Contributions to Discrete Mathematics

Volume 13, Number 1, Pages 136–149
ISSN 1715-0868

ON COMBINATORICS OF MODIFIED LATTICE PATHS AND GENERALIZED q-SERIES

M. RANA AND MEGHA GOYAL

Abstract. Recently, Agarwal and Sachdeva, 2017, proved two Rogers–Ramanujan type identities for modified lattice paths by establishing a bijection between split $(n+t)$-color partitions and the modified lattice paths. In this paper, we interpret four generalized basic series combinatorially in terms of modified lattice paths by using a similar bijection. This leads to four new Rogers–Ramanujan type identities for modified lattice paths.

1. Introduction

The theory of lattice paths and the theory of partitions are closely related. A detailed history and survey of lattice path enumerations for the last 35 years is given in [19]. MacMahon [20] used two dimensional rectangular graphs to study 'Composition of Numbers'. These rectangular graphs are called 'reticulation' and the paths are known as 'line of route'. MacMahon then studied directions of these graphs which we today call 'directed paths'. In [21] MacMahon used the word 'lattice' and showed how paths are used to study permutations, combinations, the theory of partitions and the theory of probabilities. Interestingly, he also established a bijection between Ferrers graph and lattice paths in [21] and used these objects to address the Ballot Theorem [1, 12, 14]. MacMahon further explored lattice paths and the theory of partitions in [22]. Agarwal and Andrews [4] studied $n(y,x)$-reflected lattice paths and succeeded in relating these paths with self conjugate partitions by proving that the number of $n(y,x)$-reflected lattice paths equals the number of self conjugate partitions with the largest part at most n. Agarwal [2] proved that the number of $2n(x,y)$-reflected lattice paths equals the number of partitions of $2n^2$ into at most $2n$ parts, each at most $2n$ and the parts which are strictly less than $2n$ can be paired such that the sum of each pair is $2n$. Agarwal and Bressoud [7] introduced a new class of weighted lattice paths with three steps northeast ($\nearrow$): from (i,j) to $(i+1,j+1)$,

Received by the editors March 28, 2017, and in revised form June 21, 2017.
2000 *Mathematics Subject Classification.* 05A15, 05A17, 11P81.
Key words and phrases. q-series, split $(n+t)$-color partitions, combinatorial identities, weighted lattice paths, modified lattice paths.

southeast ($\searrow$): from (i, j) to $(i+1, j-1)$, only allowed if $j > 0$ and horizontal ($\rightarrow$): from $(i, 0)$ to $(i+1, 0)$, only allowed along the x-axis. They used these paths in interpreting the multiple basic series identities found by Agarwal et al. [6]. Several elegant results on lattice paths and basic series can be found in [3, 8, 9, 16, 17, 18, 25]. In 2014, Agarwal and Sood [11] introduced and defined split $(n+t)$-color partitions and gave the combinatorial interpretations of two basic functions of Gordon–McIntosh from [15] which are given below:

$$V_0(q) = 1 + 2\sum_{n=1}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_n}, \tag{1.1}$$

$$V_1(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}(-q;q^2)_{n-1}}{(q;q^2)_n}. \tag{1.2}$$

Agarwal and Sood posed an open problem: 'Is it possible to find Rogers–Ramanujan type identities for split $(n+t)$-color partitions?' This problem was addressed by Sood and Agarwal [26]. In [26] three generalized basic series were interpreted combinatorially in terms of split $(n+t)$-color partitions and three Rogers–Ramanujan type identities for split $(n+t)$-color partitions were obtained as particular cases. Rana et al. [23] further used split $(n+t)$-color partitions to interpret the following four generalized q-series combinatorially.

Let $S = \{-1, 1, 3, 5, \ldots\}$, for $|q| < 1$, $j \in S$ and $1 \le i \le 4$, define $g_i^j(q)$ by

$$g_1^j(q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n[1+(n-1)(j+3)/2]}}{(q^4;q^4)_n (q;q^2)_n}, \tag{1.3}$$

$$g_2^j(q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n(n+1)(j+3)/2}}{(q^4;q^4)_n (q;q^2)_{n+1}}, \tag{1.4}$$

$$g_3^j(q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n[1+(n+1)(j+3)/2]}}{(q^4;q^4)_n (q;q^2)_{n+1}}, \tag{1.5}$$

$$g_4^j(q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n[1+(n+1)(j+3)/2]}}{(q^4;q^4)_n (q;q^2)_n}. \tag{1.6}$$

These generalized four q-series yield the following four Rogers–Ramanujan type identities.

$$\sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n^2}}{(q^4;q^4)_n (-q;q^2)_n} = \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} [q^5, -q^2, -q^3; q^5]_\infty, \tag{1.7}$$

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{2n(n+1)}}{(q^4;q^4)_n (q;q^2)_{n+1}} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} [q^{12}, q^3, q^9; q^{12}]_\infty, \tag{1.8}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n(n+2)}}{(q^4;q^4)_n (-q;q^2)_{n+1}} = \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} [q^5, -q^5, -q^5; q^5]_\infty, \tag{1.9}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n(n+2)}}{(q^4;q^4)_n(-q;q^2)_n} = \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}[q^5,-q,-q^4;q^5]_\infty. \tag{1.10}$$

Agarwal and Sachdeva in [24] studied a new class of lattice paths which they called 'modified lattice paths' for enumerating (1.1)–(1.2) in terms of modified lattice paths and further successfully interpreted Rogers–Ramanujan type identities in [10] using modified lattice paths. Here, in this paper, we provide the combinatorial interpretations of four generalized q-series (1.3)–(1.6) using the modified lattice paths and then further establish bijections between certain restricted classes of modified lattice paths and split $(n+t)$-color partitions. We also succeeded in obtaining Rogers–Ramanujan type identities for modified lattice paths. Before stating our main results let us recall some definitions.

Definition 1.1 ([13]). *A partition of a positive integer ν is a finite non-increasing sequence of positive integers $\lambda_1, \lambda_2, \ldots \lambda_r$ such that $\sum_{i=1}^{r} \lambda_i = \nu$, where the λ_is are called* summands of the partition. *The number of partitions of ν is denoted by $p(\nu)$.*

Definition 1.2 ([5]). *An $(n+t)$*-color partition *is a partition in which a summand of size n can come in $(n+t)$ different colors denoted by the subscripts, $n_1, n_2, n_3, \cdots, n_{n+t}$. Note that zeros are permitted, without repetition, if and only if $t \geq 1$.*

Definition 1.3. *The* weighted difference *of two summands $m_i, n_j, m \geq n$ is defined by $m-n-i-j$ and is denoted by $((m_i - n_j))$.*

Definition 1.4 ([11]). *Let m_i be a summand in an $(n+t)$-color partition of a nonnegative integer ν. Now split the color 'i' into two parts–'the green part' and 'the red part' and denote them by 'g' and 'r' respectively, such that $1 \leq g \leq i$, $0 \leq r \leq i-1$ and $i = g + r$. An $(n+t)$-color partition in which each summand is split in this manner is called a* split $(n+t)$-color partition.

Remark. In a split $(n+t)$-color partition, whenever the red part is 0, then it will not be written separately. That is, 2_{g+0} is written as 2_g only.

Example 1.5. *The split $(n+1)$-color partitions of 2 are:*

2_1	2_1+0_1	1_1+1_1	$1_1+1_1+0_1$
2_2	2_2+0_1	1_2+1_1	$1_2+1_1+0_1$
2_3	2_3+0_1	1_2+1_2	$1_2+1_2+0_1$
2_{1+1}	$2_{1+1}+0_1$	$1_{1+1}+1_1$	$1_{1+1}+1_1+0_1$
2_{2+1}	$2_{2+1}+0_1$	$1_{1+1}+1_2$	$1_{1+1}+1_2+0_1$
2_{1+2}	$2_{1+2}+0_1$	$1_{1+1}+1_{1+1}$	$1_{1+1}+1_{1+1}+0_1$

Agarwal and Sachdeva [10, 24] gave the following description of *modified lattice paths*.

Definition 1.6. *All paths will be of finite length lying in the first quadrant. They will begin on the y-axis and terminate on the x-axis. Only three moves are allowed at each step:*

Northeast ($\nearrow$)*: from* (i, j) *to* $(i + 1, j + 1)$.
Southeast ($\searrow$)*: from* (i, j) *to* $(i + 1, j - 1)$*, only allowed if* $j > 0$.
Horizontal ($\rightarrow$)*: from* $(i, 0)$ *to* $(i + 1, 0)$*, only allowed along x-axis.*

In describing modified lattice paths, the following terminology is used:

Peak*: Either a vertex on the y-axis which is followed by a southeast step or a vertex preceded by a northeast step and followed by a southeast step.*
Valley*: A vertex preceded by a southeast step and followed by a northeast step. Note that a southeast step followed by a horizontal step followed by a northeast step does not constitute a valley.*
Mountain*: A section of the path which starts on either the x-axis or y-axis, which ends on the x-axis and which does not touch the x-axis anywhere in between the end points. Every mountain has at least one peak and may have more than one.*
Plain*: A section of the path consisting of only horizontal steps which starts either on the y-axis or at a vertex preceded by a southeast step and ends at a vertex followed by a northeast step.*
Weight of a vertex*: The x-coordinate.*
Weight of a path*: The sum of the weights of its peaks.*
Height of a vertex*: The y-coordinate. Now, the height 'h' of each peak is divided into two parts–the lower part will be called a* pillar *and the upper part a* beam *and their heights are denoted by 'p' and 'b', respectively, such that* $1 \leq p \leq h$, $0 \leq b \leq h - 1$ *and* $h = p + b$. *A pillar will be represented by a 'dark line' and a* beam *by a 'light line'.*

Example 1.7. *The following path has three peaks, one valley, three mountains, and one plain.*

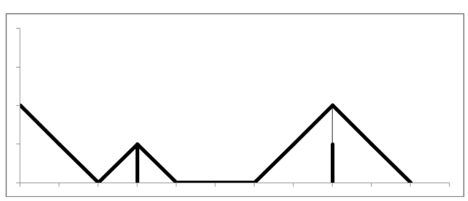


FIGURE 1. A modified lattice path.

Remark. If in the above defined lattice path, the height h is not divided into pillars and beams then the above definition reduces to the definition of the weighted lattice paths as introduced and studied in [7].

We now recall the combinatorial interpretations of (1.3)–(1.6) in terms of split $(n + t)$-color partitions. In Section 2 we give our main results and then in the next Section 3 we establish bijections between certain classes of split $(n+t)$-color partitions and modified lattice paths. As particular cases, in Section 4, we obtain combinatorial interpretations of Rogers–Ramanujan type identities (1.7)–(1.10) in terms of modified lattice paths.

In [23] Rana et al. gave the following interpretations of (1.3)–(1.6):

Theorem 1.8. *For $j \in S$, let $P_1^j(\nu)$ represent the number of split n-color partitions of ν such that*

(i) *the parts and their subscripts have the same parity;*
(ii) *the value of the red part can be 0 or 1;*
(iii) *if m_i is the least or only summand of the partition, then $m - i \equiv 0 \pmod 4$;*
(iv) *the weighted difference among any two consecutive summands is greater than j and is congruent to $(j+1) \pmod 4$.*

Then

$$\sum_{\nu=0}^{\infty} P_1^j(\nu) q^\nu = g_1^j(q).$$

Theorem 1.9. *For $j \in S$, let $P_2^j(\nu)$ represent the number of split $(n+1)$-color partitions of ν such that*

(i) *the parts and their subscripts have the opposite parity;*
(ii) *the value of the red part can be 0 or 1;*
(iii) *the smallest summand is of the form i_{i+1};*
(iv) *the weighted difference among any two consecutive summands is greater than j and is congruent to $(j+1) \pmod 4$.*

Then

$$\sum_{\nu=0}^{\infty} P_2^j(\nu) q^\nu = g_2^j(q).$$

Theorem 1.10. *For $j \in S$, let $P_3^j(\nu)$ represent the number of split $(n+2)$-color partitions of ν such that*

(i) *the parts and their subscripts have the same parity;*
(ii) *the value of the red part can be 0 or 1;*
(iii) *the smallest summand is of the form i_{i+2};*
(iv) *the weighted difference among any two consecutive summands is greater than j and is congruent to $(j+1) \pmod 4$.*

Then

$$\sum_{\nu=0}^{\infty} P_3^j(\nu) q^\nu = g_3^j(q).$$

Theorem 1.11. *For $j \in S$, let $P_4^j(\nu)$ represent the number of split n-color partitions of ν such that*

(i) *the parts and their subscripts have the same parity;*
(ii) *the value of the red part can be 0 or 1;*

(iii) *if* m_i *is the least or only summand of the partition, then* $m \geq (j+4)$ *and* $m - i \equiv (j+3) \pmod 4$*;*
(iv) *the weighted difference among any two consecutive summands is greater than* j *and is congruent to* $(j+1) \pmod 4$*.*

Then

$$\sum_{\nu=0}^{\infty} P_4^j(\nu) q^\nu = g_4^j(q).$$

Remark. The conditions (i), (iii), and (iv) in Theorems 1.8–1.11 are allowed for the whole subscript 'i' irrespective of green and red parts separately.

2. Main Results

Theorem 2.1. *For* $j \in S$*, let* $Q_1^j(\nu)$ *denote the number of modified lattice paths of weight* ν *which starts at* $(0,0)$*, such that*

(i) *they have no valley above height 0 if* $j = -1$ *and no valley at all if* $j > -1$*;*
(ii) *if there is a plain in the beginning of the path, then its length is congruent to* $0 \pmod 4$ *and the lengths of other plains are congruent to* $(j+1) \pmod 4$*;*
(iii) *there is no beam with height greater than* 1*.*

Then

$$P_1^j(\nu) = Q_1^j(\nu), \text{ for all } \nu.$$

Theorem 2.2. *For* $j \in S$*, let* $Q_2^j(\nu)$ *denote the number of modified lattice paths of weight* ν *which starts at* $(0,1)$*, such that*

(i) *they have no valley above height 0 if* $j = -1$ *and no valley at all if* $j > -1$*;*
(ii) *if there is a plain in the beginning of the path, then its length is congruent to* $0 \pmod 4$ *and the lengths of other plains are congruent to* $(j+1) \pmod 4$*;*
(iii) *there is no beam with height greater than* 1*.*

Then

$$P_2^j(\nu) = Q_2^j(\nu), \text{ for all } \nu.$$

Theorem 2.3. *For* $j \in S$*, let* $Q_3^j(\nu)$ *denote the number of modified lattice paths of weight* ν *which starts at* $(0,2)$*, such that*

(i) *they have no valley above height 0 if* $j = -1$ *and no valley at all if* $j > -1$*;*
(ii) *if there is a plain in the beginning of the path, then its length is congruent to* $0 \pmod 4$ *and the lengths of other plains are congruent to* $(j+1) \pmod 4$*;*
(iii) *there is no beam with height greater than* 1*.*

Then

$$P_3^j(\nu) = Q_3^j(\nu), \text{ for all } \nu.$$

Theorem 2.4. *For $j \in S$, let $Q_4^j(\nu)$ denote the number of modified lattice paths of weight ν which starts at $(0,0)$, such that*

(i) *they have no valley above height 0 if $j = -1$ and no valley at all if $j > -1$;*
(ii) *there is a plain of length $(j+3)$ in the beginning of the path and the lengths of other plains are congruent to $(j+1) \pmod 4$;*
(iii) *there is no beam with height greater than 1;*
(iv) *the weight of the first peak is greater than or equal to $(j+4)$.*

Then

$$P_4^j(\nu) = Q_4^j(\nu), \text{ for all } \nu.$$

2.1. Proof of Theorem 2.1.

Proof. We shall prove that

$$\sum_{\nu=0}^{\infty} Q_1^j(\nu) q^\nu = \sum_{m=0}^{\infty} \frac{q^{m[1+(m-1)(j+3)/2]}(-q;q^2)_m}{(q^4;q^4)_m (q;q^2)_m}. \tag{2.1}$$

In

$$\frac{q^{m[1+(m-1)(j+3)/2]}(-q;q^2)_m}{(q^4;q^4)_m (q;q^2)_m}$$

the factor $q^{m[1+(m-1)(j+3)/2]}$ generates modified lattice paths having m peaks starting at $(0,0)$ and terminating at $(2+(j+3)(m-1),0)$, a plain of length $(j+1)$ between every two consecutive mountains, and each peak is supported by a pillar of height 1.

For $m = 4$, the path begins as

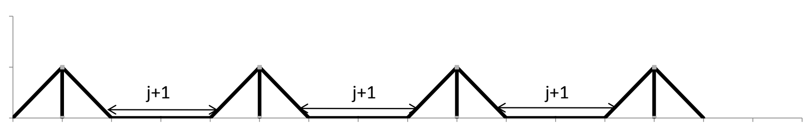


FIGURE 2. Modified lattice path when $m = 4$.

In the above figure we consider two successive peaks, say, ith and $(i+1)$th and denote them by P_i and P_{i+1} respectively.

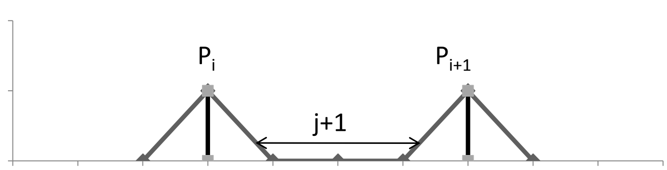


FIGURE 3. ith and $(i+1)$th peaks

The factor $1/(q^4;q^4)_m$ generates m nonnegative multiples of 4, say $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m \geq 0$, which are encoded by inserting α_m horizontal steps in front of the first mountain and $\alpha_i - \alpha_{i+1}$ horizontal steps in front of the $(m-i+1)$th mountain, $1 \leq i \leq m$. Figure 3 now becomes Figure 4:

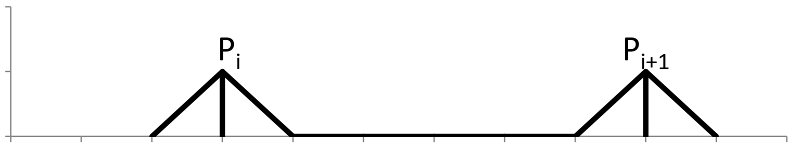
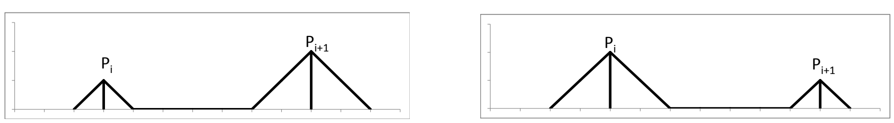


FIGURE 4. ith and $(i+1)$th peaks

The factor $1/(q;q^2)_m$ generates m nonnegative multiples of $(2i-1)$, $1 \leq i \leq m$, say, $p_1 \times 1, p_2 \times 3, \cdots, p_m \times (2m-1)$. This is encoded by increasing the height of the ith pillar to height $p_{m-i+1}+1$. Each increase by one in the height of a given peak increases its weight by one and the weight of each subsequent peak by two.

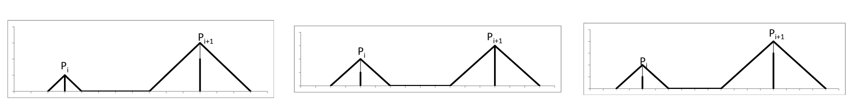
Figure 4 becomes Figure 5A or Figure 5B, depending on whether $p_{m-i} > p_{m-i+1}$ or $p_{m-i} < p_{m-i+1}$. In the case when $p_{m-i} = p_{m-i+1}$, the new graph will look like Figure 4.



(A) ith and $(i+1)$th peaks (B) ith and $(i+1)$th peaks

FIGURE 5

The factor $(-q;q^2)_m$ generates m nonnegative distinct multiples of $(2i-1)$, $1 \leq i \leq m$, say, $b_1 \times 1,\ b_2 \times 3, \cdots, b_m \times (2m-1)$, where each b_i $(1 \leq i \leq m)$ is 0 or 1. This is encoded by putting a beam of height b_{m-i+1} on the ith pillar. Figure 5A (or Figure 5B) will either not change or may change to three possible shapes. For example, Figure 5A may look like:



(A) (B) (C)

FIGURE 6

In Figure 6A, Figure 6B, and Figure 6C

$$\begin{aligned}P_i \equiv \Big(&1 + (i-1)(j+3) + \alpha_{m-i+1} + 2(p_m + p_{m-1} + \cdots + p_{m-i+2})\\&+ p_{m-i+1} + 2(b_m + b_{m-1} + \cdots + b_{m-i+2})\\&+ b_{m-i+1}, 1 + p_{m-i+1} + b_{m-i+1}\Big),\\P_{i+1} \equiv \Big(&1 + i(j+3) + \alpha_{m-i} + 2(p_m + p_{m-1} + \cdots + p_{m-i+1})\\&+ p_{m-i} + 2(b_m + b_{m-1} + \cdots + b_{m-i+1})\\&+ b_{m-i}, 1 + p_{m-i} + b_{m-i}\Big).\end{aligned}$$

Each modified lattice path enumerated by $Q_1^j(\nu)$ is uniquely generated in this manner. This proves (2.1).

3. Bijective Proofs

In this section, we establish a 1–1 correspondence between the modified lattice paths enumerated by $Q_1^j(\nu)$ and the split n-color partitions enumerated by $P_1^j(\nu)$.

We do this by encoding each modified lattice path as the sequence of weights of the peaks with each weight subscripted by the height of the respective peak considered as the height of the supporting pillar which corresponds to the green part plus the height of the supporting beam which corresponds to the red part. Thus, if we denote the ith and $(i+1)$th peaks in the final graph by A_x and B_y $(B \geq A)$, respectively, then

$$\begin{aligned}A &= 1 + (i-1)(j+3) + \alpha_{m-i+1} + 2(p_m + p_{m-1} + \cdots + p_{m-i+2}) + p_{m-i+1}\\&\quad + 2(b_m + b_{m-1} + \cdots + b_{m-i+2}) + b_{m-i+1},\\x &= 1 + p_{m-i+1} + b_{m-i+1},\\B &= 1 + i(j+3) + \alpha_{m-i} + 2(p_m + p_{m-1} + \cdots + p_{m-i+1}) + p_{m-i}\\&\quad + 2(b_m + b_{m-1} + \cdots + b_{m-i+1}) + b_{m-i},\\y &= 1 + p_{m-i} + b_{m-i}.\end{aligned}$$

Clearly, the parity of both A and x depends upon $p_{m-i+1} + b_{m-i+1}$. If $p_{m-i+1}+b_{m-i+1}$ is odd then both A and x are even and when $p_{m-i+1}+b_{m-i+1}$ is even then both A and x are odd. This confirms that the parts and their subscripts have same parity.

The weighted difference of these two parts is

$$((B_y - A_x)) = B - A - x - y = \alpha_{m-i} - \alpha_{m-i+1} + j + 1 > j,$$

which is nonnegative and congruent to $(j+1) \pmod 4$. Since the height of any beam cannot exceed 1, the red part in the corresponding split n-color partition is at most 1.

To see the reverse implication, we consider two summands of a partition enumerated by $P_1^j(\nu)$, say, C_u and D_v with $D \geq C$. (Note that there is no need to consider the split subscripts.) Clearly $u \leq C$ and $v \leq D$. Let $Q_1 \equiv (C, u)$ and $Q_2 \equiv (D, v)$ be the corresponding peaks in the modified lattice path.

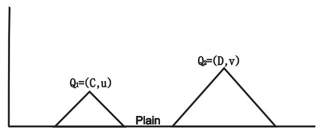


FIGURE 7. Two peaks Q_1 and Q_2 separated by a plain.

If there is a plain between Q_1 and Q_2, its length would be $D - C - u - v$ which is the weighted difference between the two parts C_u and D_v and is therefore greater than j and congruent to $(j+1) \pmod 4$ (by condition (iv) of the Theorem 1.8). We thus conclude that the corresponding path can have plains with minimal length congruent to $(j + 1) \pmod 4$.

If C_u were the smallest part of the partition, the corresponding peak in the modified lattice path would be the first peak preceded by a plain of length α, where α is a nonnegative multiple of 4, since

$$C = 1 + \alpha + p + b,$$
$$u = 1 + p + b,$$

where $b = 0$ or 1. Thus

$$C - u = \alpha \equiv 0 \pmod 4 .$$

Finally, we show that there cannot be a valley above height 0 for $j = -1$ and no valley at all for $j > -1$.

Suppose there is a valley V at height r between the peaks Q_1 and Q_2.

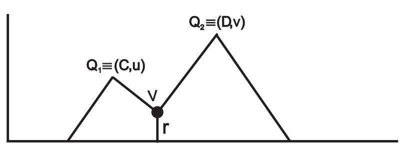


FIGURE 8. A valley at height r between two peaks.

In this case there is a descent of $u - r$ from Q_1 to V and an ascent of $v - r$ from V to Q_2. This implies that $D = C + (u-r) + (v-r)$ and so this implies $D - C - u - v = -2r$. However since the weighted difference is nonnegative, $r = 0$.

Now for $j > -1$, we know there is always a plain of minimal length congruent to $(j + 1) \pmod 4$ which is always positive for $j > -1$. Hence there cannot be a valley at all in the corresponding paths for $j > -1$. This completes the bijection between the modified lattice paths enumerated by $Q_1^j(\nu)$ and the split n-color partitions enumerated by $P_1^j(\nu)$. □

3.1. **Outline of the Proofs of Theorems 2.2–2.4.** Here, the changes required to prove the remaining theorems are discussed briefly.

Theorem 2.2. An appeal to Theorem 2.1; the extra factor $q^{m(j+2)}$ puts a southeast step from (0,1) to (1,0) followed by a plain of length $(j+1)$ at the front of the lattice path. So in this case the path begins with $(m+1)$ peaks starting from $(0,1)$ and ending at $(m(j+3)+1,0)$ with a plain of minimal length $(j+1) \pmod 4$ between each pair of peaks. Also, the extra factor $1/(1-q^{2m+1})$ introduces a nonnegative multiple of $2m+1$, say $p_{m+1} \times (2m+1)$. This is encoded by having the first peak grow to height $p_{m+1}+1$ in the northeast direction. Clearly, it will correspond to the i_{i+1} part of the corresponding colored partition.

Theorem 2.3. An appeal to Theorem 2.1; the extra factor $q^{m(j+3)}$ puts two southeast steps (0,2) to (1,1) and (1,1) to (2,0) followed by a plain of length $(j+1)$ at the front of the lattice path. So in this case the path begins with $(m+1)$ peaks starting from $(0,2)$ ending at $(m(j+3)+2,0)$ and with a plain of minimal length $(j+1) \pmod 4$ between each pair of peaks. Also, the factor $1/(1-q^{2m+1})$ introduces a nonnegative multiple of $2m+1$, say $p_{m+1} \times (2m+1)$. This is encoded by having the first peak grow to height $p_{m+1}+2$ in the northeast direction. Clearly, it will correspond to the i_{i+2} part of the corresponding colored partition.

Theorem 2.4. An appeal to Theorem 2.1; the extra factor $q^{m(j+3)}$ puts a plain of length $(j+3)$ in front of the first peak. This causes a total increase of $m(j+3)$ in the weight of the path and makes the weight of the first peak greater than or equal to $(j+4)$.

4. Particular Cases

Identity (1.7), in conjunction with Theorems 1.8 and 2.1, for $j=-1$, leads to the following theorem:

Theorem 4.1. *Let $E_1(\nu)$ denote the number of n-color partitions of ν such that parts are distinct and first two copies of parts congruent to $5 \pmod{10}$ are used and only the first copy of parts congruent to $\pm1 \pmod{10}$ are used. Let $F_1(\nu)$ denote the number of n-color partitions of ν such that the first two copies of parts congruent to $\pm2 \pmod{10}$ are used. Further let*

$$A_1(\nu)=\sum_{k=0}^{\nu} E_1(k)F_1(\nu-k),$$

then

$$A_1(\nu)=P_1^{-1}(\nu)=Q_1^{-1}(\nu), \text{ for all } \nu. \tag{4.1}$$

Example 4.2. *We can verify Theorem 4.1 by showing that $A_1(4)=P_1^{-1}(4)=Q_1^{-1}(4)=3$. The relevant split n-color partitions corresponding to $P_1^{-1}(4)$*

are: $4_4, 4_{3+1}, 3_1 + 1_1$. *Also,* $Q_1^{-1}(4) = 3$ *and the relevant modified lattice paths are:*

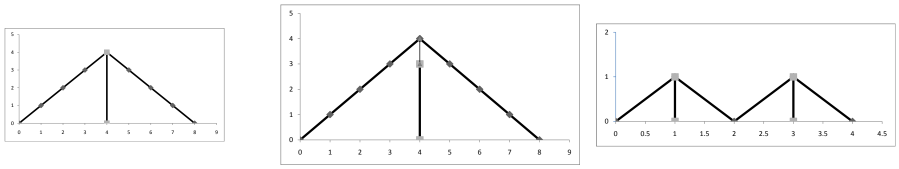


FIGURE 9

The relevant partitions corresponding to $E_1(\nu)$ *and* $F_1(\nu)$, $0 \leq \nu \leq 4$, *are given in the table below:*

TABLE 1. Number of partitions enumerated by $E_1(\nu)$ and $F_1(\nu)$ for $0 \leq \nu \leq 4$.

ν	$E_1(\nu)$	partitions enumerated by $E_1(\nu)$	$F_1(\nu)$	partitions enumerated by $F_1(\nu)$
0	1	empty partition	1	empty partition
1	1	1_1	0	-
2	0	-	2	2_1, 2_2
3	0	-	0	-
4	0	-	3	$2_1 2_1$, $2_2 2_1$, $2_2 2_2$

Hence, $A_1(4) = \sum_{k=0}^{4} E_1(k)F_1(4-k) = E_1(4)F_1(0) + E_1(3)F_1(1) + \cdots + E_1(0)F_1(4) = 3$.

Identity (1.8), in conjunction with Theorems 1.9 and 2.2, for $j = 1$, leads to the following theorem:

Theorem 4.3. *Let* $E_2(\nu)$ *denote the number of partitions of* ν *into distinct parts congruent to* $\pm 1, \pm 5 \pmod{12}$ *and let* $F_2(\nu)$ *denote the number of partitions of* ν *into parts congruent to* $\pm 2, \pm 4 \pmod{12}$. *Further let* $A_2(\nu) = \sum_{k=0}^{\nu} E_2(k)F_2(\nu-k)$, *then*

$$A_2(\nu) = P_2^1(\nu) = Q_2^1(\nu), \text{ for all } \nu.$$

Identity (1.9), in conjunction with Theorems 1.10 and 2.3, for $j = -1$, leads to the following theorem:

Theorem 4.4. *Let* $E_3(\nu)$ *denote the number of partitions of* ν *into parts congruent to* $\pm 1, \pm 3 \pmod{10}$ *and let* $F_3(\nu)$ *denote the number of partitions of* ν *into parts congruent to* $\pm 2, \pm 4 \pmod{10}$. *Further let* $A_3(\nu) = \sum_{k=0}^{\nu} E_3(k)F_3(\nu-k)$, *then*

$$A_3(\nu) = P_3^{-1}(\nu) = Q_3^{-1}(\nu), \text{ for all } \nu.$$

Identity (1.10), in conjunction with Theorems 1.11 and 2.4, for $j = -1$, leads to the following theorem:

Theorem 4.5. *Let $E_4(\nu)$ denote the number of n-color partitions of ν into distinct parts such that first two copies of parts congruent to* $5 \pmod{10}$ *are used and only first copy of parts congruent to* $\pm 3 \pmod{10}$ *are used and let $F_4(\nu)$ denote the number of n-color partitions of ν such that first two copies of parts congruent to* $\pm 4 \pmod{10}$ *are used. Further let* $A_4(\nu) = \sum_{k=0}^{\nu} E_4(k)F_4(\nu - k)$, *then*

$$A_4(\nu) = P_4^{-1}(\nu) = Q_4^{-1}(\nu), \text{ for all } \nu.$$

Acknowledgement

The first author M. Rana acknowledges the research support provided by Council of Scientific and Industrial Research, New Delhi, India, through project No. 25(0256)/16/EMR-II. The authors are also thankful to the anonymous referee for his/her valuable suggestions.

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School of Mathematics, Thapar University, Patiala-147001, India
E-mail address: mrana@thapar.edu

Department of Mathematical Sciences, I.K. Gujral Punjab Technical University, Jalandhar, Main Campus, Kapurthala-144603, India
E-mail address: meghagoyal2021@gmail.com