, \dots, \frac{(n-k+1)!}{n-k+2}\bigg)\frac{t^n}{n!} &= \frac{1}{k!}\Bigg(\sum_{m=1}^{\infty} \frac{t^m}{m+1}\Bigg)^k \\ = \frac{(-1)^k}{k!}\bigg[\frac{\ln(1-t)}{t}+1\bigg]^k &= \frac{(-1)^k}{k!}\sum_{i=0}^{k}\binom{k}{i}\bigg[\frac{\ln(1-t)}{t}\bigg]^i \\ &= (-1)^k\frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i}\frac{i!}{t^i}\sum_{\ell=i}^{\infty}(-1)^\ell s(\ell,i)\frac{t^\ell}{\ell!} \\ &= (-1)^k\sum_{i=0}^{k}\frac{1}{(k-i)!}\sum_{\ell=i}^{\infty}(-1)^\ell s(\ell,i)\frac{t^{\ell-i}}{\ell!}. \end{aligned}$$

This implies that

$$\mathrm{B}_{n,k}\bigg(\frac{1!}{2}, \frac{2!}{3}, \dots, \frac{(n-k+1)!}{n-k+2}\bigg) = n!(-1)^k\sum_{i=0}^{k}\frac{(-1)^{n+i}}{(k-i)!}\frac{s(n+i,i)}{(n+i)!}$$

$$= (-1)^{n-k}\frac{1}{k!}\sum_{i=0}^{k}\frac{\binom{k}{i}}{\binom{n+i}{i}}(-1)^i s(n+i,i),$$

from here Equation (1.3) follows.

Substituting Equation (1.3) into (2.5) leads to

$$\mathrm{B}_{n+k,k}(0,1!,2!\dots,n!) = (-1)^{n-k}\binom{n+k}{k}\sum_{i=0}^{k}(-1)^i\frac{\binom{k}{i}}{\binom{n+i}{i}}s(n+i,i),$$

which may be rewritten as Equation (1.4).

By virtue of Equation (2.4), we have

$$\begin{aligned}\mathrm{B}_{n-k,m}&\left(-\frac{1}{2},\frac{2}{3},\dots,\frac{(-1)^{n-k-m+1}(n-k-m+1)!}{n-k-m+2}\right)\\ &= (-1)^{n-k}\mathrm{B}_{n-k,m}\left(\frac{1!}{2},\frac{2!}{3},\dots,\frac{(n-k-m+1)!}{n-k-m+2}\right).\end{aligned}\tag{2.8}$$

After substituting Equation (1.3) into (2.8), then into (2.7), and finally simplifying, we find that when $2k\ge n\ge k\ge 1$,

$$s(n,k) = \sum_{m=1}^{n-k}\sum_{\ell=1}^{m}(-1)^{m+\ell}\binom{n}{k-\ell}\binom{k-\ell}{m-\ell}s(n-k+\ell,\ell),\tag{2.9}$$

and when $n>2k>0$,

$$s(n,k) = \sum_{m=1}^{k}\sum_{\ell=1}^{m}(-1)^{m+\ell}\binom{n}{k-\ell}\binom{k-\ell}{m-\ell}s(n-k+\ell,\ell).\tag{2.10}$$

Using the convention that $s(n,k)=0$ for $0\le n<k$, we can unify Equations (2.9) and (2.10) into

$$s(n,k) = \sum_{m=1}^{n}\sum_{\ell=1}^{m}(-1)^{m+\ell}\binom{n}{k-\ell}\binom{k-\ell}{k-m}s(n-k+\ell,\ell),\tag{2.11}$$

which can be further formulated as Equation (1.5).

Interchanging the two sums in (1.5) and computing the inner sum yields

$$\begin{aligned}s(n,k) &= (-1)^k\sum_{\ell=k-n}^{k-1}(-1)^\ell\binom{n}{\ell}\left[\sum_{m=k-\ell}^{n}(-1)^m\binom{\ell}{k-m}\right]s(n-\ell,k-\ell)\\ &= (-1)^{n-k}\sum_{\ell=k-n}^{k-1}(-1)^\ell\binom{n}{\ell}\binom{\ell-1}{k-n-1}s(n-\ell,k-\ell),\end{aligned}$$

which can be rewritten as Equation (1.6). The proof of Theorem 1.1 is complete.

3. Remarks

Remark 3.1. The recurrence relations (1.5) and (1.6) are neither "triangular", nor "vertical", nor "horizontal" recurrence relations as listed in [1, pp. 214–215, Theorems A, B, and C], so we call them "diagonal" recurrence relations for the Stirling numbers of the first kind $s(n,k)$.

Remark 3.2. Equation (1.5) is also true if we change the sum over m from 1 to k instead of from 1 to n.

Remark 3.3. From [11, Corollary 2.3], we may compute the Stirling numbers of the first kind, $s(n,k)$, for $2 \le k \le n$ by the equation

$$s(n,k) = (-1)^{n-k}(n-1)! \sum_{\ell_1=k-1}^{n-1} \frac{1}{\ell_1} \sum_{\ell_2=k-2}^{\ell_1-1} \frac{1}{\ell_2} \cdots \sum_{\ell_{k-2}=2}^{\ell_{k-3}-1} \frac{1}{\ell_{k-2}} \sum_{\ell_{k-1}=1}^{\ell_{k-2}-1} \frac{1}{\ell_{k-1}}.$$

This equation may be reformulated as

$$(-1)^{n-k}\frac{s(n,k)}{(n-1)!} = \sum_{m=k-1}^{n-1} \frac{1}{m}\left[(-1)^{m-(k-1)}\frac{s(m,k-1)}{(m-1)!}\right].$$

Remark 3.4. By applying the integral representation described in Equation (2.1), some properties for the Stirling numbers of the first kind $s(n,k)$, including the logarithmic convexity with respect to $n \ge 0$ of the sequence

$$\left\{\frac{|s(n+k,k)|}{\binom{n+k}{k}}\right\}_{n\ge 0}$$

for any fixed $k \in \mathbb{N}$ (see [10, Corollary 5.1]), were established in [10, Section 5].

Remark 3.5. It is well known in combinatorics that

$$\mathrm{B}_{n,k}(1!, 2!, \dots, (n-k+1)!) = \binom{n}{k}\binom{n-1}{k-1}(n-k)! \tag{3.1}$$

for $n \ge k \ge 1$; see [1, p. 135, Theorem B]. We now recover this identity in an alternative way.

In [14, Theorems 2.1 and 2.2], it was inductively obtained, for $i \in \mathbb{N}$ and $t \ne 0$,

$$\frac{\mathrm{d}^i e^{1/t}}{\mathrm{d}t^i} = (-1)^i e^{1/t}\frac{1}{t^{2i}}\sum_{k=0}^{i-1} a_{i,k}t^k \tag{3.2}$$

and

$$\frac{\mathrm{d}^i e^{-1/t}}{\mathrm{d}t^i} = \frac{e^{-1/t}}{t^{2i}}\sum_{k=0}^{i-1}(-1)^k a_{i,k}t^k, \tag{3.3}$$

where

$$a_{i,k} = \binom{i}{k}\binom{i-1}{k}k!$$

for all $0 \le k \le i-1$ and $a_{n,n-k}$ are the Lah numbers $L(n,k)$; see also [13, Equations (1.3) and (1.4)]. For more information on the Lah numbers $L(n,k)$, please refer to the recent references [2, 5, 6] and related references therein.

From Equations (2.2) and (2.4), it follows that, for $i \in \mathbb{N}$ and $t \ne 0$,

$$
\begin{aligned}
\frac{\mathrm{d}^i e^{1/t}}{\mathrm{d} t^i} &= e^{1/t} \sum_{k=1}^{i} \mathrm{B}_{i,k}\left(-\frac{1!}{t^2}, \frac{2!}{t^3}, \dots, (-1)^{i-k+1}\frac{(i-k+1)!}{t^{i-k+2}}\right) \\
&= (-1)^i e^{1/t} \sum_{k=1}^{i} \frac{1}{t^{i+k}} \mathrm{B}_{i,k}(1!, 2!, \dots, (i-k+1)!)
\end{aligned}
\tag{3.4}
$$

and

$$
\begin{aligned}
\frac{\mathrm{d}^i e^{-1/t}}{\mathrm{d} t^i} &= e^{-1/t} \sum_{k=1}^{i} \mathrm{B}_{i,k}\left(\frac{1!}{t^2}, -\frac{2!}{t^3}, \dots, (-1)^{i-k}\frac{(i-k+1)!}{t^{i-k+2}}\right) \\
&= e^{-1/t} \sum_{k=1}^{i} (-1)^k \mathrm{B}_{i,k}\left(-\frac{1!}{t^2}, \frac{2!}{t^3}, \dots, (-1)^{i-k+1}\frac{(i-k+1)!}{t^{i-k+2}}\right) \\
&= e^{-1/t} \sum_{k=1}^{i} \frac{(-1)^{i+k}}{t^{i+k}} \mathrm{B}_{i,k}(1!, 2!, \dots, (i-k+1)!).
\end{aligned}
$$

Combining Equation (3.2) with (3.4) and Equation (3.3) with the above equation, respectively, shows

$$
(-1)^i \frac{1}{t^{2i}} \sum_{k=0}^{i-1} a_{i,k} t^k = (-1)^i \sum_{k=1}^{i} \frac{1}{t^{i+k}} \mathrm{B}_{i,k}(1!, 2!, \dots, (i-k+1)!)
$$

and

$$
\frac{1}{t^{2i}} \sum_{k=0}^{i-1} (-1)^k a_{i,k} t^k = \sum_{k=1}^{i} \frac{(-1)^{i+k}}{t^{i+k}} \mathrm{B}_{i,k}(1!, 2!, \dots, (i-k+1)!).
$$

As a result,

$$
\sum_{k=1}^{i} a_{i,i-k} t^k = \sum_{k=1}^{i} \mathrm{B}_{i,k}(1!, 2!, \dots, (i-k+1)!) t^k,
$$

which implies

$$
\begin{aligned}
\mathrm{B}_{n,k}(1!, 2!, \dots, (n-k+1)!) &= a_{n,n-k} \\
&= (n-k)! \binom{n}{n-k} \binom{n-1}{n-k} = (n-k)! \binom{n}{k} \binom{n-1}{k-1}.
\end{aligned}
$$

Thus we have reestablished the identity described in Equation (3.1).

Remark 3.6. We can find Equation (1.3) in [15].

Remark 3.7. In [3, 4, 12, 15] and the related references therein, several special values of the Bell polynomials of the second kind $\mathrm{B}_{n,k}$ are discovered, collected, and applied.

Remark 3.8. This paper is a revised version of the preprint [9].

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