

ON COPS AND ROBBERS ON G^{Ξ} AND COP-EDGE
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ABSTRACT. Cops and Robbers is a two player game played on an undirected graph. In this game the cops try to capture a robber moving on the vertices of a graph. The cop number of a graph, denoted by $c(G)$, is the least number of cops needed to guarantee that the robber will be caught. In this paper we present results concerning games on G^{Ξ} , that is the graph obtained by connecting the corresponding vertices in G and its complement \bar{G} . In particular we show that for planar graphs $c(G^{\Xi}) \leq 3$. Furthermore we investigate the cop edge-critical graphs, i.e. graphs that for any edge e in G we have either $c(G - e) < c(G)$ or $c(G - e) > c(G)$. We show a couple of examples of cop edge-critical graphs having cop number equal to 3.

1. INTRODUCTION

In a graph G , a set S of vertices is a *dominating set* if every vertex not in S has a neighbor in S . The minimum cardinality of a dominating set is the *domination number* of G , denoted by $\gamma(G)$.

The cop number of a graph is a graph parameter related to the domination number $\gamma(G)$ of a graph. The domination number can be seen as a variant of Cops and Robbers. This variant is a vertex pursuit game played on a graph G . There are two players, a set of k cops (or searchers), where $k > 0$ is a fixed integer, and a robber. At the beginning of the game cops place themselves on a set of up to k vertices (more than one cop is allowed to occupy a single vertex), next the robber chooses one vertex. The game proceeds as follows: first the cops move, that is, switch from their vertices to adjacent ones, or

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pass, that is; remain on their current vertex, then the robber moves in the same way as cops (remaining at a vertex or moving to an adjacent one). The game continues with alternating moves by the cops and the robber, which we will call *rounds* or *steps*. The cops win and the game ends if at the end of their round one of the cops occupies the same vertex as the robber; otherwise the robber wins. Players know each others actual position (that is, the game is played with complete information). The minimum number of cops required to catch the robber (regardless of robber's strategy) is called the *cop number* of G , and is denoted $c(G)$. This parameter is well studied for several types of graphs (see [2, 3, 7, 8, 18]). Since the existence of loops or parallel edges has no influence in the results of this paper, throughout the text we consider only simple graphs which are herein called graphs.

We call a graph G *cop win* if a single cop wins the Cop and Robber game on G . A graph G is said to be *cop vertex-critical* if for any vertex v in G either $c(G - v) < c(G)$ or $c(G - v) > c(G)$. Similarly a graph G is *cop edge-critical* if for any edge e in $E(G)$ either $c(G - e) < c(G)$ or $c(G - e) > c(G)$. It is immediate that every totally disconnected graph, i.e., a graph where all its vertices are isolated, of order $n > 1$ is cop vertex-critical. On the other hand, every graph where each connected component is K_2 is cop edge-critical. Another trivial example of a cop edge-critical graph is a tree of order $n > 1$.

If a graph has cop number k and is cop edge-critical we call it k -cop edge-critical. S. L. Fitzpatrick [11] characterized edge critical planar graphs with cop number 2 whose cop number decreases after removal of any edge. A more general study of 2-cop edge-critical graphs is due to N. E. Clarke et al. [10]. To our knowledge, the only known example of a 3-cop edge-critical graph is the Petersen graph and it is due to W. D. Baird et al [4]. In this paper we present new examples of cop edge-critical graphs with cop number equal to 3.

In the next section we present some preliminary results on Cops and Robbers on graphs and their complements. In the third section, particular attention is given to the games on graphs G^Ξ , that is, graphs obtained by connecting the corresponding vertices in G and its complement \overline{G} . Those graphs have already been considered in different contexts, for example in [1]. We show that for planar graphs $c(G^\Xi) \leq 3$. In the last section the cop edge-critical graphs are investigated. Among them we find examples of graphs created by taking G^Ξ . For instance, the Petersen graph is the graph C_5^Ξ . We conclude the paper with a few conjectures and remarks. We believe that in general there are many cop edge-critical graphs among graphs of the form G^Ξ .

2. PRELIMINARY RESULTS

The complement of a graph G is a graph \overline{G} on the same set of vertices such that two distinct vertices of \overline{G} are adjacent if and only if they are

not adjacent in G . In this section we focus on analysing the cop number of the graph G and its complement \bar{G} . Notice that placing a cop on each element of a dominating set ensures that the cops win in at most two rounds, thus $c(G) \leq \gamma(G)$ (see [8]). For the domination number itself, the following bounds have been proved (see [13, 14]).

Proposition 2.1 ([13, 14]). *Consider a graph G of order n , we may conclude the following upper bounds on the domination number,*

$$\begin{aligned} \gamma(G) + \gamma(\bar{G}) &\leq n + 1, \\ \gamma(G) + \gamma(\bar{G}) &\leq \frac{n}{2} + 2, \text{ if } \delta(G), \delta(\bar{G}) \geq 1, \end{aligned}$$

where $\delta(G)$ is the minimum degree of G .

Regarding the cop number, as an immediate consequence of this proposition, we have the following upper bounds.

Proposition 2.2. *Let G be a graph of order $n \geq 4$.*

- a) $c(G) + c(\bar{G}) \leq n + 1$ and the equality holds if and only if either G or \bar{G} is totally disconnected (that is, all vertices are isolated).
- b) $c(G) + c(\bar{G}) \leq n/2 + 2$ if $\delta(G), \delta(\bar{G}) \geq 1$ and the upper bound is attained if G is C_4 .

Proof. Let us prove each of the cases as follows.

- a) Since $c(G) \leq \gamma(G)$, from Proposition 2.1, we may conclude the inequalities

$$c(G) + c(\bar{G}) \leq \gamma(G) + \gamma(\bar{G}) \leq n + 1.$$

Furthermore, it is immediate that if G is totally disconnected, then $c(G) = n$ and $c(\bar{G}) = 1$ and thus the equality holds. Conversely, let us assume that G and \bar{G} are not totally disconnected and the equality holds.

Case 1. Consider the case that either G or \bar{G} contains at least one isolated vertex v . Assume that $d_{\bar{G}}(v) = n - 1$, then $c(\bar{G}) = 1$ and since G has at least one edge we have $c(G) \leq n - 1$. It follows that $c(G) + c(\bar{G}) \leq n - 1 + 1 = n$, which is a contradiction.

Case 2. Now, assume that there exists no isolated vertex either in G or in \bar{G} . This implies that $\delta(G), \delta(\bar{G}) \geq 1$ and from Proposition 2.1,

$$c(G) + c(\bar{G}) \leq \gamma(G) + \gamma(\bar{G}) \leq \frac{n}{2} + 2 \leq n,$$

which is contradiction.

- b) This part is direct consequence of Proposition 2.1.

□

Proposition 2.3. *If G is a disconnected graph, then $c(\bar{G}) \leq 2$.*

Proof. The domination number $\gamma(\bar{G})$ is equal to 2 (since we can consider two vertices from different components of G as a dominating set). □

By $\text{diam}(G)$ we denote the diameter of G which is the greatest distance between any pair of vertices in G .

Lemma 2.4. *If G is a graph with $\text{diam}(G) \geq 3$, then $c(\overline{G}) \leq 2$.*

Proof. Take two vertices w, v such that the length of shortest path connecting this two vertices is greater than 3. We know that $N[w] \cap N[v] = \emptyset$ so in the complement v is adjacent to every vertex in $N[w]$ and the same is true for w and $N[v]$. All the other vertices in \overline{G} are adjacent to both w and v . Thus the domination number of \overline{G} is equal to 2, implying that its cop number is bounded by 2. \square

3. COPS AND ROBBER ON G^Ξ

We define a graph G^Ξ as the graph obtained by the disjoint union of G with its complement \overline{G} and adding a perfect matching between the corresponding vertices of G and \overline{G} . That is, considering the graph G and its complement \overline{G} , every vertex v in G is adjacent to its copy (herein also called a *mirror vertex*) v' in \overline{G} . From now on, we denote each vertex of G by a letter and the corresponding mirror vertex in \overline{G} by the same letter with an apostrophe. Recall that, as previously mentioned, the Petersen graph is isomorphic to C_5^Ξ .

Lemma 3.1. *For every graph G , the graph G^Ξ is connected.*

Proof. It is well known that at least one of G, \overline{G} is connected, and this immediately implies that G^Ξ is connected. \square

Consider a graph G of order n , a vertex ordering (v_1, \dots, v_n) is a cop win ordering (or dismantling ordering) if for each $i < n$, there is $j > i$ such that $N_i[v_i] \subseteq N_i[v_j]$, where $N_i[v_j]$ is the closed neighborhood of the vertex v_j in the subgraph of G induced by the vertices in (v_i, \dots, v_n) (see [9, 16]). A graph is cop win if and only if it has a cop win ordering [16]. According to Bandelt and Prisner [5], the cop-win graphs were introduced by Poston [17] and Quilliot [19] under the name *dismantlable graphs* defined recursively as follows: the trivial graph with just one vertex is dismantlable and a graph G with at least two vertices is dismantlable if there exists two vertices x and y such that $N[x] \subseteq N[y]$ and $G - \{x\}$ is dismantlable.

An induced subgraph H of a graph G is called a *retract* of G if there is a homomorphism ϕ from $V(G)$ onto $V(H)$ such that $\phi(v) = v$ for every $v \in V(H)$; that is ϕ is an identity function on $V(H)$ (see [8]). If G is a connected graph and H is a retract of G , then $c(H) \leq c(G)$ (see [6]).

Proposition 3.2. *Let G be any connected graph of order $n \geq 2$. Then G^Ξ is cop win if and only if either G or \overline{G} is a complete graph.*

Proof. If G is the complete graph K_n , then it is immediate that G^Ξ is cop win. Let us prove the “only if” implication by contraposition, assuming that

G is not complete but (wlog) it is connected. Consider the partition of the vertex set of G^Ξ into the four vertex subsets $K_G \cup K_{\overline{G}} \cup V_G \cup V_{\overline{G}} = V(G^\Xi)$, where K_G is the set of vertices of G with degree equal to $|V(G)|$ in G^Ξ , $K_{\overline{G}}$ is the set of vertices of degree 1 in G^Ξ , $V_G = V(G) - K_G$ and $V_{\overline{G}} = V(\overline{G}) - K_{\overline{G}}$. Retracting all the vertices of the set $K_G \cup K_{\overline{G}}$ into just one vertex z , we obtain a new graph H such that $V(H) = V_G \cup V_{\overline{G}} \cup \{z\}$. Notice that since G is not complete, $|V_G \cup V_{\overline{G}}| \geq 4$ and then H has at least five vertices. Now we show that in H no pair of vertices dominate each other and this implies that H is not dismantlable. In fact, we have that:

- z cannot dominate vertices from $V_{\overline{G}}$ because it does not have neighbors in $V_{\overline{G}}$ and cannot dominate vertices from V_G because every vertex in V_G has one neighbor in $V_{\overline{G}}$,
- each pair of adjacent vertices $x \in V_G$ and $x' \in V_{\overline{G}}$ cannot dominate each other because x has neighbors in $V_G \cup \{z\}$ and x' has neighbors in $V_{\overline{G}}$,
- each pair of adjacent vertices $x, y \in V_G$ ($x, y \in V_{\overline{G}}$) cannot dominate each other because $\exists x' \in N(x) \setminus N(y)$ (resp. $\exists x \in N(x') \setminus N(y')$) and $\exists y' \in N(y) \setminus N(x)$ (resp. $\exists y \in N(y') \setminus N(x')$).

Therefore, H is not dismantlable and thus G is also not dismantlable which is equivalent to saying that G is not cop win. \square

Proposition 3.3. *Let T be a tree of order $n \geq 3$. Then $c(T^\Xi) = 2$.*

Proof. Let v be a leaf of T . Consider the vertex v' corresponding to v in T^Ξ . Place the first cop c_1 on v' . The cop c_1 can guard vertices in $N[v']$, the closed neighborhood of v' . Remaining vertices in $T^\Xi - N[v']$ forms a tree. Place the second cop on this tree. We know from [8] that every tree is cop win. Thus, two cops are always sufficient to defeat a robber in this graph. Moreover, by Proposition 3.2, $c(T^\Xi) \geq 2$. \square

Proposition 3.4. *Let C_n be the cycle of order $n \geq 5$. Then $c(C_n^\Xi) = 3$.*

Proof. Let v' be any vertex among the vertices of the $\overline{C_n}$ part of C_n^Ξ . Place the first cop c_1 on the vertex v' . Now c_1 can guard the vertices in $N[v']$, the closed neighborhood of v' . Notice that remaining vertices in $C_n^\Xi - N[v']$ forms a cycle C_i of order $i \geq 5$ and, since $c(C_i) = 2$, it follows that $c(C_n^\Xi) \leq 3$.

For the reverse inequality let us consider the following two cases.

Case 1: The cycle C_n for $n = 5$.

In this case, C_5^Ξ is the Petersen graph and we know that its cop number is 3 (see [4]).

Case 2: The cycle C_n for $n \geq 6$.

We have just two cops in C_n^Ξ . Let $V = \{v_1, v_2, \dots, v_n\}$ be the vertices in C_n and $V' = \{v'_1, v'_2, \dots, v'_n\}$ be the vertices in $\overline{C_n}$. If both cops starts on vertices inside $\overline{C_n}$ we can place the robber on any vertex of C_n not adjacent to a vertex occupied by a cop. Assume that the cop c_2 is placed on v'_i , $1 \leq i \leq n$. Now we place the robber either on v'_{i+1} or v'_{i-1} by using

the following strategy: If c_1 is on v_{i+1} , then place the robber on v'_{i-1} . If c_1 is on v_{i-1} , then place the robber on v'_{i+1} . Let $M = V - \{v_{i+1}, v_{i-1}\}$. If c_1 is on any vertex from M , then we put the robber on v'_{i+1} or v'_{i-1} arbitrarily. If both cops move (or place themselves) inside $\overline{C_n}$, then the robber is forced to move to a vertex from C_n . As long as the cops stay in $\overline{C_n}$, the robber can move through the cycle C_n (or maintain their position) without being caught. If both c_1 and c_2 moves to the vertices of C_n , the robber immediately moves back to an adjacent vertex from $\overline{C_n}$. If the robber is on C_n while c_1 is on some v_j and c_2 is on some v'_i , the robber can move along the cycle C_n up to the point when they can move to one of the vertices v'_{i+1} or v'_{i-1} . Notice that any of those operations take the robber from a “safe” position to another “safe” position.

This implies that there is no configuration of locations for the cops and the robber in which the robber cannot escape in his next step. Thus we have $c(C_n^\Xi) > 2$. □

It is an easy exercise to show that $c(C_3^\Xi) = 1$ (notice that C_3 is a complete graph) and that $c(C_4^\Xi) = 2$.

Proposition 3.5. *Let G be a graph which is not cop win (that is, $c(G) \geq 2$), then*

$$c(G^\Xi) \leq \max \{c(G), c(\overline{G})\} + 1$$

and the bound is attained if G is C_5 .

Proof. We want to prove that in every case $\max \{c(G), c(\overline{G})\} + 1$ cops suffice to catch the robber on G^Ξ .

Our strategy will be as follows: first we force the robber to go to the \overline{G} part of the graph and then using one cop we prevent the robber from returning to G . In the next step, $\max \{c(G), c(\overline{G})\}$ cops will catch the robber on \overline{G} .

To force the robber to go to \overline{G} , in the first round of the game we place all the cops except for one (let us denote this cop by c_0) inside G . Assume that the robber starts in the vertex $v \in G$. Since $\max \{c(G), c(\overline{G})\} \geq c(G)$ after some number of rounds the robber will be forced to go to \overline{G} (or lose). Let us assume that the robber moves from the vertex $r \in V(G)$ to the mirror vertex r' and also that c_0 occupies a vertex of \overline{G} not adjacent to r' (since otherwise the robber is caught). Then, after switching the two groups of cops (all cops from G moves to \overline{G} and c_0 moves from \overline{G} to G), the cop c_0 is located on a vertex adjacent to r . This implies that in the next round robber cannot return to the mirror vertex r .

We claim that there is either an easy strategy for c_0 to prevent the robber from returning to the graph G or $\gamma(\overline{G}) = 2$.

Assume that the robber moves from vertex r' to t' in \overline{G} and that c_0 stays at vertex c . If $(c, t) \in E(G)$, then c_0 can stay at their position and block

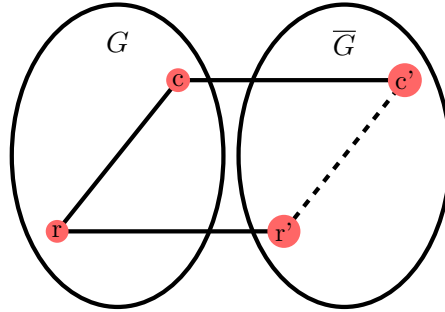


FIGURE 1. The cop c_0 moves from c' to c when the robber appears on r' .

the robber from going back to G . In the second case there is a vertex g connected to current position of the cop such that $(g, t) \in E(G)$ (see Figure 2). In the third case assume that c_0 is on vertex c and there is no vertex g that satisfies our conditions, then every vertex in \overline{G} is connected either to c' or t' . This means that the domination number of \overline{G} is equal to 2 (i.e. t' and c' dominate all of \overline{G}).

In the first two cases, as c_0 will prevent the robber from going back to G and $\max\{c(G), c(\overline{G})\} \geq c(\overline{G})$, the number of cops inside \overline{G} is enough to catch the robber.

In the third case (when $\gamma(\overline{G}) = 2$), we have at least three cops in the game. Let us denote them by c_0, c_1 , and c_2 . When the robber moves, they either end up on a vertex adjacent to a cop and the game is finished or they move to a vertex which is not adjacent to the vertices occupied by any cop. Thus, in the next round, we move c_0 back to the graph \overline{G} and move c_1 onto G . The cop c_1 prevents the robber from moving to G . Next, we move c_2 toward one of the dominating vertices in \overline{G} . In the next round we do the same but with c_2 going onto graph G , c_1 onto graph \overline{G} and c_0 towards one of the vertices from the dominating set (other than the one occupied by c_2). It is easy to see that after few rounds we will obtain a state where both c_0 and c_2 are positioned on vertices of dominating set in \overline{G} and since in every step we prevent the robber from returning to G , they will remain on \overline{G} . Finally we have reached a state where c_1 stops the robber from being able to move to G , and c_0 and c_2 dominates all vertices of \overline{G} . Thus in next step cops will win the game.

□

Proposition 3.6. *For graphs G and \overline{G} with $c(G) \neq c(\overline{G})$, and $c(\overline{G}), c(G) \geq 2$ we have*

$$c(G^\Xi) \leq \max\{c(G), c(\overline{G})\}.$$

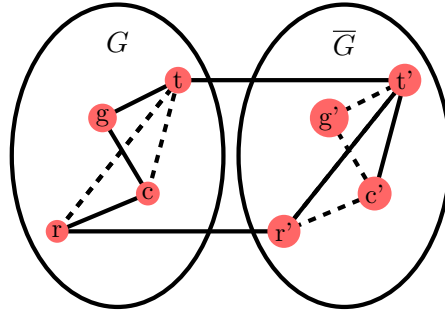


FIGURE 2. The cop c_0 moves from c to g when the robber moves from r' to t' .

Proof. Our goal is to prove that in every case $\max\{c(G), c(\overline{G})\}$ cops suffice to catch the robber on G^Ξ . Without loss of generality assume that $c(G) > c(\overline{G})$.

We use almost the same strategy as described in the previous proof. First (as in the previous proof) we force the robber to go to \overline{G} and then (using a single cop) we prevent the robber from returning to G . In order to force the robber to go to \overline{G} , in the first round of the game, we place all cops on the graph G . If the robber is placed on \overline{G} , then we move all cops on \overline{G} to try to catch the robber there. Since $c(G)$ cops are sufficient to catch the robber on graph G , at some point the robber will be forced to move to the graph \overline{G} (the robber can also make the transition any time earlier when they are not in any danger from the cops). Let r' be the vertex currently occupied by the robber. We take any cop from a vertex $u \in V(G)$ and move them onto its adjacent vertex u' . Let us denote this cop by c_0 . The remaining cops follow the optimal strategy to catch the robber as if the robber stayed on vertex r . There are two possible options for the robber:

- a) The robber returns to G . Then the cops continue the pursuit on this graph. Notice that from the point of view of the cops, the robber remained in the same position for two rounds and all the cops apart from c_0 are one step closer to catching the robber following the optimal strategy on G . As there are at least three cops in play we can position them in a way that the cops reach a game state where the robber cannot return to G without being caught. That is we move $c(G) - 1$ cops toward the optimal strategy and have the remaining cop maintain their position. If robber repeats their behavior of entering and leaving \overline{G} , we alternate the vertices such that after $c(G)$ rounds every cop is one step closer to catching the robber on G . At this stage only option b) remains.
- b) The robber moves from r' onto another vertex $t' \in V(\overline{G})$. Then either $t'u' \in E(\overline{G})$ and the game is over or we move c_0 to the vertex $u \in G$. Notice that when c_0 moves to $u \in G$ they are adjacent to t . This implies that in the next round the robber cannot return to G .

At this stage, we move the remaining $c(G) - 1$ cops onto the graph \overline{G} and then we have the following two possibilities (this analysis is similar to the one used in the previous proposition):

- (i) The diameter of G is equal to 2. This means that we can move c_0 in such a way that c_0 blocks the robber from returning to G . In this case as $c(\overline{G}) < c(G)$ the rest of the cops can catch the robber on \overline{G} and thus winning the game.
- (ii) The diameter of G is greater than or equal 3. From Lemma 2.4, it follows that $c(\overline{G}) \leq 2$ and since we have at least three cops in play the cops are able to win.

□

Proposition 3.7. *Let G be a graph with connected components $\{G_1, G_2, \dots, G_n\}$, $\max\{c(G_1), \dots, c(G_n)\} \neq c(\overline{G})$, and $c(\overline{G}) \geq 2$. Then*

$$c(G^\Xi) \leq \max\{c(G_1), c(G_2), \dots, c(G_n), c(\overline{G})\}.$$

Proof. We can follow the strategy from the previous proposition. First assume that $\max\{c(G_1), \dots, c(G_n)\} > c(\overline{G})$. Moreover assume that the robber starts at some component G_i of G . We then move all the cops through graph \overline{G} onto the component G_i and try to capture the robber there. Notice that the robber cannot move onto another component of G without going through \overline{G} . As soon as the robber moves to \overline{G} and makes a move within this graph we play on \overline{G} with the same strategy as in previous proof. Previous arguments shows that eventually the cops will win the game.

Let us now consider the case when $\max\{c(G_1), \dots, c(G_n)\} < c(\overline{G})$. We start the game on \overline{G} forcing the robber to move onto G . Assume that the robber goes to some component G_i of G (or is already there in the first step). Then after moving some cop, say c_0 , to a vertex $v' \in V(\overline{G})$ which is a copy of a vertex $v \in V(G_j)$ such that $j \neq i$, the robber cannot return to \overline{G} . Therefore, moving the remaining cops to G_i is enough to catch the robber after a finite number of steps. □

Notice that as stated earlier either G or \overline{G} is connected, thus the above proposition covers all possible cases (we cannot have multiple components in both G and \overline{G}). Now, using the above arguments together with Proposition 3.5, we get to the following corollary.

Corollary 3.8. *Let G be a graph with connected components $\{G_1, G_2, \dots, G_n\}$ and $\max\{c(G_1), \dots, c(G_n), c(\overline{G})\} \geq 2$, then*

$$c(G^\Xi) \leq \max\{c(G_1), c(G_2), \dots, c(G_n), c(\overline{G})\} + 1.$$

Furthermore we can bound the cop number of the graphs G^Ξ in terms of the minimum degree of G .

Proposition 3.9. *Let G be a simple graph with $c(G) = k$ and $\delta(G) = m$. Then $c(G^\Xi) \leq \max\{k, m + 1\} + 1$.*

Proof. Consider $V(G) = \{v_1, v_2, \dots, v_n\}$ and $V(\overline{G}) = \{v'_1, v'_2, \dots, v'_n\}$. Assume that v_1 is a vertex in G with minimum degree m and neighbors v_2, \dots, v_{m+1} . Now place a cop on v'_1 . This cop can guard the vertices in $N[v'_1] = \{v_1, v'_1, v'_{m+2}, \dots, v'_n\}$. Now in \overline{G} we have m vertices which remain unguarded. We can then use m cops to guard them. Thus the cop number of \overline{G} is at most $m + 1$. Hence the total number of cops we need to win in this game is at most $\max\{m + 1, k\} + 1$. \square

Before proceeding, it is worth introducing the following proposition.

Proposition 3.10. *Let G be a planar graph and \overline{G} its complement with components $\{\overline{G}_1, \overline{G}_2, \dots, \overline{G}_n\}$. Then $\max\{c(\overline{G}_1), \dots, c(\overline{G}_n)\} \leq 3$.*

Proof. Since G is a planar graph, it has a vertex u of degree less than or equal to five. Let us assume that $u \in V(G)$ is such that $N_G(u) = \{v_1, v_2, \dots, v_t\}$, with $t \leq 5$ and consider the graph \overline{G} which is the complement of G . Then, the vertex u' is adjacent to each vertex of \overline{G} apart from itself and the vertices v'_1, v'_2, \dots, v'_t . If we put the first cop on u' , then it prevents the robber from moving onto any vertex of \overline{G} apart from v'_1, v'_2, \dots, v'_t . Since we know that the Petersen graph is the smallest graph with cop number equal to three (see [4]) two cops are enough to win the game on every connected graph built on the remaining t vertices. Therefore, together with a cop placed on vertex u this implies that three cops can protect any connected component of \overline{G} . \square

Recall the bound obtained in [8] for planar graphs which will be used in the proof of Proposition 3.12 below and in some additional results throughout the paper.

Proposition 3.11 ([8]). *If G is a planar graph, then $c(G) \leq 3$.*

Proposition 3.12. *If G is planar, then $c(G^\Xi) \leq 3$.*

Proof. Assume that G is a planar graph, according to [8], $c(G) \leq 3$ and we have two cases.

Case 1: The diameter of G is greater or equal to 3.

In this case, \overline{G} has domination number 2 and hence $c(\overline{G}) \leq 2$. Thus,

- (1) if $c(G) = 3$, by Proposition 3.7, $c(G^\Xi) \leq 3$ when $c(\overline{G}) = 2$ and hence it is immediate that this inequality also holds when $c(\overline{G}) = 1$;
- (2) if $\max\{c(G), c(\overline{G})\} = 2$, by Corollary 3.8, the inequality holds;
- (3) finally, if $\max\{c(G), c(\overline{G})\} = 1$, the inequality is immediate.

Case 2: The diameter of G is less than three.

We may apply the result of Goddard and Henning [12] which states that every planar graph of diameter 2 has a domination number of at most 2 except for the graph F of Figure 3 which has domination number 3. Since, as it is easy to check, the graph F has cop number equal to 2, it follows that all planar graphs of diameter 2 has a cop number of

at most two. Moreover, by Proposition 3.10 all the components in the complement of G has a cop number of at most 3, thus again, due to Proposition 3.7 and Corollary 3.8, we have $c(G^\Xi) \leq 3$. □

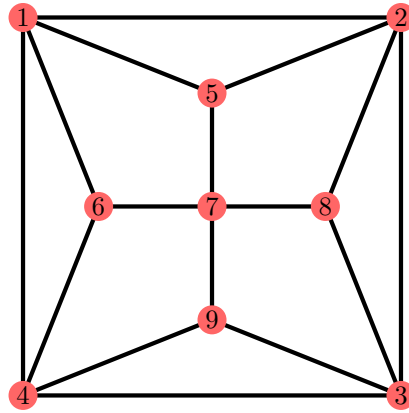


FIGURE 3. The unique planar graph F of diameter 2 with $\gamma(F) = 3$ and $c(F) = 2$.

The *Cartesian product* $G \square H$ of two graphs G and H is the graph with vertex set equal to the Cartesian product $V(G) \times V(H)$, where two vertices (u, v) and (x, y) are adjacent in $G \square H$ if, and only if, $u = x$ and v is adjacent to y in H or $v = y$ and u is adjacent to x in G .

A *two dimensional grid graph* is the graph obtained by the Cartesian product $P_n \square P_m$, where m and n are integers.

Proposition 3.13. *Let P_n be the path of order $n \geq 3$ and let $G = P_n \square K_2$. Then $c(G^\Xi) \leq 3$.*

Proof. Let $G = P_n \square K_2$. Let v_1, v_2, \dots, v_n be the vertices of P_n in G and $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n$ be the vertices in the second copy of P_n in G (let us denote it by \widetilde{P}_n). Also let v'_1, v'_2, \dots, v'_n be the copies of the vertices of P_n in \overline{G} and $\bar{v}'_1, \bar{v}'_2, \dots, \bar{v}'_n$ be the copies of the vertices of \widetilde{P}_n in \overline{G} .

Consider the vertex v'_1 and place the first cop c_0 on the vertex v'_1 . This cop can guard the vertices in $N[v'_1] = \{v_1, v'_1, v'_3, v'_4, \dots, v'_n, \bar{v}'_2, \dots, \bar{v}'_n\}$. The remaining vertices in G^Ξ create a graph with cop number equal to two. □

4. COP EDGE-CRITICAL GRAPHS

According to [8], every tree is cop win. Furthermore, taking into account that a wheel graph W_n is a graph with n vertices ($n \geq 4$), formed by connecting a single vertex to all vertices of an $(n - 1)$ -cycle, we have the following result.

Lemma 4.1 ([8]). *For every integer $n \geq 4$ we have that $c(W_n) = c(P_n) = c(K_n) = 1$ and $c(C_n) = 2$.*

There are graphs which are both cop vertex-critical and cop edge-critical, as it is the case of C_n , with $n \geq 4$. Notice that $c(C_n) = 2$ and both $C_n - \{e\}$ and $C_n - \{v\}$ are paths, which implies that $c(C_n - \{e\}) = c(C_n - \{v\}) = 1$. Furthermore, it is immediate that the cop number of a unicyclic graph with a cycle C_n such that $n \geq 4$ is 2.

Proposition 4.2. *Let G be a unicyclic graph and C_n a cycle in G . If $n \geq 4$, then G is cop edge-critical.*

Proof. Let G be a unicyclic graph and e one of its edges. If we remove e , then we have one of the following cases:

- (1) $G - e$ is a tree (when edge $e \in C_n$) and then $c(G - e) = c(T) = 1$.
- (2) $G - e$ is the sum of a unicyclic graph G_1 and a tree (when $e \notin C_n$).
Therefore, $c(G - e) = c(G_1 \cup T) = c(G_1) + c(T) = 2 + 1 = 3$.

□

Lemma 4.3. *Removing a single edge or a single vertex from a graph G can decrease its cop number by at most one.*

Proof. If we put a single cop onto a removed vertex v (or on the end-vertex of a removed edge e) we get $c(G) \leq c(G - v) + 1$ ($c(G) \leq c(G - e) + 1$). □

As mentioned earlier S. L. Fitzpatrick [11], characterized edge-critical planar graphs whose cop number changes from 2 to 1. A more general study of this problem is due to N. E. Clarke et al. [10], they examined when the cop number of the graph grows from 1 to 2 after addition, deletion, subdivision, or contraction of edges. W. D. Baird et al. [4] proved that the Petersen graph, C_5^Ξ , is the minimum order 3-cop edge-critical graph. The following proposition states that the next two graphs of the family, C_n^Ξ , with $n \geq 5$, have the same property but since $|V(C_n^\Xi)| = 2n$ they have two more and four more vertices, respectively.

Proposition 4.4. *The graphs C_6^Ξ and C_7^Ξ are 3-cop edge-critical.*

Proof. By Proposition 3.4, $c(C_7^\Xi) = c(C_6^\Xi) = 3$. Let $V = \{v_1, v_2, \dots, v_n\}$ be the vertices in C_n and $V' = \{v'_1, v'_2, \dots, v'_n\}$ be the vertices in \overline{C}_n as usual. We will prove that for any $e \in E(C_n^\Xi)$, $c(C_n^\Xi - e) = 2$ when $n \in \{6, 7\}$.

We consider the following three cases for the deleted edge e :

- 1) $e = v_{i-1}v_i$ (an edge connecting two vertices in C_n),
- 2) $e = v'_i v_i$ (an edge between C_n and \overline{C}_n),
- 3) $e = v'_i v'_j$ (an edge connecting two vertices in \overline{C}_n).

Without loss of generality assume that $i = 2$. Consider two cops. In both cases 1) and 2) we start the game by putting these cops on vertices v'_1 and v'_2 .

Case 1: $e = v_{i-1}v_i$ (an edge connecting two vertices in C_n).

The vertices v'_1 and v'_2 dominate all vertices in \overline{C}_n , so the robber has to start the game in C_n . In particular, the robber has to choose one of the vertices in $\{v_3, v_4, \dots, v_n\}$. Let us assume that the robber starts the game on v_k such that $3 \leq k \leq n$. Then we move the cops onto v'_k and v'_{k-1} ; this forces the robber to move to v_{k+1} . We move the cop from v'_{k-1} to v'_{k+1} and so on, until the robber will reach v_1 , where they will be caught (as they cannot move to v_2).

Case 2: $e = v'_i v_i$ (an edge between C_n and \overline{C}_n).

We start the game similarly to the first case and force the robber to move to v_2 , while cops stay on v'_n and v'_1 . Then we move the cops on v'_1 and v'_3 , this forces the robber to stay on v_2 . Next we move the cop on v'_1 to v_1 and the cop on v'_3 to v_3 , then the robber will be caught.

Case 3: $e = v'_i v'_j$ (an edge connecting two vertices in \overline{C}_n).

If we delete an edge of this type from C_6^Ξ (C_7^Ξ), we can find two vertices v'_k and v'_{k+1} of degree three (four) in \overline{C}_6 (\overline{C}_7) whose mirror vertices are adjacent in C_6 (C_7) and such that v'_k and v'_{k+1} dominates all vertices in \overline{C}_6 (\overline{C}_7). Those vertices would be the starting points for cops. We begin the game similarly to Cases 1 and 2 by placing the cops on v'_k and v'_{k+1} and force the robber to move to v_2 . Once the robber moves to v_2 we get a state where the cops stay on v'_3 and v'_1 . Now the robber has two choices: a) the robber can remain at their position or b) the robber can move to v'_2 .

- a) Suppose that the robber remains on the vertex v_2 . Then we move one cop, say c_1 , to v_1 and the other cop, say c_2 , will stay on v'_3 . Now the robber must move to v'_2 and we move c_2 in order to prevent the robber from reaching $N[v'_2]$ (jointly with c_1). Thus, the robber will be captured in the next round.
- b) Suppose the robber moves to v'_2 . Then we move one cop, say c_1 , to the vertex v_1 and the other cop, say c_2 , is moved in order to prevent the robber from reaching $N[v'_2]$ (jointly with c_1). Thus, the robber will be captured in the next round.

□

Notice that a similar result to the above proposition is not valid for the graphs C_n^Ξ such that $n \geq 8$. In fact, by deleting the edge between the vertices $v'_1, v'_{\lfloor n/2 \rfloor + 1} \in V(\overline{C}_n)$, the cop number of C_n^Ξ does not decrease when $n \geq 8$. Hence those graphs are no longer cop edge-critical. Although removing any edge of C_n or any edge between a vertex in C_n and a vertex in \overline{C}_n still do decrease the cop number of C_n^Ξ to 2.

The girth of a graph is the length of a shortest cycle contained in the graph. The minimum degree of G is denoted by $\delta(G)$. In [2] the following elementary but useful result is presented.

Proposition 4.5 ([2]). *If G has girth of at least 5, then $c(G) \geq \delta(G)$.*

Proposition 4.6. *Let G be the dodecahedron graph. Then G is 3-cop edge-critical.*

Proof. Let G be the dodecahedron graph. Then G is a planar 3-regular graph with girth 5. Therefore, from Proposition 4.5 combined with Proposition 3.11, it follows that $c(G) = 3$. Suppose we delete an edge e from G (see, for instance, the graph depicted in Figure 4). By symmetry playing the game on the RHS is equivalent to playing the game on the LHS. Moreover, since the dodecahedron is edge transitive, if we remove any edge e from it, then we would get the same figure as depicted in Figure 4 (consider the dodecahedron embedded in the sphere).

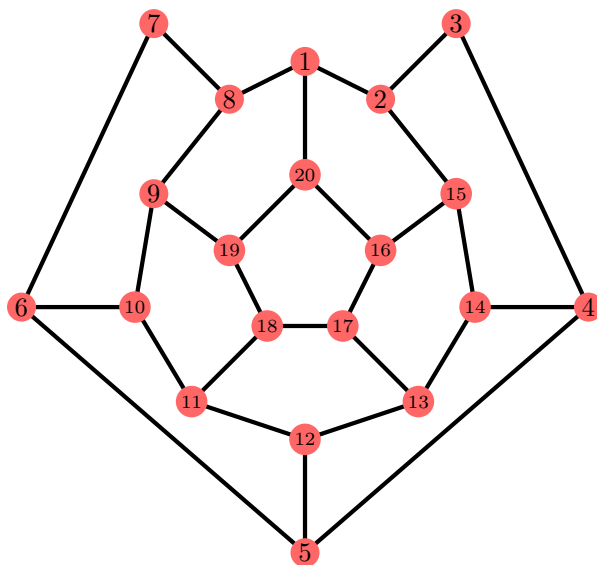


FIGURE 4. The dodecahedron graph with an edge removed.

Claim. $c(G - e) = 2$ for any edge $e \in G$.

We always start the game by placing the cop c_1 on vertex 12 and the cop c_2 on vertex 20. These two cops dominate vertices 1, 16, 19, 11, 13, 5, 20, and 12. Now the remaining vertices on the right hand side of the axis of symmetry are 17, 2, 15, 14, 3, and 4. We play the game by placing the robber at each of these vertices and we get the following strategy for each of the following cases:

Case I: The robber is on vertex 17.

The cops can catch the robber in at most six rounds.

R1. Move c_1 vertex 11 and move c_2 to vertex 16. The robber must move to vertex 13.

R2. Move c_1 to vertex 12. The robber must move to vertex 14.

- R3. Move c_2 to vertex 15. The robber must move to vertex 4.
- R4. Move c_1 to vertex 5. The robber must move to vertex 3.
- R5. Move c_1 to vertex 4 and move c_2 to vertex 2. These two vertices form a dominating set of the robber's neighborhood, so the cops win.

Case II: The robber is on vertex 2.

The cops can catch the robber in at most five rounds.

- R1. Move c_1 to the vertex 13 and move c_2 to vertex 1. Now the robber can move to vertex 3 or 15. Suppose the robber moves to vertex 15.
 - R2. Move c_1 to vertex 14 and move c_2 to vertex 20. The robber must move to vertex 2.
 - R3. Move c_2 to vertex 1. The robber must move to vertex 3.
 - R4. Move c_1 to vertex 4 and move c_2 to vertex 2. The game finishes in the next round.
- Now suppose that the robber instead moved to vertex 3 during the first round.
- R2. Move c_1 to vertex 14 and move c_2 to vertex 2. The game finishes in the next round.

Case III: The robber is on vertex 15.

The cops can catch the robber in at most four rounds.

- R1. Move c_1 to vertex 13. The robber either stays on vertex 15 or moves to vertex 2. Suppose the robber stays on vertex 15.
 - R2. Move c_1 to vertex 14. The robber must move to vertex 2.
 - R3. Move c_2 to vertex 1. The robber must move to vertex 3.
 - R4. Move c_1 to vertex 4. The cop c_2 moves to vertex 2. The game finishes in the next round.
- Now suppose that the robber instead moved to vertex 2 during the first round.
- R2. Move c_1 to vertex 14 and move c_2 to vertex 1. The robber must move to vertex 3.
 - R3. Move c_1 to vertex 4 and move c_2 to vertex 2. The game finishes in the next round.

Case IV: The robber is on the vertex 14.

The cops can catch the robber in at most four rounds.

- R1. Move cop c_2 to vertex 16. The robber either stays on the vertex 14 or moves to vertex 4. Suppose the robber stays on vertex 14.
 - R2. Move c_2 to vertex 15. The robber must move to vertex 4.
 - R3. Move c_1 to vertex 5. The robber must move to vertex 3.
 - R4. Move c_1 to vertex 4 and move c_2 to vertex 2. The game finishes in the next round.
- Now suppose that the robber instead moved to vertex 4 during the first round.

R2. Move c_1 to vertex 5, and move c_2 to vertex 15. The robber must move to vertex 3.

R3. Move c_1 to vertex 4 and move c_2 to vertex 2. The game finishes in the next round.

Case V: The robber is on vertex 3.

The cops can catch the robber in at most two rounds.

R1. Move c_1 to vertex 5 and move c_2 to vertex 1. The robber only can stay on vertex 3.

R2. Move c_1 to vertex 4 and move c_2 to vertex 2. The game finishes in the next round.

Case VI: The robber is on vertex 4.

The cops can catch the robber in at most four rounds.

R1. Move c_2 to vertex 16. The robber either stays on vertex 4, moves to vertex 3, or moves to vertex 14.

R2. Move c_2 to vertex 15. The robber must be on vertex 4 or 3.

R3. Move c_1 to vertex 5. The robber must be on vertex 3.

R4. Move c_2 to vertex 2 and move c_1 to vertex 4. Then the game finish in the next round.

Thus we need only two cops to win the game on the graph depicted in Figure 4. \square

The Heawood graph is the point or line incidence graph on the Fano plane. It has 14 vertices, 21 edges, it is cubic, and all cycles in this graph have six or more edges.

Proposition 4.7. *Let G be the Heawood graph. Then G is 3-cop edge-critical.*

Proof. Let G be the Heawood graph. Then G has girth 6. Therefore from Proposition 4.5 we have $c(G) \geq 3$. If we show that $c(G - e) = 2$, from Lemma 4.3 we get the conclusion that $c(G) = 3$. Suppose we delete an edge e from G see, for instance, the graph depicted in Figure 5. By symmetry playing the game on the LHS is equivalent to playing the game on the RHS. Moreover as the Heawood graph is edge transitive, if we remove any edge e from it, we get the same figure as depicted in Figure 5.

Claim. $c(G - e) = 2$ for any edge $e \in G$.

We always start the game by placing the cop c_1 on vertex 10 and the cop c_2 on vertex 7. These two cops dominate the vertices 1, 2, 6, 7, 8, 9, 10, and 11. Now the remaining vertices on the right hand side of the axis of symmetry are 3, 4, and 5. We play the game by placing the robber at each of these vertices and we get the following strategy for each of the following cases:

Case I: The robber is on vertex 5.

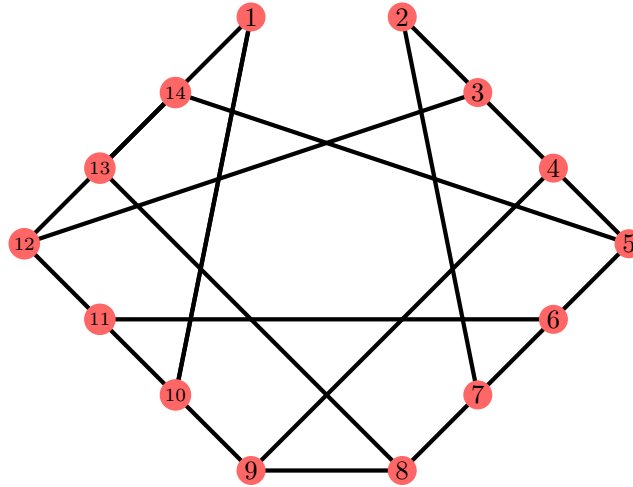


FIGURE 5. The Heawood graph with an edge removed.

The cops can catch the robber in at most five rounds.

- R1. Move c_1 to vertex 1 and move c_2 to vertex 6. The robber must move to vertex 4.
- R2. Move c_1 to vertex 10 and move c_2 to vertex 5. The robber must move to vertex 3.
- R3. Move c_1 to vertex 11 and move c_2 to vertex 4. The robber must move to vertex 2.
- R4. Move c_1 to vertex 6 and move c_2 to vertex 3. In this case we can observe that there is no opportunity for the robber to escape. If the robber moves they will be caught by c_1 and when they stay in their position they will be caught by c_2 . The game finishes in the next round.

Case II: The robber is on vertex 4.

The cops can catch the robber in at most five rounds.

- R1. Move c_2 to vertex 6. The robber can stay on vertex 4 (then after this round we are in a situation described in Case I, R2) or move to vertex 3 (we will investigate this case).
- R2. Move c_1 to vertex 11 and move c_2 to vertex 5. The robber can stay on vertex 3 (then after this round we are in the situation described in Case I R3) or move to vertex 2 (we investigate this case).
- R3. Move c_1 to vertex 6 and move c_2 to vertex 4. The robber has to stay on vertex 2 (then after this round we are in a situation described in Case I R4) or the cops win in the next round. Thus, in both cases the cops win the game.

Case III: The robber is on vertex 3.

The cops can catch the robber in at most five rounds.

R1. Move c_1 to vertex 11 and move c_2 to vertex 6. The robber can stay on vertex 3 (we investigate this as subcase R2a), move to vertex 2 (we investigate this case as subcase R2b), or move to vertex 4 (then after this round we are in a situation described in Case I R2).

R2a. Move c_2 to vertex 5. The robber stays on vertex 3 or moves to vertex 2 (both possibilities are described in Case II R2).

R2b. Move c_1 to vertex 12 and move c_2 to vertex 7. The robber can stay on vertex 2 (in next round they will be caught by c_2) or move to vertex 3 (then in the next round they will be caught by c_1).

Thus, we only need two cops to win the game on the graph depicted in Figure 5.

□

5. OPEN PROBLEMS

There are several interesting questions regarding families of 3-cop-edge critical graphs. The most general one is the following.

Problem 5.1. *Give an example of an infinite family of 3-cop edge-critical graphs or prove that such family does not exist.*

We believe that such a family might be “hidden” among the graphs G^Ξ .

Given a positive integer s , we say that v is an s -trap if one can place s cops on the vertices of $G - v$ such that v and all neighbors of v are adjacent to the vertices occupied by the cops. The next conjecture follows naturally from the given examples of 3-cop-edge critical graphs (since all of them preserve this property).

Conjecture. *If G is a graph such that $c(G) = 3$ and removing any edge would create a 2-trap, then G is 3-cop edge-critical.*

Problem 5.2. *Does there exist a graph G with diameter 2 and $c(G) \geq 3$ such that $c(\overline{G}) = c(G)$? If this is the case, is G^Ξ cop edge-critical?*

Notice that it is conjectured that graphs of order n and diameter 2 have a cop number bounded by \sqrt{n} (see [15, 20]).

Problem 5.3. *Characterize graphs G for which $c(\overline{G}) = c(G)$.*

Problem 5.4. *Characterize graphs G for which $c(G^\Xi) = c(G)$.*

In particular, we know that there are graphs G such that $c(\overline{G}) = k = c(G)$ with $k \leq 2$, as it is the case for P_4 and C_5 , and on the other hand $c(C_4^\Xi) = c(C_4) = 2$. However, the above problems remain open.

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REFERENCES

1. A. Abiad, P. DeOrsey, L. Hogben, K. Hogenson, F. Kenter, J. C.-H. Lin, S. Loeb, H. Smith, and M. Young, *Zero forcing number on the counterprism of graphs*, manuscript.
2. M. Aigner and M. Fromme, *A game of cops and robbers*, Discrete Appl. Math. **8** (1984), 1–11.
3. W. D. Baird, *Cops and robbers on graphs and hypergraphs*, 2011, Theses and dissertations. Paper 821.
4. W. D. Baird, A. Beveridge, P. Codenotti, A. Maurer, J. McCauley, and S. Valeva, *On the minimum order of k -cop-win graphs*, Contrib. Discrete Math. **9** (2014), 70–84.
5. H.-J. Bandelt and E. Prisner, *Clique graphs and Helly graphs*, J. Combin. Theory B **51** (1991), 34–45.
6. A. Berarducci and B. Intrigrila, *On the cop number of a graph*, Adv. Appl. Math. **14** (1993), 389–403.
7. A. Beveridge, A. Maurer, J. McCauley, and S. Valeva, *Cops and robbers on planar graphs*, <https://apps.carleton.edu/curricular/math/assets/PursuitEvasionReport3.pdf>, report.
8. A. Bonato and R. J. Nowakowski, *The game of cops and robbers on graphs*, Student Mathematical Library, vol. 61, American Mathematical Society, 2011.
9. A. Brandstädt, V. B. Le, and J. P. Spinrad, *Graph classes : a survey*, Monographs on Discrete Mathematics and Applications, vol. 3, SIAM, 1999.
10. N. E. Clarke, S. L. Fitzpatrick, A. Hill, and R. J. Nowakowski, *Edge critical cops and robber*, Discrete Math. **310** (2010), 2299–2309.
11. S. L. Fitzpatrick, *Edge-critical cops and robber in planar graphs*, Discrete Math. **329** (2014), 1–11.
12. W. Goddard and M. A. Henning, *Domination in planar graphs with small diameter*, J. Graph Theory **40** (2002), 1–25.
13. F. Jaeger and C. Payan, *Relations du type Nordhais-Gaddum pour le nombre d'absorption d'un graphe simple*, C. R. Acad. Sci. Paris S'er. A **274** (1972), 728–730.
14. J. P. Joseph and S. Arumugam, *Domination in graphs*, Internat. J. Management Systems **11** (1995), 177–182.
15. G. MacGillivray and K. Seyffarth, *Domination numbers of planar graphs*, J. Graph Theory **22** (1996), 213–229.
16. R. Nowakowski and P. Winkler, *Vertex-to-vertex pursuit in a graph*, Discrete Math. **43** (1983), 235–239.
17. T. Poston, *Fuzzy geometry*, Ph.D. thesis, University of Warwick, 1971.
18. P. Pralat, *When does a random graph have constant cop number?*, Australian Journal of Combinatorics **46** (2010), 285–296.
19. A. Quilliot, *Homomorphisms, points fixes, rétractions et jeux de poursuite dans les graphs, les ensembles ordonnés et les espaces métriques*, Ph.D. thesis, Université de Paris VI, 1983.
20. Z. A. Wagner, *Cops and robbers on diameter two graphs*, arXiv:1312.0755v3, 2014.

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