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THE NON-EXISTENCE OF DISTANCE-2 OVOIDS IN $\mathsf{H}(4)^D$

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Abstract. We give a computer-assisted proof for the non-existence of distance-2 ovoids in the dual split Cayley hexagon $\mathsf{H}(4)^D$. Furthermore, we obtain bounds on the size of a partial distance-2 ovoid in $\mathsf{H}(4)^D$.

1. Introduction

The study of distance-j ovoids in generalized polygons was initiated by Thas, who investigated the existence of distance-2 ovoids in generalized quadrangles and distance-3 ovoids in generalized hexagons (which are simply known as ovoids) [10]. The existence of distance-j ovoids is related to the existence of particular perfect codes, cores of symmetric graphs, and various other topics.

In this work, we show the non-existence of distance-2 ovoids in the *dual split Cayley hexagon* $\mathsf{H}(4)^D$. For the *split Cayley hexagon* $\mathsf{H}(q)$ itself, the existence of distance-2 ovoids is already known for $q = 2, 3, 4$ [12, 13, 14] and the ovoids are completely classified [7, Sec. 18.3]. For $\mathsf{H}(q)^D$, only the non-existence for $q = 2$ [3] and the existence for $q = 3$ [12] is known. Note that we have $\mathsf{H}(q)$ isomorphic to $\mathsf{H}(q)^D$ if and only if q is a power of 3 [11, Cor. 3.5.7]. Here we present a computer-assisted proof for the next open case, $\mathsf{H}(4)^D$.

Theorem 1.1. *The dual split Cayley hexagon $\mathsf{H}(q)^D$ does not possess a distance-2 ovoid for $q \in \{2, 4\}$.*

We note that Theorem 1.1 has been used in [1] to prove that there does not exist any semi-finite generalized hexagon containing $\mathsf{H}(q)^D$ as a full subgeometry for $q \in \{2, 4\}$, thus answering specific cases of a version of the question raised by Tits on existence of semi-finite generalized polygons. In fact, non-existence of distance-2 ovoids in any given finite generalized hexagon implies that every generalized hexagon containing the hexagon as a full subgeometry must be finite [1, Cor. 3.9]. We believe that the computational techniques that we describe below might be useful in "small" cases of other finite geometrical problems that have a similar flavour, and where brute force or direct symmetry breaking does not work.

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2. Proof of the theorem

We refer to the standard text [11] for an introduction to generalized polygons. The number of points in a generalized hexagon of order (s,t) is $(1+s)(1+st+s^2t^2)$. We denote the *Desarguesian projective plane* over $\mathbb{F}_q$ by $\mathrm{PG}(2,q)$. Then $\mathsf{H}(q,1)$ denotes the generalized hexagon of order $(q,1)$ whose points are the incident point-line pairs of $\mathrm{PG}(2,q)$, lines are the points and lines of $\mathrm{PG}(2,q)$, and incidence is reverse containment. The *dual split Cayley hexagon* $\mathsf{H}(q)^D$ is a generalized hexagon of order (q,q) (see [11] for a definition). The following is a well known result on the relationship between these generalized hexagons.

Lemma 2.1. *The dual split Cayley hexagon $\mathsf{H}(q)^D$ contains a subhexagon $\mathcal{H}$ of order $(q,1)$ isomorphic to $\mathsf{H}(q,1)$. Moreover, for every pair of lines $\ell_1,\ell_2 \in \mathsf{H}(q)^D$ that are at distance 6 from each other in the incidence graph, there is a unique $\mathsf{H}(q,1)$-subhexagon of $\mathsf{H}(q)^D$ containing both ℓ_1 and ℓ_2.*

Definition 2.2. *Let $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$ be a generalized polygon.*

(*a*) *A* partial distance-2 ovoid *of $\mathcal{S}$ is a set of points $\mathcal{O}$ such that every line contains at most one point of $\mathcal{S}$.*

(*b*) *A* distance-2 ovoid *of $\mathcal{S}$ is a set of points $\mathcal{O}$ such that every line contains exactly one point of $\mathcal{S}$.*

From double counting, it follows that the size of a distance-2 ovoid in a generalized hexagon of order (s,t) is equal to s^2t^2+st+1. The following Lemma follows directly from the definition of a distance-2 ovoid.

Lemma 2.3. *Let $\mathcal{H}$ be a generalized hexagon of order (s,t). Let $\mathcal{H}'$ be a subhexagon of order (s,t') of $\mathcal{H}$ and let $\mathcal{O}$ be a distance-2 ovoid of $\mathcal{H}$. Then $\mathcal{H}'\cap\mathcal{O}$ is a distance-2 ovoid of $\mathcal{H}'$ and*

$$|\mathcal{O}\cap\mathcal{H}'| = s^2t'^2+st'+1.$$

In our case, we have $\mathsf{H}(4,1)$ contained in $\mathsf{H}(4)^D$, and thus $s=t=4$ and $t'=1$. Therefore, we want to determine if we can find a subset of size 273 which intersects each of the 1365 lines in a unique point from the 1365 points of $\mathsf{H}(4)^D$. While –as far as we know– it is not possible to solve this problem by brute force or by standard symmetry breaking arguments (see for example [7]) in any reasonable amount of time; in view of Lemmas 2.1 and 2.3, we can use the following idea:

> First classify all distance-2 ovoids in $\mathsf{H}(4,1)$ up to isomorphism under the action of the stabilizer of $\mathsf{H}(4,1)$. Next, see if any of these distance-2 ovoids can be extended to a distance-2 ovoid of $\mathsf{H}(4)^D$.

We note that the stabilizer of a subgeometry of $\mathsf{H}(4)^D$ which is isomorphic to $\mathsf{H}(4,1)$, under the action of the automorphism group of $\mathsf{H}(4)^D$ is in fact isomorphic to the automorphism group of $\mathsf{H}(4,1)$. As the point graph of $\mathsf{H}(q,1)^D$ corresponds to the incidence graph of the projective plane $\mathrm{PG}(2,q)$, a distance-2 ovoid in $\mathsf{H}(q,1)$ corresponds to a perfect matching of the incidence graph of $\mathrm{PG}(2,q)$. It is folklore that the number of perfect matchings in a balanced bipartite graph corresponds to the permanent of the biadjacency matrix of that graph. It is easy to verify the following by calculating the corresponding permanent.

Lemma 2.4 ([8, A000794]). *The number of perfect matchings in the incidence graph of* $\mathrm{PG}(2,4)$ *is* 18534400.

Notice that a perfect matching is an exact cover, and so we can use Knuth's Dancing Links Algorithm [5] to enumerate all perfect matchings in a bipartite graph.

Proposition 2.5. *Let G be the automorphism group of $\mathsf{H}(4)^D$. Let $\mathcal{H}$ be a subhexagon of $\mathsf{H}(4)^D$ ismormorphic to $\mathsf{H}(4,1)$. Then there are exactly* 350 *non-isomorphic distance-*2 *ovoids in $\mathcal{H}$ with respect to $G_{\mathcal{H}}$, the stabilizer of $\mathcal{H}$ under the action of G.*

We used a computer to prove Proposition 2.5. The following algorithm was able to classify all 350 distance-2 ovoids in a few minutes at the time of writing.[1] We rely on Linton's algorithm `SmallestImageSet(H, S)`, which returns the lexicographically smallest element in the orbit of a set `S` under the action of a group `H` [6]. We used the implementation of Dancing Links in SAGE [2] to iterate through all perfect matchings and the implementation of `SmallestImageSet` in the GRAPE [9] package of GAP [4] to find the representatives of these 350 isomorphism classes of distance-2 ovoids.[2] For each distance-2 ovoid $\mathcal{O}'$ of $\mathcal{H}$, we can define an integer linear optimization problem (ILP) which is feasible if and only if $\mathsf{H}(4)^D$ has a distance-2 ovoid containing $\mathcal{O}'$. Then the ILP solvers easily shows that these equations are infeasible for all of the 350 cases.[3] This proves Theorem 1.1.

Remark. For the next open case, $\mathsf{H}(5)^D$, our algorithmic approach fails for several reasons:

(a) The incidence graph of $\mathrm{PG}(2,5)$ has 4598378639550 perfect matchings while the automorphism group of $\mathrm{PG}(2,5)$ has size 744000. So a classification of all non-isomorphic distance-2 ovoids of $\mathsf{H}(5,1)$ seems to be out of reach.

(b) Even for one given distance-2 ovoid of $\mathsf{H}(5,1)$, the corresponding integer linear program takes too long to solve with a state-of-the-art ILP solver.

One can use the same methods to obtain bounds on partial distance-2 ovoids in $\mathsf{H}(q)^D$.

[1] Running time: 28m37.576s with Sage Version 6.4.1 with a Intel Core i5-2400 CPU @ 3.10 GHz processor.

[2] We provide our full code online: `http://math.ihringer.org/data.php`

[3] We verified this with CPLEX (several versions), Gurobi Optimizer (several versions) and the constraint solver Minion. The 350 ILPs in 350 files in the LP format took 540.3 seconds with Gurobi Optimizer version 6.5.0 build v6.5.0rc1 (linux64) with an Intel Core i5-3550 CPU @ 3.30GHz processor.

Lemma 2.6. *Let $\mathcal{O}$ be a partial distance-2 ovoid of $\mathsf{H}(q)^D$. Suppose that no subhexagon $\mathcal{H}$ of $\mathsf{H}(q)^D$ isomorphic to $\mathsf{H}(q,1)$ contains q^2+q+1 points of $\mathcal{O}$. Then $|\mathcal{O}| \leq (q^2-q+1)(q^2+q)$.*

Proof. Let $\mathcal{P}$ be the set of points of $\mathsf{H}(q)^D$. We double count pairs $(p, \mathcal{H})$, where $\mathcal{H}$ a subhexagon of $\mathsf{H}(q)^D$ isomorphic to $\mathsf{H}(q,1)$ and $p \in \mathcal{O} \cap \mathcal{H}$. From Lemma 2.1 we can see that each point is contained in $(1+q)q^3/2$ subhexagons isomorphic to $\mathsf{H}(q,1)$ which tells us that there are $|\mathcal{O}|(1+q)q^3/2$ such pairs. Again by Lemma 2.1, there are $q^3(1+q)(q^2-q+1)/2$ subhexagons of $\mathsf{H}(q)^D$ which are isomorphic to $\mathsf{H}(q,1)$. Under the condition $|\mathcal{O} \cap \mathcal{H}| \leq q^2+q$, this yields $|\mathcal{O}| \leq (q^2-q+1) \cdot (q^2+q)$. □

To prove bounds *for all* partial distance-2 ovoids of $\mathsf{H}(q)^D$, $q \in \{2,4\}$, we can use the following computational approach. If the corresponding ILP does not have a solution larger than some integer $b > (q^2-q+1)(q^2+q)$ for all of the non-isomorphic distance-2 ovoids of $\mathsf{H}(q,1)$, then by Lemma 2.6 we obtain b as an upper bound on the size of a partial distance-2 ovoids. Using this approach we have proved the following.

Theorem 2.7. *A maximum partial distance-*2 *ovoid $\mathcal{O}$ of $\mathsf{H}(q)^D$ satisfies the following:*
(a) $|\mathcal{O}| = 19$ *for* $q = 2$*,*
(b) $261 \leq |\mathcal{O}| \leq 265$ *for* $q = 4$*.*

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