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BALL HULLS, BALL INTERSECTIONS, AND 2-CENTER PROBLEMS FOR GAUGES

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Abstract. The notions of ball hull and ball intersection of finite sets, important in Banach space theory, are extended from normed planes to generalized normed planes, i.e., to (possibly asymmetric) convex distance functions which are also called gauges. Related to this, we extend the known 2-center problem and a modified version of it from the Euclidean situation to norms and gauges. We also derive algorithmical results on the construction of ball hulls and ball intersections, yielding computational approaches to 2-center problems.

1. Introduction and preliminaries

We denote by $\mathbb{M}^d = (\mathbb{R}^d, \|\cdot\|)$ a (*generalized*) *normed space*, namely, the d-dimensional Euclidean space endowed either with a *norm* or with a *generalized convex distance function* (which can be *asymmetric*), also called a *gauge*. We write B for the *unit ball* of $\mathbb{M}^d$, which is a compact, convex set with the origin o as interior point. The set $x + rB = B(x,r)$ is the *ball with center* x *and radius* r, and the *spheres* S and $S(x,r)$ are the boundaries of B and $B(x,r)$, respectively. The set $\hat{B} = (-B)$ is in general, the unit ball of another gauge; $\hat{S}$ is the boundary of $\hat{B}$, $\hat{B}(x,r)$ the set $x + r\hat{B}$, and $\hat{S}(x,r)$ its boundary. We use the usual abbreviations $\mathrm{conv}(K)$ and $\mathrm{diam}(K)$ for the convex hull and the diameter of a set K, $\overline{pq}$ for the *line segment* connecting p and q, and $\langle p,q\rangle$ for its affine hull. A generalized normed space is *strictly convex* if its unit sphere contains no nondegenerate segment.

Given a set of points K in $\mathbb{R}^d$ and $r > 0$, the rB-*ball hull* $\mathrm{bh}_B(K,r)$ and the rB-*ball intersection* $\mathrm{bi}_B(K,r)$ of K are the sets

$$\mathrm{bh}_B(K,r) = \bigcap_{K \subset B(x,r)} B(x,r) \quad , \qquad \mathrm{bi}_B(K,r) = \bigcap_{x \in K} B(x,r).$$

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Clearly, the boundary of $\mathrm{bi}_B(K,r)$ in a (generalized) normed plane consists of circular arcs. Theorem 2.3 below describes the boundary structure of $\mathrm{bh}_B(K,r)$ and Theorem 2.4 describes the relationship between $\mathrm{bh}_B(K,r)$ and $\mathrm{bi}_{\hat{B}}(K,r)$.

We have $\mathrm{bh}_B(K,r) \neq \emptyset$ if and only if $r \geq r_K$, where r_K is the B-*circumradius of* K, i.e., the smallest number such that K is contained in a translate of $r_K B$. It is easy to see that even for gauges, $\mathrm{bi}_{\hat{B}}(K,r_K)$ is the set of centers of B-minimal enclosing discs of K. Therefore $\mathrm{bi}_B(K,r) \neq \emptyset$ if and only if r is greater than or equal to the $\hat{B}$-circumradius of K. For rB-ball hulls (respectively, the rB-ball intersections) of K, we always mean that $r \geq r_K$ (respectively, r equals at least the $\hat{B}$-circumradius of K).

If K_1 and K_2 are bounded sets of points in $\mathbb{M}^d$, then

(P1) $K_1 \subseteq K_2 \Rightarrow \mathrm{bi}_B(K_1,r) \supseteq \mathrm{bi}_B(K_2,r)$ and $\mathrm{bh}_B(K_1,r) \subseteq \mathrm{bh}_B(K_2,r)$,

(P2) $r_1 \leq r_2 \Rightarrow \mathrm{bi}_B(K,r_1) \subseteq \mathrm{bi}_B(K,r_2)$ and $\mathrm{bh}_B(K,r_1) \supseteq \mathrm{bh}_B(K,r_2)$.

The proof of the second inclusion of (P2) presented in [20] for norms is extended to gauges. The definitions imply the rest of the properties.

The notions of ball hull and ball intersection are important in Banach space theory; they are basic for investigations on circumballs and Chebyshev sets (see [4] and [20]), complete sets and bodies of constant width (cf. [21] and [22]), and ball polytopes ([9], [6], [7, Chapter 6], [18], and [8, Chapter 5]). For finite sets K, both the ball hulls and the ball intersections can be obtained in $O(n \log n)$ time in the Euclidean subcase ([15]) and for a general norm ([19]), and we prove in Section 3 that this holds also for a gauge (Algorithms I and II in Theorem 3.1). In a more applied sense, ball hulls and ball intersections also played an important role for solving certain clustering problems. One example, well known also in computational geometry, is the so-called *planar 2-center problem*, which asks how to cover a given set K in the plane with two congruent balls of minimal radii. Sharir ([25]) achieves the crucial first subquadratic solution (taking ($O(n \log^9 n)$ time; see also [10], [11]). Approaching the 2-center problem requires a procedure referring to the *fixed radius problem*, asking whether a set K of n points in the plane can be covered by two discs of two fixed radii. On the other hand, the *fixed radius problem with constrained circles* requires the centers of the circles to be from K ([15], [3]). As far as we know, there are not many results on non-Euclidean norms, not even for L_p spaces apart from $p = \{1, 2, \infty\}$ ([5], [15], [17], and [16]). We justify in Section 3 why Sharir's operational framework for the fixed radius problem does not work for every gauge. We adapt the Euclidean quadratic approach of Hershberger ([14]), which is the inspiration of the procedure presented in [1], in order to obtain an almost quadratic solution (Theorem 3.5) when B defines a general gauge. We also show that the *fixed radius problem with constrained circles* can be computed for every gauge in $O(n^2)$ time (see Algorithm III in Theorem 3.1).

2. The ball hull structure and planar gauges

Our objective in this section is to describe the geometric structures of, and the relationship between the rB-ball hull and the $r\hat{B}$-ball intersection of a finite set K (see Theorem 2.3 and Theorem 2.4) for gauges.

For the following lemma we refer to [23, § 3.3] and [4].

Lemma 2.1. *Let $\mathbb{R}^2$ be the Euclidean plane and $B \subset \mathbb{R}^2$ be a convex body. If $u, v \in \mathbb{R}^2$ and $r > 0$, then $S(u,r) \cap S(v,r)$ is the union of two nonempty connected components, A_1 and A_2, which may degenerate to the same set or to the empty set.*

Suppose that $S(u,r) \cap S(v,r)$ consists of two different nonempty connected components. Then the two lines parallel to the line of translation and supporting $B(u,r) \cap B(v,r)$ intersect $B(u,r) \cap B(v,r)$ exactly in A_1 and A_2.

Let us choose $p_i \in A_i$, $i = 1, 2$. Let $u_i = p_i - (v-u)$ and $v_i = p_i + (v-u)$ for $i = 1, 2$. Let $S_1(u,r)$ be the part of $S(u,r)$ on the same side of the line $\langle p_1, p_2\rangle$ as u_1 and u_2; let $S_2(u,r)$ be the part of $S(u,r)$ on the side of $\langle p_1, p_2\rangle$ opposite to u_1 and u_2. Let $S_1(v,r)$ be the part of $S(v,r)$ on the same side of the line $\langle p_1, p_2\rangle$ as v_1 and v_2; let $S_2(v,r)$ be the part of $S(v,r)$ on the side of $\langle p_1, p_2\rangle$ opposite to v_1 and v_2. Then $S_2(u,r) \subseteq \mathrm{conv}(S_1(v,r))$ and $S_2(v,r) \subseteq \mathrm{conv}(S_1(u,r))$.

Having in mind Lemma 2.1 if $\hat{B}(p_1,r) \cap \hat{B}(p_2,r)$ has two different connected components (two segments), A_1 and A_2, we let $u \in A_1$ and $v \in A_2$ be extreme points of A_1 and A_2, respectively. If $\hat{B}(p_1,r) \cap \hat{B}(p_2,r)$ has only one connected component (one segment), we let u and v be extreme points of this segment. In both cases, $S(u,r)$ and $S(v,r)$ determine two arcs $S_2(u,r)$ and $S_2(v,r)$ meeting p_1 and p_2 (eventually only one if they degenerate to the same set). We call each of these arcs *rB-minimal (with center u or v) meeting p_1 and p_2.*

Lemma 2.2. *Let $w \in \mathbb{M}^2$, $r > 0$, and p_1, p_2 be two points from $B(w,r)$. Then*

1. *there exist only two rB-minimal arcs meeting p_1 and p_2 (which may degenerate to the segment $\overline{p_1p_2}$ if B is not strictly convex), both contained in $B(w,r)$, and $\langle p_1, p_2\rangle$ separates them if they are different,*
2. *for every $r' \geq r$, any $r'B$-minimal arc of p_1, p_2 is contained in $B(w,r)$,*
3. *if for $x \in \mathbb{R}^2$ there is an arc on $S(x,r)$ meeting p_1 and p_2, contained in $B(w,r)$, and containing interior points of $B(w,r)$, then it is rB-minimal.*
4. *Let $p_1, p_2 \in S(u,r) \cap S(v,r)$ for some $u \neq v$, and $p \neq \{p_1, p_2\}$ be from the rB-minimal arc with center u. Then v and the support line L of $B(u,r) \cap B(v,r)$ at p are separated from u by the line $(1/2)(u+v)+L$.*

Proof. (1) It is easy to check (see Figure 1) that

$$\{w \in \mathbb{R}^2 / \{p_1, p_2\} \in B(w,r)\} = \hat{B}(p_1,r) \cap \hat{B}(p_2,r),$$

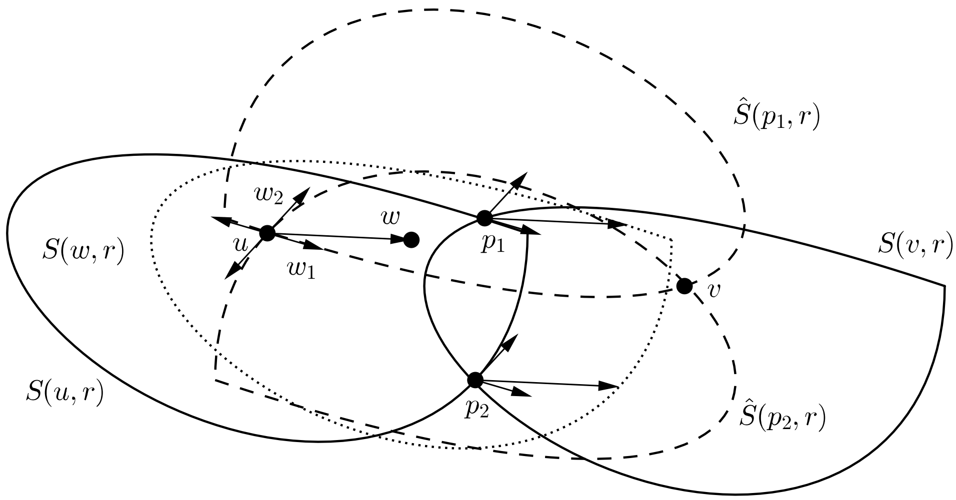

FIGURE 1. $B(w, r)$ contains the minimal arcs meeting p_1 and p_2.

and the boundary of this set is the union of the $r\hat{B}$-minimal arcs with centers in p_1 and p_2. By Lemma 2.1, $\hat{S}(p_1, r) \cap \hat{S}(p_2, r) = \{A_1, A_2\}$, where A_1 and A_2 can be segments or points. Let us consider $w \in \hat{B}(p_1, r) \cap \hat{B}(p_2, r)$.

Suppose that $A_1 = u$, $A_2 = v$ for different $u, v \in \mathbb{R}^2$, and assume $u = o$. We prove that $S_2(u, r) \subset B(w, r)$; $S_2(v, r) \subset B(w, r)$ is verified analogously.

Choose two lines l_1 and l_2 such that $p_i + l_i$ supports $B(u, r)$ at p_i. Due to symmetry, $u + l_i$ $(i = 1, 2)$ supports $\hat{B}(p_1, r) \cap \hat{B}(p_2, r)$ at $u = o$. Choose $w_i \in \mathbb{R}^2$ parallel to l_i such that $\langle p_1, p_2 \rangle$ leaves $p_i + w_i$ and $S_2(u, r)$ in the same hyperplane.

For every $w \in \hat{B}(p_1, r) \cap \hat{B}(p_2, r)$, also $w = \alpha w_1 + \beta w_2$ holds with $\alpha, \beta \geq 0$. For positive values α and β we have

$$S_2(u, r) \subset \operatorname{conv}(p_1, p_2, w + S_2(u, r)) \subset B(w, r).$$

Let A_1 and A_2 be different segments. We choose u_1 and v_1 such that $A_1 = \overline{u_1 v_1}$ with $\vec{u_1 v_1} = \alpha \vec{p_1 p_2}$ for some $\alpha \geq 0$. Consider $S(u_1, r) \cap S(v_1, r)$ like in Lemma 2.1. The segment $p_1 + A_1$ belongs to $S(u_1, r)$ and is parallel to $\overline{p_1 p_2}$. Therefore, $\overline{p_1 p_2}$ itself belongs to $S(u_1, r)$, and $S_2(u_1, r) = \overline{p_1 p_2} \subset B(w, r)$. Similarly, we prove that $S_2(v_1, r) = \overline{p_1 p_2} \subset B(w, r)$. If $A_2 = \overline{u_2 v_2}$, analogously $\overline{p_1 p_2}$ is the rB-minimal arc with center at u_2 or v_2 meeting p_1 and p_2.

In order to prove the case when A_1 is a segment and A_2 is only a point, we can combine the arguments managed in both cases above.

Assume that $A_1 = A_2 = \overline{uv}$ for some $u, v \in \mathbb{R}^2$. If $u = v$, there is nothing to prove. For $u \neq v$, $\overline{uv}$ is parallel to the support line of $B(u, r) \cap B(v, r)$ (Lemma 2.1). Hence for every $w = u + \alpha(v - u)$ with $0 \leq \alpha \leq 1$ we have that

$$S_2(u,r) \subset \mathrm{conv}(\overline{p_1p_2}, \alpha(v-u) + S_2(u,r) \subset S(w,r).$$

Similarly, $S_2(v,r) \subset S(w,r)$.

(2) By (1), $\mathrm{bh}_B(\{p,q\},r)$ is bounded by the two rB-minimal arcs meeting p and q. If $r' \geq r$, $\mathrm{bh}_B(\{p,q\},r') \subseteq \mathrm{bh}_B(\{p,q\},r)$ by (P2), and (2) holds.

(3) We have $x \neq w$ because the arc on $S(x,r)$ contains interior points of $B(w,r)$. There are two arcs on $S(x,r)$ meeting p and q. By Lemma 2.1 and (1), one of them is rB-minimal and contained in $B(w,r)$, and the other one is not from $B(w,r)$. Thus, the conditions force the arc in (3) to be the first one.

(4) By Lemma 2.1 and the convexity of $B(u,r) \cap B(v,r)$, $1/2(u+v)+L$ separates u and L. And, obviously, u and v are separated by $1/2(u+v)+L$. □

Theorem 2.3. *Let $K = \{p_1, p_2, \ldots, p_n\}$ be a finite set, and let $r \geq r_K$. Then*

$$\mathrm{bh}_B(K,r) = \bigcap_{K \subset B(x_s,r)} B(x_s,r) = \mathrm{conv}\left(\bigcup_{i,j=1}^{n} \widehat{p_ip_j}\right),$$

where x_s are extreme points of the components $\hat{S}(p_i,r) \cap \hat{S}(p_j,r)$, and $\widehat{p_ip_j}$ are rB-minimal arcs with centers x_s that meet pi and p_j.

Proof. We denote by $\widehat{p_ip_j}$ the rB-minimal arc meeting (clockwise) p_i and p_j. Since $r \geq r_K$, there exists $B(x_1,r)$ such that $K \subset B(x_1,r)$. After translating suitably, we may assume that $S(x_1,r)$ contains two points $p_1, p_2 \in K$, and that $\widehat{p_1p_2}$ is the largest rB-minimal arc on $S(x_1,r)$ meeting points of K.

If the arc on $S(x_1,r)$ from p_2 to p_1 (clockwise) is also minimal, then $\mathrm{bh}_B(K,r) = B(x_1,r)$ (Lemma 2.2). Otherwise, we move a point z clockwise along $\hat{S}(p_2,r)$, and we observe the arcs on $S(z,r)$ starting (clockwise) in p_2. Let x_2 denote the first position of z such that a point of K is reached by one of these arcs (more than one point can be reached at the same time). Statement (3) in Lemma 2.2 guarantees that the arc on $S(x_2,r)$ starting in p_2 and ending (clockwise) in a new point from K is rB-minimal. We consider $A = B(x_1,r) \cap B(x_2,r)$. Since z moves continuously in $\hat{S}(p_2,r)$, A contains K. If $p_1 \in S(x_2,r)$, then the arc meeting (clockwise) p_2 and p_1 on $S(x_2,r)$ is minimal and $A = \bigcap_{K \subset B(x,r)} B(x,r)$. If $p_1 \notin S(x_2,r)$, let p_3 be the new point on $K \cap S(x_2,r)$ such that no other rB-minimal arc on $S(x_2,r)$ meets K and is larger than $\widehat{p_2p_3}$. We repeat the operation and now move a point z clockwise along $\hat{S}(p_3,r)$ starting in $z = x_2$. As above, x_3 be the first value of z such that one of the following is verified: either $p_1 \in S(x_3,r)$ or there is a new $p_4 \in K$ such that $\widehat{p_3p_4}$ (clockwise) is the largest rB-minimal arc on $S(x_3,r)$ starting in p_3 and ending in a point of K. In both cases, $A = B(x_1,r) \cap B(x_2,r) \cap B(x_3,r)$ contains K since z moves continuously in $\hat{S}(p_3,r)$. Besides this, we have $\bigcap_{K \subset B(x,r)} B(x,r) \subset A$. If $p_1 \in S(x_3,r)$, the boundary of A is generated by rB-minimal arcs meeting points of K, and by Lemma 2.2 we have $A = \bigcap_{K \subset B(x,r)} B(x,r)$. If $p_1 \notin S(x_3, r)$, the process continues similarly, and it is clearly finite. Starting with p_1 and p_2, we cannot get a previous point $p_i \in \{p_2, ..., p_{i-1}\}$ in a new step, because any new point (except when the process ends in p_1) cannot be from the convex hull of the union of the previous minimal arcs.

At the end we obtain the set $A = \bigcap_{s=1}^{k} B(x_s, r)$, where x_s are extreme points of the components $\hat{S}(p_i, r) \cap \hat{S}(p_{i+1}, r)$. Clearly, A contains the ball hull, and the boundary of A is generated by rB-minimal arcs meeting points of K. Thus, by Lemma 2.2, both sets are equal to $\operatorname{conv}(\bigcup_{i,j=1}^{n} \widehat{p_i p_j})$. □

Theorem 2.3 shows that the boundary of a planar ball hull consists of minimal arcs meeting points from K. Their endpoints are extreme and called *vertices of ball hulls*. The boundary of a planar ball intersection consists of circular arcs, whose endpoints are called *vertices of ball intersections*.

Theorem 2.4. *Let $K = \{p_1, p_2, \ldots, p_n\}$ be a finite set in a generalized normed plane $\mathbb{M}^2$ and $r \geq r_K$. Then every arc of the boundary of $\mathrm{bi}_{\hat{B}}(K, r)$ has a vertex of $\mathrm{bh}_B(K, r)$ as center. Moreover, every vertex of $\mathrm{bi}_{\hat{B}}(K, r)$ is the center of an arc belonging to the boundary of $\mathrm{bh}_B(K, r)$.*

Proof. Let us consider the constructive process described in Theorem 2.3. The points $x_1, x_2, ..., x_k$ are the centers of the rB-minimal arcs $\widehat{p_1 p_2}$, $\widehat{p_2 p_3}$, ..., $\widehat{p_k p_1}$ whose union is the boundary of $\mathrm{bh}_B(K, r)$. Define $x_{k+1} := x_1$ and consider the arcs $\widehat{x_i x_{i+1}}$ on $\hat{S}(p_i, r)$ meeting (clockwise) at x_i and x_{i+1}. The process assures that every rB-disc whose center belongs to $\widehat{x_i x_{i+1}}$ contains K, and therefore $\widehat{x_i x_{i+1}} \subset \mathrm{bi}_{\hat{B}}(K, r)$. Besides this, the union of the arcs $\widehat{x_i x_{i+1}}$ is the boundary of the intersection of some $r\hat{B}$-balls whose centers are points of K. Consequently, this is the boundary of $\mathrm{bi}_{\hat{B}}(K, r)$, and $x_1, x_2, ..., x_k$ are its vertices. Moreover, the centers p_i of the arcs $\widehat{x_i x_{i+1}}$ are the vertices of $\mathrm{bh}_B(K, r)$. □

3. Some applications

Now we present applications to computational geometry in generalized normed planes, all of them based on the results of Section 2. Restrictions (like strict convexity) are explicitly mentioned. The running time refers to the cost of elementary operations, like computing the intersection of two convex curves. Since the unit balls B are general convex bodies, in our computation model B is given via an "oracle" as it is described in Section 3.3 of [13] or on page 316 in [24].

The *fixed 2-center problem with constrained circles* and the computation of ball hulls and ball intersection of K will be solved now in generalized normed planes, extending the results for the Euclidean case ([15]) and for normed planes ([19]).

Theorem 3.1. *Let $\mathbb{M}^2$ be a generalized normed plane. Let K be a set of n points and $r > 0$. Then the following algorithms can be designed:*

- *Algorithm I, that constructs* $\mathrm{bi}_B(K,r)$ *in* $O(n\log n)$ *time.*
- *Algorithm II, that constructs* $\mathrm{bh}_B(K,r)$ *in* $O(n\log n)$ *time.* $\mathrm{bi}_B(K,r)$ *can be constructed in* $O(n\log n)$ *time.*
- *Algorithm III, that solves the fixed 2-center problem with constrained circles in* $O(n^2)$ *time for two radii* $r \geq r_1$.

Proof. We fix a Euclidean orthonormal background system with basis $\{v_1, v_2\}$. The points of K can be ordered by the lexicographical ordering. Likewise, we say that an arc a_1 *of* $\mathrm{bi}_B(K,r)$ *is on the left with respect to an arc* a_2 if the leftmost point of a_1 has an x-coordinate smaller than the x-coordinate of the leftmost point of a_2, breaking the ties.

We consider the two lines parallel to v_2 that support $\mathrm{bi}_B(K,r)$, and the two corresponding supporting sets, namely, the intersections of these lines and $\mathrm{bi}_B(K,r)$. We choose two points, one from each supporting set. The line through these points separates the boundary of $\mathrm{bi}_B(K,r)$ into an *upper* and a *lower chain*.

Item (4) in Lemma 2.2 proves that if K is a set of two points, the left-to-right order of the arcs along the upper (lower) chain of $\mathrm{bi}_B(K,r)$ is just the reverse of the left-to-right order of the centers of these arcs. If a connected piece of $S(x^i,r)\cap S(x^{i+1},r)$ belongs to the upper chain of the boundary of $\mathrm{bi}_B(K,\lambda)$, then it also belongs to the upper chain of $\mathrm{bi}_B(\{x^i,x^{i+1}\},r)$, and their common arcs are located in the same arc order. Applying this repeatedly for every pair (x^i,x^{i+1}) from K, we prove that the centers $x^1, x^2, \ldots, x^m$ of the arcs of the boundary of $\mathrm{bi}_B(K,r)$ are ordered conversely to the sequence of these arcs.

Algorithm I. 1) Sort the points of K from left to right in $O(n\log n)$ time; 2) Start with the leftmost arc and its center, consider the centers at the left side to find the arc following the right one. Thus, the upper (lower) chain of $\mathrm{bi}_B(K,\lambda)$ can be constructed in $O(n)$ time.

Algorithm II. 1) Build $\mathrm{bi}_{\hat{B}}(K,r)$ in $O(n\log n)$ time (Algorithm I); 2) Consider the set K' of sorted vertices $\{x_1,...,x_k\}$ of $\mathrm{bi}_{\hat{B}}(K,r)$ from 1) and build $\mathrm{bh}_B(K,\lambda)=\mathrm{bi}_B(K',r)$ (Theorem 2.4 and Algorithm I) in $O(n)$ time.

Algorithm III. 1) Sort the points of K from left to right in $O(n\log n)$ time (x-coordinate). 2) For each $p\in K$ define $U := \{x\in K : x\notin B(p,r)\}$; obtain $\mathrm{bi}_{\hat{B}}(U,r_1)$ in $O(n)$ time. 3) Test if $\mathrm{bi}_{\hat{B}}(U,r_1)\cap K\neq\emptyset$ in $O(n)$ time, march through K from left to right, maintaining the two arcs of the boundary of $\mathrm{bi}_{\hat{B}}(U,r_1)$ that overlap the x-coordinate of the current point. □

Now we deal with the *fixed 2-center problem.* Given $r\geq r_1>0$, we ask whether a set K of n points in the plane can be covered by two discs of radius r and r_1, respectively. Without loss of generality, we can assume that $r_1=1$, and of course $\mathrm{diam}(K)>r$.

Sharir [25] solved the Euclidean fixed-radius problem in $O(n\log^3 n)$ time. For this he assumed that two covering r-discs exist, and their possible centers c_1 and c_2 are searched in two cases: when they are well separated ($\|c_1-c_2\|>$

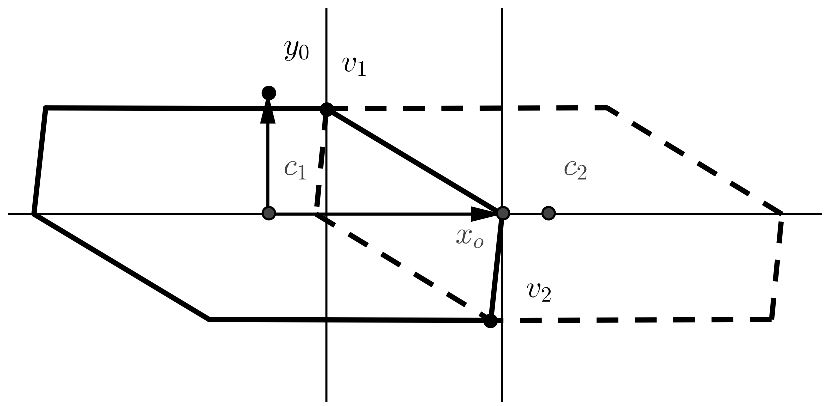

FIGURE 2

r), and when they are close to each other ($\|c_1 - c_2\| \leq r$). We can find in constant time an orthogonal basis such that the orientation of $c_1 - c_2$ is *almost parallel* to the x-axis for some of them. If $\|c_1 - c_2\| > r$, the orthogonal projections of the centers on the x-axis (denoted by $\|x(c_i)\|$) are at a distance close to r (namely, $\|x(c_1) - x(c_2)\| > 0.99r$) in such an orientation. As a consequence, if $r < \|c_1 - c_2\| < 3r$ and v_1 is the left most point from $S(c_1, r) \cap S(c_2, r)$, the projection of v_1 on the x-axis (denoted by $x(v_1)$) is far away from the projection of c_1 (namely, $\|x(v_1) - x(c_1)\| > 0.4r$). This allows to draw a constant number of vertical lines (separated by a constant distance smaller than $0.4r$), such that at least one of them separates c_1 and v_1. All these arguments above are used in Sharir's algorithm for searching the centers when $r < \|c_1 - c_2\| \leq 3r$.

Figure 2 shows a basis $\{x_0, y_0\}$ and two r-discs of a hexagonal normed plane. The centers of the discs are c_1 and c_2, respectively. Without loss of generality, we can assume that c_1 is the origin. Since $\|x_0\| \leq \|x_0 + ry_0\|$ for every $r \in \mathbb{R}$, x_0 is *Birkhoff orthogonal* to y_0. (Birkhoff orthogonality, defined precisely by this condition, is a generalization of Euclidean orthogonality for normed planes, see [23]). The vector $c_1 - c_2$ is parallel to x_0, and $\|c_1 - c_2\| = 1.2r$. If we write $x(v_1)$ for the y_0-projection of v_1 on the x_0-axis, then $\|x(v_1) - c_1\| = 0.25r < 0.4r$. If we translate c_2 closer to $S(c_1, r)$ along the line rx_0 (but maintaining $\|c_1 - c_2\| > r$), we achieve a y_0-projection closer to c_1. Besides this, if c_2 remains fixed but v_1 is translated by $-\epsilon x_0$ ($\epsilon > 0$), it is easy to design new discs of other normed planes centered at c_1 and c_2, whose r-circles pass across v_1 and satisfy that $x(v_1)$ is closer to c_1. *Rounding* slightly the boundaries of the discs, we obtain strictly convex discs with the same properties. Since we do not know the distance $\|c_1 - c_2\|$ in advance, and since we cannot assure a minimum value for $\|x(v_1) - c_1\|$, it is not possible to apply a procedure similar to Sharir's one that works correctly for *every* strictly convex gauge, and not even for norms.

But we can apply the results presented in Section 2 and show that Hershberger's $O(n^2)$ Euclidean algorithm works also for every strictly convex gauge. There, the full arrangement of discs of radius r (shortly called *r-discs*) centered at points of K is built. For each r-circle of the arrangement it is explored whether the points not covered by the r-disc can be covered by a separated unit disc.

Now, for gauges and every $\hat{S}(p_j, r)$ with $p_j \in K$, let us choose a parametrization $p(\theta)$ ($\theta \in [0, 2\pi)$) and points q_1, q_2, q_3, q_4 (clockwise ordered) on $\hat{S}(p_j, r)$ such that the support line of $\hat{S}(p_j, r)$ through q_1 and q_3 is the same, and that one through q_2 and q_4 is parallel to $\langle q_1, q_3\rangle$. If $q_5 := q_1$, the arcs (clockwise) meeting q_i and q_{i+1} are $r\hat{B}$-minimal. Let us consider the four disjoints sweeps in the discs determined by these arcs. Steps 1 and 2 below describe the global structure of the algorithm:

Step 1. Build the arrangement of $r\hat{B}$-discs centered at the points of K.

Step 2. For each circle $\hat{S}(p_j, r)$, move $p(\theta)$ along each of the four arcs of the sweep that cover $\hat{S}(p_j, r)$. For every arc, define $F_\theta := \overline{S(p(\theta), r)} \cap K$. Consider the set D_θ of points of F_θ that do not belong to the rB-discs centered at any other previous (meant in the oriented sense of the parametrization of the circle) points of the arc; and consider $A_\theta := F_\theta \setminus D_\theta$.

Step 2(a). Find the order of insertions and deletions to A_θ and D_θ in $O(n)$ time by walking along the boundary of $\hat{S}(p_j, r)$.

Step 2(b). Process the insertions to A_θ in sequence, maintaining $\mathrm{bi}_{\hat{B}}(A_\theta, 1)$. Record the changes to $\mathrm{bi}_{\hat{B}}(A_\theta, 1)$ in a transcript.

Step 2(c). Partition the initial set D_θ into a static set Z of points that will not be deleted during the sweep, and a dynamic set Y_θ that will be deleted. Compute a change-transcript for $\mathrm{bi}_{\hat{B}}(Y_\theta, 1)$, working in time-reversed order; combine this with Z to get a change-transcript for $\mathrm{bi}_{\hat{B}}(D_\theta, 1)$.

Step 2(d). Play the transcripts for A_θ and D_θ simultaneously, both in forward time order (the reverse of the construction order for D_θ). Test whether $\mathrm{bi}_{\hat{B}}(A_\theta, 1) \cap \mathrm{bi}_{\hat{B}}(D_\theta, 1) \neq \emptyset$ during the playback.

The point $p(\theta)$ and every point of this (eventually) non-empty intersection become the centers of a solution for the 2-center problem.

Lemma 3.2. *During any sweep from q_i to q_{i+1}, there are no deletions from A_θ.*

Proof. Let $\theta_1 \leq \theta \leq \theta_2$ be such that $\{p(\theta_1), p(\theta), p(\theta_2)\}$ belong to the sweep from q_i to q_{i+1}. The piece of the sweep on $\hat{S}(p_j, r)$ from $p(\theta_1)$ to $p(\theta_2)$ is an $r\hat{B}$-minimal arc (see (3) in Lemma 2.2). If $x \in B(p(\theta_1), r) \cap B(p(\theta_2), r)$, then $\{p(\theta_1), p(\theta_2)\} \in \hat{B}(x, r)$, and the $r\hat{B}$-minimal arc on $\hat{S}(p_j, r)$ going from

$p(\theta_1)$ to $p(\theta_2)$ is contained in $\hat{B}(x, r)$ (see (2) in Lemma 2.2). Therefore we have $x \in B(p(\theta), r)$, and this means that A_θ does not admit any deletion. □

Lemma 3.3. *If K is a set of n points in a strictly convex generalized normed plane $\mathbb{M}^2$ then, building the arrangement of $r\hat{B}$-circles with $r > 0$ and centered at the points of K, takes $O(n\lambda_4(n))$ time and $O(n^2)$ space.*

Proof. Each pair of curves can have at most two intersection points, therefore the total construction time of the arrangement is $O(n\lambda_4(n))$ ([12]), where $\lambda_\sigma(k)$ denotes the maximal length of a Davenport–Schinzel sequence[1], while the complexity of the arrangement is of course $O(n^2)$. □

Lemma 3.4. *Let $B(x_1, 1)$ and $B(x_2, 1)$ be two strictly convex different discs whose intersection has nonempty interior, and let $t_3 \in S(x_1, 1) \cap S(x_2, 1)$. If S is a third circle of radius 1 with $t_3 \in S$, then S cannot pass simultaneously through points of both arcs that form the boundary of $B(x_1, 1) \cap B(x_2, 1)$.*

Proof. Suppose that $t_1 \in S(x_1, 1)$ and $t_2 \in S(x_2, 1)$ are boundary points of $B(x_1, 1) \cap B(x_2, 1)$, and that there is a circle S that simultaneously contains t_1, t_2, and $t_3 \in S(x_1, 1) \cap S(x_2, 1)$. Consider the clockwise order over S, $S(x_1, 1)$, and $S(x_2, 1)$, and assume that t_1, t_2, t_3 are clockwise on S. Either the arc meeting t_3 and t_1 on S is B-minimal, or the arc meeting t_1 and t_3 on S is B-minimal. In the first case, this B-minimal arc must be equal to the B-minimal arc meeting t_3 and t_1 on $S(x_1, 1)$ (see (1) in Lemma 2.2). Similarly, in the second case the B-minimal arc meeting t_2 and t_3 on S must be equal to the B-minimal arc meeting t_2 and t_3 on $S(x_2, 1)$. Thus, either $S = S(x_1, 1)$ or $S = S(x_2, 1)$. □

Theorem 3.5. *For any strictly convex generalized normed plane, the fixed-radius 2-center problem can be solved in $O(n\lambda_4(n))$ time and $O(n^2)$ space.*

Proof. Having proved Lemma 3.2, we can rewrite the proof of this statement and the strategy for $\mathbb{R}^2$ (presented in [14]) also for strictly convex gauges: properties (P1) and (P2) (of ball hulls and ball intersections), Theorem 2.4, and Lemma 3.4 are useful in order to prove that $\mathrm{bi}_{\hat{B}}(A_\theta, 1)$ (in Step 2.b) and $\mathrm{bi}_{\hat{B}}(Y_\theta, 1)$ (in Step 2.c) can be maintained in $O(n)$ time; Theorem 2.3, Theorem 2.4, and the time cost of $\mathrm{bi}_{\hat{B}}(A_\theta, 1)$ and $\mathrm{bi}_{\hat{B}}(Y_\theta, 1)$ allow to compute a change-transcript for $\mathrm{bi}_{\hat{B}}(D_\theta, 1)$ (Step 2.c) in $O(n)$ time; properties (P1) and (P2) together with the fact that the structure of the boundary of the ball intersection of a finite set K in the strictly convex case is similar to the Euclidean case (it consists of circular arcs of balls with radius r and centers belonging to the set) are used to test Step 2.d in $O(n)$ time. The total cost is bounded by Lemma 3.3. □

[1] $\lambda_4(n) = \Theta(n\,2^{\alpha(n)})$, where $\alpha(n)$ is the inverse of the Ackermann function, grows very slowly and is less than 5 for any practical input size n, e.g., $\alpha(9876!) = 5$ (see `http://www.gabrielnivasch.org/fun/inverse-ackermann`). Thus, $\lambda_4(n)$ is almost linear ([2]).

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