Contributions to Discrete Mathematics

Volume 15, Number 2, Pages 27–41
ISSN 1715-0868

DIAMETER 3

MIODRAG SOKIĆ

ABSTRACT. Let $\mathcal{M}$ be the class of finite metric spaces with nonzero distances in the set $\{1, 2, 3\}$ omitting triangles of the form $(2, 3, 3)$ and $(3, 3, 3)$. We prove that the class of linearly ordered structures from $\mathcal{M}$ satisfies the Ramsey property.

1. INTRODUCTION

Let $\mathcal{M}$ be the class of finite metric spaces (A, d^A) with the property that for all $x, y, z \in A$ we have:

$$d^A(x, y) \in \{0, 1, 2, 3\},$$
$$d^A(x, y) = d^A(x, z) = 3 \Rightarrow d^A(y, z) = 1.$$

We may define $\mathcal{M}$ as the class of finite metric spaces with nonzero distances in $\{1, 2, 3\}$ which avoid equilateral triangles with side 3 and isosceles triangles with two sides equal to 3 and the third side equal to 2. We denote by $\mathcal{OM}$ the class of structures of the form $(A, d^A, \leq^A)$ where $(A, d^A) \in \mathcal{M}$ and $\leq^A$ is a linear ordering on the set A. We prove the following result about the Ramsey property, see Section 3 for more details.

Theorem 1.1. *$\mathcal{OM}$ is a Ramsey class.*

The proof of this theorem is based on the partite construction, see [7]. We use only one partite construction in contrast with the multiple application of partite constructions in the proof of the Ramsey property for the finite ordered metric spaces in [6]. Class $\mathcal{M}$ appears on the list given in [1]. In the following section, we introduce the basic notation. Section 3 is dedicated to the preparation for the main proof given in Section 4. We recall a definition of the ordering property and prove the corresponding statement in Section 5. We conclude with the connection to a different class of metric spaces with distances in $\{0, 1, 2, 3\}$.

Received by the editors March 29, 2017, and in revised form October 1, 2019.
2000 *Mathematics Subject Classification.* Primary: 05D10; Secondary 05C75.
Key words and phrases. Ramsey property, graphs, metric space.
I would like to thank Gregory Cherlin for discussions on the class of metric spaces of diameter 3 during the fall of 2013.

2. Notation

For a natural number n we write $[n]$ instead of $\{1, 2, ..., n\}$. We denote the cardinality of a given set A by $|A|$. For sets A and B, we denote by A^B the collection of all maps from B to A. In particular if $B = [n]$ for some natural number $n \geq 1$ then we write A^n instead of $A^{[n]}$. We denote the disjoint union of sets by $\sqcup$. Let $L = \{R_i : i \in I\}$ be a relational signature with respective arities $(n_i)_{i\in I}$. If $\mathbb{A}$ is a structure in L then we write $\mathbb{A} = (A, (R_i^A)_{i\in I})$ where A is the underlying set and each R_i^A is the interpretation of the relational symbol R_i in the structure $\mathbb{A}$. Let $\mathbb{B} = (B, (R_i^B)_{i\in I})$ also be a structure in L. We say that a map $f : A \to B$ is an *embedding* from $\mathbb{A}$ into $\mathbb{B}$ if for all $i \in I$ and all $x_1, ..., x_{n_i} \in A$ we have

$$R_i^A(x_1, ..., x_{n_i}) \Leftrightarrow R_i^B(f(x_1), ..., f(x_{n_i})).$$

If there is an embedding from $\mathbb{A}$ into $\mathbb{B}$ then we write $\mathbb{A} \hookrightarrow \mathbb{B}$. If an embedding from $\mathbb{A}$ into $\mathbb{B}$ is given by the identity map then we say that $\mathbb{A}$ is a *substructure* of $\mathbb{B}$ and we write $\mathbb{A} \leq \mathbb{B}$. An embedding from $\mathbb{A}$ into $\mathbb{B}$ which is a bijection is called an *isomorphism* from $\mathbb{A}$ to $\mathbb{B}$ and we say that $\mathbb{A}$ and $\mathbb{B}$ are *isomorphic* and write $\mathbb{A} \cong \mathbb{B}$. We denote negation of the relations $\hookrightarrow$, $\leq$, $\cong$, respectively, by $\not\hookrightarrow$, $\not\leq$, $\not\cong$. We use the binomial notation for $\binom{\mathbb{B}}{\mathbb{A}} = \{\mathbb{A}' \leq \mathbb{B} : \mathbb{A}' \cong \mathbb{A}\}$. For $S \subseteq A$ we write $R_i^A \restriction S$ instead of $R_i^A \cap S^{n_i}$ as well as $\mathbb{A} \restriction S$ instead of $(S, (R_i^A \restriction S)_{i\in I})$. For $S \subseteq A$ we say that $\mathbb{A} \restriction S$ is a *substructure of* $\mathbb{A}$ *induced by* S. For $L' = \{R_i : i \in I'\} \subseteq L$ we denote by $\mathbb{A} \restriction L'$ the structure $(A, (R_i^A)_{i\in I'})$. We say that $\mathbb{A} \restriction L'$ is the *reduct* of $\mathbb{A}$ to L' or that $\mathbb{A}$ is an *expansion* of $\mathbb{A} \restriction L'$ in L. Let $\mathcal{K}$ be a class of structures in L. Then we write $\mathcal{K} \restriction L' = \{\mathbb{A} \restriction L' : \mathbb{A} \in \mathcal{K}\}$ and we say that $\mathcal{K} \restriction L'$ is the *reduct* of $\mathcal{K}$ to L' or that $\mathcal{K}$ is an *expansion* of $\mathcal{K} \restriction L'$ to L. If $\leq \notin L$ is a binary relational symbol then we denote by $\mathcal{OK}$ the class of structures $(A, (R_i^A)_{i\in I}, \leq^A)$ with the property that $(A, (R_i^A)_{i\in I}) \in \mathcal{K}$ and $\leq^A$ is a linear ordering on the set A. In particular, $\mathcal{K} = \mathcal{OK} \restriction L$. If $\leq$, $\preceq$, $\sqsubseteq$ are linear orderings then we denote strict parts of this orderings by $<$, $\prec$, $\sqsubset$, respectively.

3. Classes

Let $\mathcal{K}$ be a class of finite relational structures. Let $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$ be structures in $\mathcal{K}$. Let r and t be nonzero natural numbers. If for every coloring $\chi : \binom{\mathbb{C}}{\mathbb{A}} \to [r]$ there is $\mathbb{B}' \in \binom{\mathbb{C}}{\mathbb{B}}$ such that $\chi \restriction \binom{\mathbb{B}'}{\mathbb{A}}$ takes at most t many values, then we write $\mathbb{C} \to (\mathbb{B})^{\mathbb{A}}_{r,t}$. If t_0 is the smallest natural number such that for every natural number $r \geq 1$ and every $\mathbb{B} \in \mathcal{K}$ there is $\mathbb{C} \in \mathcal{K}$ such that $\mathbb{C} \to (\mathbb{B})^{\mathbb{A}}_{r,t_0}$, then we say that t_0 is the *Ramsey degree* of $\mathbb{A}$ in $\mathcal{K}$. We denote this by $t_0 = t_{\mathcal{K}}(\mathbb{A})$. If $t_{\mathcal{K}}(\mathbb{A}) = 1$ then we say that $\mathbb{A}$ is a *Ramsey object* in $\mathcal{K}$. If we have $t_{\mathcal{K}}(\mathbb{A}) = 1$ for all $\mathbb{A} \in \mathcal{K}$ then we say that $\mathcal{K}$ is a *Ramsey class* or that $\mathcal{K}$ satisfies the *Ramsey property* (RP).

For the rest of the paper, R_1, R_2, R_3, and $\leq$ will be binary relational symbols. We denote by $\mathcal{K}$ the class of finite structures (A, R_1^A, R_2^A, R_3^A) with the property that for all $x, y, z \in A$ and all $i, j \in [3]$ we have

$$\neg R_i^A(x,x),$$
$$R_i^A(x,y) \Rightarrow R_i^A(y,x),$$
$$(i \neq j, R_i^A(x,y)) \Rightarrow \neg R_j^A(x,y).$$

We may consider $\mathcal{K}$ as the class of simple graphs with three types of edges. Note that structures in $\mathcal{K}$ may contain vertices without an edge between them. Let $\mathbb{A} = (A, R_1^A, R_2^A, R_3^A) \in \mathcal{K}$ be such that $A = \{x, y, z\}$, $R_i^A(x,y)$, $R_j^A(y,z)$, and $R_k^A(y,z)$. Then we say that $\mathbb{A}$ is an (i,j,k) *triangle*. For $i \in [3]$ we consider structures, triangles, and $\mathbb{T}_i = (T_i, R_1^{T_i}, R_2^{T_2}, R_3^{T_3})$ in $\mathcal{K}$ given by:

$$T_1 = T_2 = T_3 = \{v_1, v_2, v_3\},$$
$$R_3^{T_1}(v_1,v_2), R_3^{T_1}(v_2,v_3), R_3^{T_1}(v_3,v_1),$$
$$R_3^{T_2}(v_1,v_2), R_1^{T_2}(v_1,v_3), R_1^{T_3}(v_3,v_2),$$
$$R_2^{T_3}(v_1,v_2), R_3^{T_3}(v_2,v_3), R_3^{T_3}(v_3,v_1).$$

We define $\mathcal{F}$ as the class:

$$\mathcal{F} = \{\mathbb{A} \in \mathcal{K} : (\forall i \in [3])(\mathbb{T}_i \not\hookrightarrow \mathbb{A})\},$$

and we define $\mathcal{R}$ as the class of structures $(A, R_1^A, R_2^A, R_3^A) \in \mathcal{F}$ with the property that for all distinct $x, y \in A$ we have

$$R_1^A(x,y) \vee R_2^A(x,y) \vee R_3^A(x,y).$$

We can say that $\mathcal{F}$ is the class of structures in $\mathcal{K}$ which forbids $(3,3,3)$, $(2,3,3)$, and $(1,1,3)$ triangles. Note that $\mathcal{R}$ is also the class of structures that forbids the same triangles, but it contains only complete graphs.

We consider a map

$$\Delta : \mathcal{M} \to \mathcal{R},$$
$$\Delta((A, d^A)) = (A, R_1^A, R_2^A, R_3^A),$$

where for $x, y \in A$ and $i \in [3]$ we have $R_i^A(x,y) \Leftrightarrow d^A(x,y) = i$. Structure $\mathbb{T}_1$ corresponds to an equilateral triangle with sides equal to 3. Structure $\mathbb{T}_3$ corresponds to an isosceles triangle with two sides equal to 3 and the third side equal to 1. Structure $\mathbb{T}_2$ corresponds to the only possible violation of the triangle inequality. Now it is easy to see that Δ is a bijection between $\mathcal{M}$ and $\mathcal{R}$ which respects embeddings. Therefore, Theorem 1.1 is equivalent to the following.

Theorem 3.1. *$\mathcal{OR}$ is a Ramsey class.*

We will use the following result.

Theorem 3.2 ([7])**.** *$\mathcal{OF}$ is a Ramsey class.*

We will use Lemma 3.3 in Section 5. When we recall Lemma 3.3 we do not refer only to the statement of the lemma but also to particular construction presented in the proof of this lemma. We refer to Lemma 3.3 (a) as a joint embedding property and to Lemma 3.3 (b) as an amalgamation property. The amalgamation property in Lemma 3.3 is obtained directly, while the amalgamation property in [1] follows from the homogeneous property for an infinite structure.

Lemma 3.3. *Let $\mathbb{A} = (A, R_1^A, R_2^A, R_3^A)$, $\mathbb{B} = (B, R_1^B, R_2^B, R_3^B)$ and $\mathbb{C} = (C, R_1^C, R_2^C, R_3^C)$ be structures in $\mathcal{R}$.*

(a) *If $A \cap B = \emptyset$ then there is $\mathbb{D} = (D, R_1^D, R_2^D, R_3^D) \in \mathcal{R}$ such that $D = A \cup B$ and $\mathbb{A} \leq \mathbb{D}$ and $\mathbb{B} \leq \mathbb{D}$.*

(b) *If $\mathbb{A} \leq \mathbb{B}$, $\mathbb{A} \leq \mathbb{C}$, and $A = B \cap C$ then there is a structure $\mathbb{D} = (D, R_1^D, R_2^D, R_3^D) \in \mathcal{R}$ such that $D = B \cup C$ and $\mathbb{A} \leq \mathbb{D}$, $\mathbb{B} \leq \mathbb{D}$, and $\mathbb{C} \leq \mathbb{D}$.*

Proof. (a) We take $R_1^D = R_1^A \cup R_1^B$, $R_3^D = R_3^A \cup R_3^B$, and $R_2^D = R_2^A \cup R_2^B \cup \{(a,b) : a \in A, b \in B\} \cup \{(b,a) : a \in A, b \in B\}$. It is easy to see that $\mathbb{A} \leq \mathbb{D}$ and $\mathbb{B} \leq D$. In order to verify that $\mathbb{D} \in \mathcal{R}$, we must check that $\mathbb{D}$ does not contain substructure isomorphic to $\mathbb{T}_i$ for some $i \in [3]$. Since $\mathbb{A}$, $\mathbb{B} \in \mathcal{R}$, if there is a triangle $\mathbb{T} \leq \mathbb{D}$ isomorphic to some $\mathbb{T}_i$, then its underlying set contains one point in A and the other two in B or vice versa. Then $\mathbb{T}$ is a $(2,2,l)$ triangle, so it is not isomorphic to any $\mathbb{T}_i$. This is a contradiction which proves $\mathbb{D} \in \mathcal{R}$.

(b) First, we define

$$R_1^D = R_1^A \bigcup R_1^B \bigcup \Big\{(b,c),(c,b) : b \in B \backslash C, c \in C \backslash B, (\exists a \in A)(R_3^B(b,a), R_3^C(c,a))\Big\}.$$

We take $R_3^D = R_3^B \cup R_3^C$ and define

$$R_2^D = R_2^A \bigcup R_2^B \bigcup \Big\{(b,c),(c,b) : b \in B \backslash C, c \in C \backslash B, \neg R_1^D(b,c), \neg R_3^D(b,c)\Big\}.$$

Clearly $\mathbb{A} \leq \mathbb{D}$, $\mathbb{B} \leq \mathbb{D}$, and $\mathbb{C} \leq \mathbb{D}$. Let $\mathbb{T} = (T, R_1^T, R_2^T, R_3^T) \leq \mathbb{D}$ be a triangle with $T = \{t_1, t_2, t_3\}$. Suppose that $\mathbb{T}$ is isomorphic to some $\mathbb{T}_i$. Since $\mathbb{B}$, $\mathbb{C} \in \mathcal{R}$, either triangle $\mathbb{T}$ has two vertices in $B \triangle C$ and one in A, or all vertices of $\mathbb{T}$ are in $B \triangle C$. Now we consider the first case. According to the definition of R_1^D, we have $\mathbb{T} \ncong \mathbb{T}_1$. Since $R_3^D = R_3^B \cup R_3^C$, $\mathbb{T}$ is not a $(3,3,2)$ triangle. Therefore, $\mathbb{T}$ must be a $(1,1,3)$ triangle. If $t_1 \in A$, $t_2 \in B \backslash C$, and $t_3 \in C \backslash B$, then we have $R_3^B(t_1, t_2)$ or $R_3^B(t_1, t_3)$ since $R_3^D = R_3^B \cup R_3^C$. We discuss $R_3^B(t_1, t_2)$; the other case is similar. We have $R_1^D(t_2, t_3)$, so there is $t \in A$ such that $R_3^B(t, t_2)$ and $R_3^C(t, t_3)$. Since $\mathbb{B} \in \mathcal{R}$, $R_3^B(t, t_2)$, and $R_3^B(t_1, t_2)$, we have $R_1^B(t, t_1)$. Since $\{t, t_1\} \subseteq C$, $R_1^C(t, t_1)$, $R_3^C(t, t_3)$, and

$R_1^C(t_1, t_3)$, we have $\mathbb{C} \restriction \{t, t_1, t_3\} \cong \mathbb{T}_2$. This is in contradiction with $\mathbb{C} \in \mathcal{R}$, so $\mathbb{T}$ is not a $(1, 1, 3)$ triangle. Now we have to consider the second case when all vertices of $\mathbb{T}$ are in $B \triangle C$. Without loss of generality we may assume that $t_1 \in B \backslash C$ and $\{t_2, t_3\} \subseteq C \backslash B$. If $\mathbb{T} \cong \mathbb{T}_i$ for some $i \in [3]$, then $R_3^C(t_2, t_3)$, $\neg R_3^D(t_1, t_2)$, and $\neg R_3^D(t_1, t_3)$ according to the definition of R_i^D for $i \in [3]$. Then $\mathbb{T}$ is neither a $(3, 3, 3)$ triangle nor a $(2, 3, 3)$ triangle so $\mathbb{T} \cong \mathbb{T}_2$. In particular, we have $R_1^D(t_1, t_2)$ and $R_1^D(t_1, t_3)$. According to the definition of R_1^D, there are $t', t'' \in A$ such that $R_3^B(t', t_1)$, $R_3^C(t', t_2)$, $R_3^B(t'', t_1)$, and $R_3^C(t'', t_3)$. Since $\{t', t_2, t_3\} \subseteq C$, $R_3^C(t', t_2)$, $R_3^C(t_2, t_3)$, and $\mathbb{C} \in \mathcal{R}$ we have $R_1^C(t', t_3)$. Moreover, we have $t' \neq t''$. Since $\{t', t'', t_1\} \subseteq B$, $R_3^B(t'', t_1)$, $R_3^B(t', t_1)$, and $\mathbb{B} \in \mathcal{R}$ we have $R_1^B(t', t'')$. Now we have $\mathbb{C} \restriction \{t', t'', t_3\} \cong \mathbb{T}_2$. This is in contradiction with $\mathbb{C} \in \mathcal{R}$, so $\mathbb{T}$ is not a $(1, 1, 3)$ triangle. This completes the second case and verification that $\mathbb{D} \in \mathcal{R}$. $\square$

Let $\mathcal{A}$ be a finite set and let $n \geq 1$ be a given natural number. Let $[n] = M \sqcup F$ be a partition such that $M \neq \emptyset$. Let $s_F = (s_i)_{i \in F}$ be a sequence in $\mathcal{A}^F$. Then we say that

$$\Sigma = \{(x_i)_{i=1}^n \in \mathcal{A}^n : (\forall i \in F)(x_i = s_i), (\forall i, i' \in M)(x_i = x_{i'})\}$$

is a *Hales–Jewett line* (HJ line) in $\mathcal{A}^n$. We say that M is the *moving part* and that F is the *fixed part* of the HJ line Σ. We write $M = M(\Sigma)$ and $F = F(\Sigma)$. We denote sequence s_F by $(s_i)_{i \in F(\Sigma)}$. Note that we allow $F = \emptyset$ which implies that the HJ line has only a moving part. We denote the collection of all HJ lines in $\mathcal{A}^n$ by $\mathrm{HJ}(\mathcal{A}^n)$. We need the following Ramsey result.

Theorem 3.4 ([3]). *For a nonempty finite set $\mathcal{A}$ and a natural number $r \geq 1$, there is a minimal natural number $N \geq 1$ such that for every natural number $n \geq N$ and for every coloring $\chi : \mathcal{A}^n \to [r]$, there is $\Sigma \in \mathrm{HJ}(\mathcal{A}^n)$ such that $\chi \restriction \Sigma$ is constant.*

We denote the number N given in the previous theorem by $\mathrm{HJ}(\mathcal{A}, r)$.

4. Proof of Theorem 3.1

We use a variant of the partite construction in the proof.

Let $\mathbb{A} = (A, R_1^A, R_2^A, R_3^A, \leq^A)$ and $\mathbb{B} = (B, R_1^B, R_2^B, R_3^B, \leq^B)$ be structures in $\mathcal{OR}$ such that $\binom{\mathbb{B}}{\mathbb{A}} \neq \emptyset$. Let $r \geq 1$ be a fixed natural number. Since $\mathcal{OR} \subseteq \mathcal{OF}$, according to Theorem 3.2 there is $\mathbb{C} = (C, R_1^C, R_2^C, R_3^C, \leq^C) \in \mathcal{OF}$ such that $\mathbb{C} \to (\mathbb{B})_r^{\mathbb{A}}$. Structure $\mathbb{C}$, as a member of the class $\mathcal{OF}$, may contain vertices which are not connected by an edge, so we need to fill these "empty edges" in a proper way. Let $(\mathbb{A}_i)_{i=1}^a$ and $(\mathbb{B}_j)_{j=1}^b$ be the lists, without repetition, of the structures in $\binom{\mathbb{C}}{\mathbb{A}}$ and $\binom{\mathbb{C}}{\mathbb{B}}$, respectively. We denote the underlying set of structures $\mathbb{A}_i$ and $\mathbb{B}_j$ by A_i and B_j, respectively.

In the following, we construct a sequence $(\mathbb{P}_i = (P_i, R_1^{P_i}, R_2^{P_i}, R_3^{P_i}))_{i=0}^a$ of structures in $\mathcal{R}$. Each P_i will be partitioned into $|C|$ many parts, which will be called *levels*. These parts are labeled by elements of the set C. A level labeled by c is called c-level and if $p \in P_i$ belongs to c-level, we write $\mathrm{lev}(p) = c$. We also write $P_{i,c} = \{p \in P_i : \mathrm{lev}(p) = c\}$. We assume linear ordering on the set of levels such that

$$P_{i,c} <^C P_{i,c'} \Leftrightarrow c <^C c',$$

but we do not have a linear ordering inside levels. A set $S \subseteq P_i$ is *transversal* if and only if $(\forall c \in C)(|S \cap P_{i,c}| \leq 1)$. For a set $C' \subseteq C$ and $L = \bigcup_{c \in C'} P_{i,c}$ we write $\mathbb{P}_i \downharpoonleft C'$ instead of $\mathbb{P}_i \upharpoonright L$. For distinct $x, y \in P_i$ with $\mathrm{lev}(x) = c$ and $\mathrm{lev}(y) = c'$ we denote by $\ddagger(x, y)$ the conjunction of the following statements:

$$\begin{gathered}
(c \neq c', R_1^C(c, c')) \Rightarrow (R_1^{P_i}(x, y) \vee R_2^{P_i}(x, y)),\\
(c \neq c', R_2^C(c, c')) \Rightarrow R_2^{P_i}(x, y),\\
(c \neq c', R_3^C(c, c')) \Rightarrow (R_3^{P_i}(x, y) \vee R_2^{P_i}(x, y)),\\
(c \neq c', (\forall t \in [3])(\neg R_t^C(c, c'))) \Rightarrow (R_1^{P_i}(x, y) \vee R_2^{P_i}(x, y)),\\
c = c' \Rightarrow R_1^{P_i}(x, y).
\end{gathered}$$

We require that for each P_i and all $x, y \in P_i$, statement $\ddagger(x, y)$ is satisfied, see Figure 1.

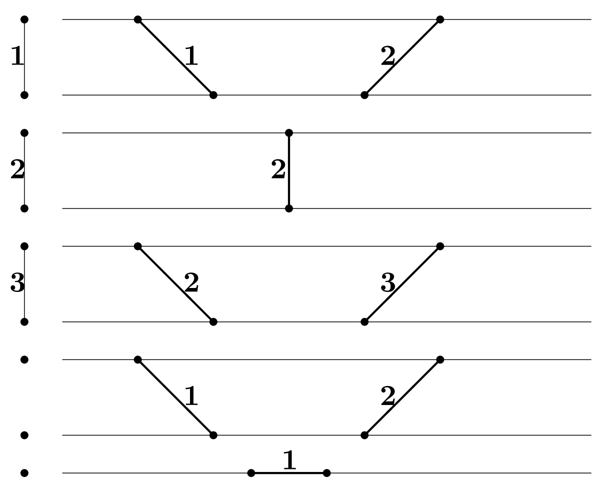

FIGURE 1. Edges between points depending on levels.

Construction of $\mathbb{P}_0$. We define P_0 such that for $j \in [b]$ we have:

$$\begin{gathered}
P_0 = \bigsqcup_{j=1}^{b} P_{0,j},\\
|P_{0,j}| = |B|,\\
P_{0,j} \text{ is transversal},\\
\{c \in C : (\exists p \in P_{0,j})(lev(p) = c)\} = B_j.
\end{gathered}$$

Each $P_{0,j}$ comes with the linear ordering $\leq^{0,j}$ which is induced from the linear ordering of the levels. Then, for each $j \in [b]$ there is a unique structure $\mathbb{P}_{0,j} = (P_{0,j}, R_1^{P_{0,j}}, R_2^{P_{0,j}}, R_3^{P_{0,j}}, \leq^{0,j}) \in \mathcal{OR}$ such that $\mathbb{P}_{0,j} \cong \mathbb{B}$. We define relations $R_1^{P_0}, R_2^{P_0}, R_3^{P_0}$ as follows:

$$R_1^{P_0} = (\bigsqcup_{j=1}^{b} R_1^{P_{0,j}}) \bigsqcup \Big\{(x,y) : (\exists j_1 \neq j_2 \in [b])$$

$$(x \in P_{0,j_1}, y \in P_{0,j_2}, \mathrm{lev}(x) = \mathrm{lev}(y))\Big\},$$

$$R_2^{P_0} = (\bigsqcup_{j=1}^{b} R_2^{P_{0,j}}) \bigsqcup \Big\{(x,y) : (\exists j_1 \neq j_2 \in [b])$$

$$(x \in P_{0,j_1}, y \in P_{0,j_2}), \mathrm{lev}(x) \neq \mathrm{lev}(y)\Big\},$$

$$R_3^{P_0} = \bigsqcup_{j=1}^{b} R_3^{P_{0,j}}.$$

It is easy to see that for all $x, y \in P_0$, statement $\ddagger(x,y)$ is satisfied. At this point we have $\mathbb{P}_0 \in \mathcal{OK}$, so we have to verify that $\mathbb{P}_0 \in \mathcal{OR}$ by examining triangles:

$\mathbb{T}_i \not\hookrightarrow \mathbb{P}_0$: According to our definition of $R_3^{P_0}$, a $(3,3,3)$ triangle can occur only as a subset of some $P_{0,j}$. Since $\mathbb{P}_{0,j} \cong \mathbb{B}$, this is impossible.

$\mathbb{T}_2 \not\hookrightarrow \mathbb{P}_0$: Suppose that $\mathbb{T} \leq \mathbb{P}_0$ is a $(1,1,3)$ triangle. According to our definition of $R_3^{P_0}$, there is some j such that at least two vertices, say v_1 and v_2, of the triangle $\mathbb{T}$ are in $P_{0,j}$. Since $\mathbb{P}_{0,j} \cong \mathbb{B} \in \mathcal{OR}$, there is $j' \neq j$ such that the third vertex, say v_3, of the triangle $\mathbb{T}$ is in $P_{0,j'}$. Now, we have $|\{v_1, v_2, v_3\}| = 3$ and our definition of $R_1^{P_0}$ implies $R_1^C(\mathrm{lev}(v_1), \mathrm{lev}(v_3))$ and $R_1^C(\mathrm{lev}(v_2), \mathrm{lev}(v_3))$. Consequently, $\mathbb{C} \restriction \{\mathrm{lev}(v_1), \mathrm{lev}(v_3), \mathrm{lev}(v_2)\} \cong \mathbb{T}_2$ which contradicts $\mathbb{T}_2 \not\hookrightarrow \mathbb{C}$.

$\mathbb{T}_3 \not\hookrightarrow \mathbb{P}_0$: Suppose that $\mathbb{T} \leq \mathbb{P}_0$ is $(2,3,3)$ triangle. According to our definition of $R_3^{P_0}$, there is some j such that $\mathbb{T} \leq \mathbb{P}_0 \restriction P_{0,j}$. This contradicts $\mathbb{T}_3 \not\hookrightarrow \mathbb{B}$.

Note that we can not use only Lemma 3.3 (a) in the construction of $\mathbb{P}_0$ since the property given by $\ddagger$ has to be satisfied. This completes the construction of $\mathbb{P}_0$. We proceed with the construction of $\mathbb{P}_{i+1}$ assuming that we have $\mathbb{P}_i$.

Construction of $\mathbb{P}_{i+1}$ *from* $\mathbb{P}_i$. Let $\mathcal{A}_{i+1}$ be the collection of all $\mathbb{U} \leq \mathbb{P}_i$ with the property that:

- $\mathbb{U}$ is transversal with the underlying set U,
- $A_{i+1} = \{c \in C : (\exists p \in U)(\mathrm{lev}(p) = c)\}$,
- if $\leq^U$ is a linear ordering induced by $\leq^U$ on U then $(\mathbb{U}, \leq^U) \cong \mathbb{A}$.

Let $N_{i+1} = \mathrm{HJ}(\mathcal{A}_{i+1}, r)$ and suppose that A_{i+1} is linearly ordered by $\leq^C$: $a_1 <^C a_2 <^C \cdots <^C a_s$. For $k \in [s]$, we consider

$$L_k = \{p \in P_i : (\exists \mathbb{U} \in \mathcal{A}_{i+1})(p \in U), \mathrm{lev}(p) = a_k\}$$

and the set $L_k^{N_{i+1}} = \{(x_1, ..., x_{N_{i+1}}) : (\forall l \in N_{i+1})(x_l \in L_k)\}$. We define the structure $\mathbb{Q} = (Q, R_1^Q, R_2^Q, R_3^Q) \in \mathcal{R}$ such that we have a partition $Q = \bigsqcup_{k=1}^s Q_k$ where $Q_k = L_k^{N_{i+1}}$ and the a_k level of Q is equal to Q_k for $k \in [s]$. In order to define relations on Q, we consider $\mathrm{HJ}(\mathcal{A}_{i+1}^{N_{i+1}})$ and introduce more technical notation. Let $\overrightarrow{\mathbb{Z}} = (\mathbb{Z}_l)_{l=1}^{N_{i+1}}$ be a sequence in $\mathcal{A}_{i+1}^{N_{i+1}}$. We assume that vertices in each $\mathbb{Z}_l$ are linearly ordered by linear ordering levels in P_i such that $z_{l,1} <^C z_{l,2} <^C \cdots <^C z_{l,s}$. Then we write $\overrightarrow{\mathbb{Z}}_{a_l} = (z_{1,l}, z_{2,l}, ..., z_{N_{i+1},l})$. In this way $\overrightarrow{\mathbb{Z}}$ determines a sequence of points in Q.

Let $\overrightarrow{x} = (x_1, ..., x_{N_{i+1}}) \in L_{j_1}^{N_{i+1}}$ and $\overrightarrow{y} = (y_1, ..., y_{N_{i+1}}) \in L_{j_2}^{N_{i+1}}$ be vertices in Q. Let $\dagger(\overrightarrow{x}, \overrightarrow{y})$ denote the following statement

$$(\exists \Sigma \in HJ(\mathcal{A}_{i+1}^{N_{i+1}}))(\exists \overrightarrow{\mathbb{X}}, \overrightarrow{\mathbb{Y}} \in \Sigma)(\exists c_1, c_2 \in A_{i+1})(\overrightarrow{\mathbb{X}}_{c_1} = \overrightarrow{x}, \overrightarrow{\mathbb{Y}}_{c_2} = \overrightarrow{y}).$$

In this case we say that $\overrightarrow{x}$ and $\overrightarrow{y}$ belongs to the same HJ line.

If $\dagger(\overrightarrow{x}, \overrightarrow{y})$ is true it is verified by HJ line Σ_1, $\overrightarrow{\mathbb{X}} \in \Sigma_1$, $\overrightarrow{\mathbb{Y}} \in \Sigma_1$, $c_1 \in A_{i+1}$, and $c_2 \in A_{i+1}$ where $\overrightarrow{\mathbb{X}}_{c_1} = \overrightarrow{x}$, $\overrightarrow{\mathbb{Y}}_{c_2} = \overrightarrow{y}$. Then for $t \in [3]$ we define

$$R_t^Q(\overrightarrow{x}, \overrightarrow{y}) \Leftrightarrow R_t^{P_i}(x_{l_0}, y_{l_0}) \text{ where } l_0 \in M(\Sigma_1).$$

Note that this definition does not depend on the choice of $l_0 \in M(\Sigma_1)$ due to the definition of HJ lines. Still, we need to verify that this definition is independent on the choice of HJ line. Note that the choice of c_1 and c_2 is independent of the choice of HJ line. Suppose that $\dagger(\overrightarrow{x}, \overrightarrow{y})$ is also verified by HJ line Σ_2, $\overrightarrow{\mathbb{Z}} \in \Sigma_2$, $\overrightarrow{\mathbb{W}} \in \Sigma_2$, $c_1 \in A_{i+1}$, and $c_2 \in A_{i+1}$ where $\overrightarrow{\mathbb{Z}}_{c_1} = \overrightarrow{x}, \overrightarrow{\mathbb{W}}_{c_2} = \overrightarrow{y}$. Depending on the mutual position of Σ_1 and Σ_2 we have the following cases:

- $M(\Sigma_1) = M(\Sigma_2), F(\Sigma_1) \neq F(\Sigma_2)$: Since relations R_t^Q depends only on the moving part of HJ lines which are the same, we have that the relation is defined in the same way by both HJ lines.
- $M(\Sigma_1) \neq M(\Sigma_2), M(\Sigma_1) \cap M(\Sigma_2) \neq \emptyset$: For $l_0 \in M(\Sigma_1) \cap M(\Sigma_2)$, relations R_t^Q are defined in the same way by both HJ lines.
- $M(\Sigma_1) \neq M(\Sigma_2), M(\Sigma_1) \cap M(\Sigma_2) = \emptyset$: We choose some $l_1 \in M(\Sigma_1)$ and $l_2 \in M(\Sigma_2)$. Since $M(\Sigma_1) \cap M(\Sigma_2) = \emptyset$, we have $l_2 \in F(\Sigma_1)$ and $l_1 \in F(\Sigma_2)$. From the definition of the HJ line we have that x_{l_2} and y_{l_2} are points in the structure $\mathbb{X}_{l_2} = \mathbb{Y}_{l_2} \cong \mathbb{A}_{i+1}$, so $R_t^Q(\overrightarrow{x}, \overrightarrow{y}) \Leftrightarrow R_t^{P_i}(c_1, c_2)$ considering HJ line Σ_2. In the same way, x_{l_1} and y_{l_1} are points in the structure $\mathbb{W}_{l_1} = \mathbb{Z}_{l_1} \cong \mathbb{A}_{i+1}$, so $R_t^Q(\overrightarrow{x}, \overrightarrow{y}) \Leftrightarrow R_t^{P_i}(c_1, c_2)$ considering HJ line Σ_1. This proves that the definition of the relation R_t^Q is the same in both cases, either we use Σ_1 or Σ_2.

Now we consider the case that $\dagger(\overrightarrow{x}, \overrightarrow{y})$ is false. Let c_1 and c_2 be such that $\mathrm{lev}(\overrightarrow{x}) = c_1$ and $\mathrm{lev}(\overrightarrow{y}) = c_2$. We define relations R_t^Q such that:

$$\begin{aligned} c_1 = c_2 &\Rightarrow R_1^Q(\overrightarrow{x}, \overrightarrow{y}),\\ c_1 \neq c_2, R_1^C(c_1, c_2) &\Rightarrow R_1^Q(\overrightarrow{x}, \overrightarrow{y}),\\ c_1 \neq c_2, R_2^C(c_1, c_2) &\Rightarrow R_2^Q(\overrightarrow{x}, \overrightarrow{y}),\\ c_1 \neq c_2, R_3^C(c_1, c_2) &\Rightarrow R_2^Q(\overrightarrow{x}, \overrightarrow{y}).\end{aligned}$$

Note that we do not need to consider the case of $c_1 \neq c_2$ with the property that $(\forall t \in [3])(\neg R_t^C(c_1, c_2))$ because c_1 and c_2 are in the copy of $\mathbb{A}$.

At this point we know that $\mathbb{Q} = (Q, R_1^Q, R_2^Q, R_3^Q) \in \mathcal{K}$. Since property $\ddagger(x, y)$ is satisfied for all $x, y \in P_i$ and according to our construction, we conclude that $\ddagger(x, y)$ is also satisfied for all $x, y \in Q$. We verify that $\mathbb{Q} \in \mathcal{R}$ by considering triangles. Let $T = \{\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}\}$ be a subset of Q such that $\mathrm{lev}(\overrightarrow{x}) = c_1$, $\mathrm{lev}(\overrightarrow{y}) = c_2$ and $\mathrm{lev}(\overrightarrow{z}) = c_3$. We examine if $\mathbb{Q} \restriction T$ is isomorphic to one of the following triangles:

- $\mathbb{T}_1$: Our construction implies that $|\{c_1, c_2, c_3\}| = 3$ and each pair of vertices from T must be contained in some HJ line. Note that from the construction, $R_3^C(\overrightarrow{x}, \overrightarrow{y})$ implies $R_3^C(c_1, c_2)$ and similarly in the other two cases. Therefore, we have $R_3^C(c_1, c_2)$, $R_3^C(c_2, c_3)$, and $R_3^C(c_3, c_1)$. Moreover, $\mathbb{T}_1 \hookrightarrow \mathbb{A}_{i+1} \cong \mathbb{A}$, so we have a contradiction with $\mathbb{A} \in \mathcal{R}$.
- $\mathbb{T}_2$: Suppose $R_3^C(\overrightarrow{y}, \overrightarrow{z})$, so $c_2 \neq c_3$ and $R_3^C(c_2, c_3)$. Since $\ddagger(\overrightarrow{y}, \overrightarrow{z})$ is satisfied we have $c_1 \neq c_2$ and $c_1 \neq c_3$. Property $\ddagger$ also implies $R_1^C(c_1, c_2)$ and $R_1^C(c_1, c_3)$. Therefore $\mathbb{C} \restriction \{c_1, c_2, c_3\} \cong \mathbb{T}_2$, contradicting $\mathbb{A} \in \mathcal{R}$.
- $\mathbb{T}_3$: Suppose $R_3^C(\overrightarrow{x}, \overrightarrow{y})$ and $R_3^C(\overrightarrow{x}, \overrightarrow{z})$. Due to the above discussion, we have $c_1 \neq c_2$, $c_1 \neq c_3$, $R_3^C(c_1, c_2)$, and $R_3^C(c_3, c_1)$. If $c_2 = c_3$ then by our construction we must have $R_1^C(c_2, c_3)$. Therefore $|\{c_1, c_2, c_3\}| = 3$. Let $\overrightarrow{x} = (x_i)_{i=1}^{N_{i+1}}$, $\overrightarrow{y} = (y_i)_{i=1}^{N_{i+1}}$, and $\overrightarrow{z} = (z_i)_{i=1}^{N_{i+1}}$. We consider relation $R_3^C(\overrightarrow{x}, \overrightarrow{y})$. There is a HJ line Σ and there are $\overrightarrow{\mathbb{X}}, \overrightarrow{\mathbb{Y}} \in \Sigma$ such that $\overrightarrow{\mathbb{X}}_{c_1} = \overrightarrow{x}$ and $\overrightarrow{\mathbb{X}}_{c_2} = \overrightarrow{y}$. Then for $l_0 \in M(\Sigma)$ we have $R_3^{P_i}(x_{l_0}, y_{l_0})$. Since $R_3^C(c_1, c_2)$, we have $R_3^{P_i}(x_{l_0}, y_{l_0})$ for $l_0 \in F(\Sigma)$. In particular, $R_3^{P_i}(x_{l_0}, y_{l_0})$ for all $l_0 \in [N_{i+1}]$. In a similar way we obtain $R_3^{P_i}(x_{l_0}, z_{l_0})$ for all $l_0 \in [N_{i+1}]$. Since $R_1^C(c_2, c_3)$ and $R_2^C(\overrightarrow{y}, \overrightarrow{z})$, there is a HJ line Σ' and there are $\overrightarrow{\mathbb{Z}}, \overrightarrow{\mathbb{W}} \in \Sigma'$ such that $\overrightarrow{\mathbb{Z}}_{c_2} = \overrightarrow{y}$ and $\overrightarrow{\mathbb{W}}_{c_3} = \overrightarrow{z}$. Then for $l' \in M(\Sigma')$ we have $R_2^{P_i}(y_{l'}, z_{l'})$. We also have $R_3^{P_i}(x_{l'}, y_{l'})$ and $R_3^{P_i}(x_{l'}, z_{l'})$, so $\mathbb{P}_i \restriction \{x_{l'}, y_{l'}, z_{l'}\} \cong \mathbb{T}_3$. This is in contradiction with $\mathbb{P}_i \in \mathcal{R}$.

Having structure $\mathbb{Q} \in \mathcal{R}$, for each $\Sigma \in HJ(\mathcal{A}_{i+1}^{N_{i+1}})$ we consider the set $Q_\Sigma = \{\overrightarrow{\mathbb{Z}}_k \in \mathcal{A}_{i+1}^{N_{i+1}} : \overrightarrow{\mathbb{Z}} \in \Sigma, k \in [s]\}$ and the structure $\mathbb{Q}_\Sigma = \mathbb{Q} \restriction Q_\Sigma$. Moreover, we have $\mathbb{Q}_\Sigma \cong \mathbb{P}_i \restriction (\bigcup_{k=1}^s L_k)$. For each $\Sigma \in HJ(\mathcal{A}_{i+1}^{N_{i+1}})$ we consider the set J_Σ, which is also partitioned into levels labeled by elements of C, such that:

$$|J_\Sigma| = |P_i|,$$
$$J_\Sigma \cap Q = Q_\Sigma,$$
$$(\forall c \in C)(|J_{\Sigma,c}| = |P_{i,c}|).$$

Then there is a bijection $\pi_\Sigma : P_i \to J_\Sigma$ such that $\pi_\Sigma \restriction P_{i,c} : P_{i,c} \to J_\Sigma$ is a bijection for all $c \in C$. For distinct $\Sigma \neq \Sigma'$, we require that $J_\Sigma \cap J_{\Sigma'} = Q_\Sigma \cap Q_{\Sigma'} \subseteq Q$. On each J_Σ we put relations $R_1^\Sigma, R_2^\Sigma, R_3^\Sigma$ such that π_Σ is an isomorphism between $\mathbb{P}_i$ and $(J_\Sigma, R_1^\Sigma, R_2^\Sigma, R_3^\Sigma)$. We introduce the set P_{i+1} as

$$P_{i+1} = \bigcup\nolimits_{\Sigma \in HJ(\mathcal{A}^{N_{i+1}})} J_\Sigma.$$

Since each element of Q belongs to some $Q_\Sigma \subseteq J_\Sigma$, we have $Q \subseteq P_{i+1}$. In the following, we introduce relations $R_1^{P_{i+1}}, R_2^{P_{i+1}}, R_3^{P_{i+1}}$. Our first requirement is that for $t \in [3]$ we have

$$\bigcup\nolimits_{\Sigma \in HJ(\mathcal{A}^{N_{i+1}})} R_t^\Sigma \subseteq R_t^{P_{i+1}}.$$

Let x and y be vertices in P_{i+1} such that $\mathrm{lev}(x) = c$ and $\mathrm{lev}(y) = c'$. Let Σ and Σ' be HJ lines such that $x \in J_\Sigma$ and $y \in J_{\Sigma'}$. To satisfy property ‡ in P_{i+1}, we put

$$c \neq c', R_2^C(c, c') \Rightarrow R_2^{P_{i+1}}(x, y),$$
$$c = c' \Rightarrow R_1^{P_{i+1}}(x, y).$$

Since property ‡ is satisfied in P_i, relations defined so far are well-defined. It remains to consider the case: $\Sigma \neq \Sigma'$, $\{x, y\} \nsubseteq Q$, $c \neq c'$, and $\neg R_2^C(c, c')$. Assuming this, we take (see Lemma 3.3 (b)):

$$(\exists z \in J_\Sigma \cap J_{\Sigma'})(R_3^\Sigma(z, x), R_3^{\Sigma'}(z, y)) \Rightarrow R_1^{P_{i+1}}(x, y),$$
$$\neg(\exists z \in J_\Sigma \cap J_{\Sigma'})(R_3^\Sigma(z, x), R_3^{\Sigma'}(z, y)) \Rightarrow R_2^{P_{i+1}}(x, y).$$

We need to verify the property ‡ in $\mathbb{P}_{i+1}$. Depending on relation between c and c' we have the following cases:

$R_3^C(c, c')$: We need to show that $\neg R_1^{P_{i+1}}(x, y)$. If we have $R_1^{P_{i+1}}(x, y)$ then there is $z \in J_\Sigma \cap J_{\Sigma'}$ such that $R_3^\Sigma(z, x)$ and $R_3^{\Sigma'}(z, y)$. Our construction implies that $R_3^C(\mathrm{lev}(z), c)$ and $R_3^C(\mathrm{lev}(z), c')$. Consequently, $\mathbb{C} \restriction \{c, c', \mathrm{lev}(z)\} \cong \mathbb{T}_1$, so we have a contradiction with $\mathbb{C} \in \mathcal{F}$.

$R_2^C(c, c')$: We need to show that $\neg R_1^{P_{i+1}}(x, y)$. If we have $R_1^{P_{i+1}}(x, y)$ then there is $z \in J_\Sigma \cap J_{\Sigma'}$ such that $R_3^\Sigma(z, x)$ and $R_3^{\Sigma'}(z, y)$. Now, we must have $R_3^C(\mathrm{lev}(z), c)$ and $R_3^C(\mathrm{lev}(z), c')$. Consequently, $\mathbb{T}_3 \cong \mathbb{C} \restriction \{c, c', \mathrm{lev}(z)\}$, so we have a contradiction with $\mathbb{C} \in \mathcal{F}$.

$R_1^C(c,c')$: This case follows directly from the fact that we add only edges with labels R_1 or R_2.

$(\forall t \in [3])(\neg R_t^C(c,c'))$: We need to show that $\neg R_3^{P_{i+1}}(x,y)$. This follows directly from the fact that for $\{x,y\} \not\subseteq Q$ we may have only $R_1^{P_{i+1}}(x,y)$ or $R_2^{P_{i+1}}(x,y)$.

This completes verification of the property ‡ in $\mathbb{P}_{i+1}$; it remains to verify that $\mathbb{P}_{i+1} \in \mathcal{R}$. We fix vertices $v_1, v_2, v_3 \in P_{i+1}$ with $\mathrm{lev}(v_1) = c_1$, $\mathrm{lev}(v_2) = c_2$, $\mathrm{lev}(v_3) = c_3$. Let $\Sigma_1, \Sigma_2, \Sigma_3$ be HJ lines such that $v_1 \in J_{\Sigma_1}, v_2 \in J_{\Sigma_2}, v_3 \in J_{\Sigma_3}$. We discuss the fact that $\mathbb{T}$ is equivalent to:

$\mathbb{T}_1$: Since the property ‡ is satisfied in $\mathbb{P}_{i+1}$, we have $|\{c_1, c_2, c_3\}| = 3$ which implies $R_3^C(c_1,c_2), R_3^C(c_2,c_3), R_3^C(c_1,c_3)$. Then $\mathbb{C} \restriction \{c_1,c_2,c_3\} \cong \mathbb{T}_1$, so we have a contradiction with $\mathbb{C} \in \mathcal{F}$.

$\mathbb{T}_2$: Without loss of generality we may assume $R_3^{P_{i+1}}(v_2,v_3)$. Note that relation R_3 in $\mathbb{P}_{i+1}$ can occur only between points in some J_Σ. Therefore, there is a HJ line Σ such that $\{v_2,v_3\} \subseteq J_\Sigma$. Since $\mathbb{P}_{i+1} \restriction J \cong \mathbb{P}_i \in \mathcal{R}$, we have $v_1 \notin J_\Sigma$, so there is a HJ line Σ' such that $v_1 \in J_{\Sigma'} \backslash J_\Sigma$. Similarly, without loss of generality we may assume that $v_3 \notin J_{\Sigma'}$. Property ‡ implies $|\{c_1,c_2,c_3\}| = 3$. For the vertex v_2 we have two options: $v_2 \in J_\Sigma \cap J_{\Sigma'}$ or $v_2 \notin J_{\Sigma'}$. Depending on v_2 we discuss these two options:

$v_2 \in J_\Sigma \cap J_{\Sigma'}$: There is a vertex $v \in J_\Sigma \cap J_{\Sigma'}$ such that $R_3^\Sigma(v,v_1)$ and $R_3^{\Sigma'}(v,v_3)$. Since $R_1^\Sigma(v_1,v_2)$, we have $v \neq v_2$. From relation $R_3^{\Sigma'}(v,v_3)$, $R_3^{\Sigma'}(v_2,v_3) \Leftrightarrow R_3^{P_{i+1}}(v_2,v_3)$, and $\mathbb{P}_{i+1} \restriction J_{\Sigma'} \cong \mathbb{P}_i \in \mathcal{R}$. We obtain $R_1^{P_{i+1}}(v,v_2) \Leftrightarrow R_1^\Sigma(v,v_1)$. At this point, we have $\mathbb{P}_{i+1} \restriction \{v,v_1,v_2\} \cong \mathbb{T}_2 \leq \mathbb{P}_{i+1} \restriction J_\Sigma \cong \mathbb{P}_i \in \mathcal{R}$, so we have a contradiction.

$v_2 \in J_\Sigma \backslash J_{\Sigma'}$: Then there is $v' \in J_\Sigma \cap J_{\Sigma'}$ such that $R_3^\Sigma(v',v_1)$ and $R_3^{\Sigma'}(v',v_2)$. From $R_3^{\Sigma'}(v_2,v_3)$, $R_3^{\Sigma'}(v',v_2)$, and $\mathbb{P}_{i+1} \restriction J_{\Sigma'} \cong \mathbb{P}_i \in \mathcal{R}$. We obtain $R_1^{P_{i+1}}(v',v_3) \Leftrightarrow R_1^{\Sigma'}(v',v_3)$. We consider the triangle given by vertices v', v_1, and v_3. Now we may obtain a contradiction in the same way as in the previous case when $v_2 \in J_\Sigma \cap J_{\Sigma'}$.

$\mathbb{T}_3$: Without loss of generality we may assume $R_2^{P_{i+1}}(v_2,v_3)$. Note that R_3 relation in $\mathbb{P}_{i+1}$ can occur between points in some J_Σ. Therefore, there are HJ lines Σ and Σ' such that $\{v_1,v_2\} \subseteq J_\Sigma$ and $\{v_1,v_3\} \subseteq J_{\Sigma'}$. Since $\mathbb{P}_i \in \mathcal{R}$ we have $\Sigma \neq \Sigma'$. According to the definition of relations between points in $J_\Sigma \triangle J_{\Sigma'}$, we must have $R_1^{P_{i+1}}(v_2,v_3)$. This is a contradiction.

This completes the definition of the structure $\mathbb{P}_{i+1}$. Let $\leq^a$ be a linear ordering on the set P_a such that it respects linear ordering of levels and it has no restriction inside each level. We claim that $(\mathbb{P}_a, \leq^a) \to (\mathbb{B})^{\mathbb{A}}_r$.

Final verification. Let $\chi : \binom{(\mathbb{P}_a, \leq^a)}{\mathbb{A}} \to [r]$ be a given coloring. We construct recursively a sequence of structures $(\mathbb{P}'_i)_{i=0}^a$ such that $\mathbb{P}'_0 \leq \mathbb{P}'_1 \leq \cdots \leq \mathbb{P}'_a =$

$\mathbb{P}_a$ and $\mathbb{P}'_i \cong \mathbb{P}_i$ for all $0 \leq i \leq a$. Suppose we have a structure $\mathbb{P}'_{i+1}$ and we want to find a structure $\mathbb{P}'_i$. Without loss of generality we may assume that $\mathbb{P}'_i = \mathbb{P}_i$. We assume that the points in A_{i+1} are linearly ordered by $\leq^C$ as $c_1 <^C c_2 <^C \cdots <^C c_s$. We consider an induced coloring

$$\chi_{i+1} : \mathcal{A}_{i+1}^{N_{i+1}} \to [r],$$
$$\chi_{i+1}(\overrightarrow{\mathbb{X}}) = \chi(\mathbb{P}_i \restriction \{\overrightarrow{\mathbb{X}}_{c_1}, \overrightarrow{\mathbb{X}}_{c_2}, ..., \overrightarrow{\mathbb{X}}_{c_s}\}).$$

According to the choice of N_{i+1}, there is $\Sigma \in \mathrm{HJ}(\mathcal{A}_{i+1}^{N_{i+1}})$ such that $\chi_{i+1} \restriction \Sigma = \mathrm{const}_{i+1}$, the constant map. We take $\mathbb{P}'_i = \mathbb{P}_i \restriction J_\Sigma$. Let $\mathbb{A}' \cong \mathbb{A}$ be transversal inside $\mathbb{P}'_i$ placed exactly on levels given by A_{i+1} and with ordering inherited from the ordering of levels. Then we have $\chi(\mathbb{A}') = \mathrm{const}_{i+1}$. Therefore, at the end of this recursive construction we obtain $\mathbb{P}'_0$ and a coloring

$$\chi' : \binom{\mathbb{C}}{\mathbb{A}} \to [r],$$
$$\chi'(\mathbb{A}_i) = \mathrm{const}_i\,.$$

Without loss of generality we may assume $\mathbb{P}'_0 = \mathbb{P}_0$. According to the choice of the structure $\mathbb{C}$, there is $j{\in}[b]$ such that $\chi' \restriction \binom{\mathbb{B}_j}{\mathbb{A}} = \mathrm{const}$. Consequently, we have that $\chi \restriction \binom{\mathbb{P}_{0,j}}{\mathbb{A}} = \mathrm{const}$. This completes verification of the RP for the class $\mathcal{OR}$.

5. Ordering property

We say that the class $\mathcal{OR}$ satisfies the *ordering property* (OP) with respect to $\mathcal{R}$ if for every $\mathbb{A} \in \mathcal{R}$ there is a structure $\mathbb{B} \in \mathcal{R}$ such that for all linear orderings $\leq^A$ and $\leq^B$ satisfying $(\mathbb{A}, \leq^A) \in \mathcal{R}$ and $(\mathbb{B}, \leq^B) \in \mathcal{R}$ we have $(\mathbb{A}, \leq^A) \hookrightarrow (\mathbb{B}, \leq^B)$. We prove the following.

Lemma 5.1. *$\mathcal{OR}$ satisfies the OP with respect to $\mathcal{R}$.*

Proof. Since $\mathcal{R}$ satisfies the joint embedding property, it is enough to prove that for a given $(\mathbb{A}, \leq^A) \in \mathcal{OR}$, $\mathbb{A} = (A, R_1^A, R_2^A, R_3^A)$, there is $\mathbb{B} \in \mathcal{R}$ such that for every $\leq^B$ with $(\mathbb{B}, \leq^B) \in \mathcal{OR}$ we have $(\mathbb{A}, \leq^A) \hookrightarrow (\mathbb{B}, \leq^B)$. Let $\leq^{opA}$ be a linear ordering on A such that for all $x, y \in A$ we have $x \leq^A y \Leftrightarrow y \leq^{opA} x$. According to Lemma 3.3 (a) there is a structure $\mathbb{A}' = (A', R_1^{A'}, R_2^{A'}, R_3^{A'}) \in \mathcal{R}$ such that $A' = A \sqcup A_{op}$, $\mathbb{A}' \restriction A = \mathbb{A}$ and $\mathbb{A}' \restriction A_{op} \cong \mathbb{A}$. There is a linear ordering $\leq^{A'}$ on A' such that $\leq^{A'}\restriction A = \leq^A$ and $(\mathbb{A}' \restriction A_{op}, \leq^{A'}\restriction A_{op}) \cong (\mathbb{A}, \leq^{opA})$. Suppose that the vertices of A' are linearly ordered according to $\leq^{A'}$ as $a_1 <^{A'} a_2 <^{A'} \cdots <^{A'} a_s$. According to Lemma 3.3 (b) there is $\mathbb{A}'' = (A'', R_1^{A''}, R_2^{A''}, R_3^{A''}) \in \mathcal{R}$ such that $A'' = A' \sqcup \{b_1, b_2, ..., b_{s-1}\}$ and for all $i \in [s-1]$ we have

$$\mathbb{A}'' \restriction \{a_i, b_i\} \cong \mathbb{A}'' \restriction \{b_i, a_{i+1}\} \cong \mathbb{X}.$$

The structure $\mathbb{X}=(X, R_1^X, R_2^X, R_3^X)$ is given by $X=\{x_1,x_2\}$ and $R_2^X(x_1,x_2)$. We denote by $\leq^X$ a linear ordering on X such that $x_1 <^X x_2$. We consider a linear ordering $\leq^{A''}$ on A'' such that for all $i \in [s-1]$ we have $a_i <^{A''} b_i <^{A''} < a_{i+1}$. Since $\mathcal{OR}$ is a Ramsey class there is $(\mathbb{B}, \leq^B) \in \mathcal{OR}$ such that $(\mathbb{B}, \leq^B) \to (\mathbb{A}'', \leq^{A''})_2^{(\mathbb{X}, \leq^X)}$. Let $\preceq^B$ be a given linear ordering such that $(\mathbb{B}, \preceq^B) \in \mathcal{OR}$. We consider the following coloring

$$\chi : \binom{(\mathbb{B}, \leq^B)}{(\mathbb{X}, \leq^X)} \to [2],$$
$$\chi((\mathbb{X}', \leq^{X'})) = 1 \Leftrightarrow \leq^B \restriction X' = \preceq^B \restriction X',$$

where X' is the underlying set of $\mathbb{X}'$. There is $(\mathbb{A}''', \leq^{A'''}) \in \binom{(\mathbb{B}, \leq^B)}{(\mathbb{A}'', \leq^{A''})}$ such that $\chi \restriction \binom{(\mathbb{A}''', \leq^{A'''})}{(\mathbb{X}, \leq^X)} = \text{const}$. Without loss of generality we may assume that $(\mathbb{A}''', \leq^{A'''}) = (\mathbb{A}'', \leq^{A''})$. If const $= 1$ then $\leq^B \restriction A'' = \preceq^B \restriction A''$, so we have $(\mathbb{A}, \leq^A) \hookrightarrow (\mathbb{B}, \preceq^B)$. If const $= 2$ then for $x, y \in A''$ we have $x \leq^B y \Leftrightarrow y \preceq^B x$. Moreover, we have $(\mathbb{A}', \preceq^B \restriction A') \cong (\mathbb{A}, \leq^A)$, so $(\mathbb{A}, \leq^A) \hookrightarrow (\mathbb{B}, \preceq^B)$. This completes verification of the OP. $\square$

From Lemma 5.1 and the fact that $\mathcal{OR}$ is a Ramsey class, we may calculate the Ramsey degree for structures in $\mathcal{R}$, see [4] for details. Let $\mathbb{A}$ be a structure in $\mathcal{R}$ with cardinality n. Then we have

$$t_{\mathcal{R}}(\mathbb{A}) = \frac{n!}{|\operatorname{Aut}(\mathbb{A})|}.$$

In particular, vertex and edges are Ramsey objects in $\mathcal{R}$. Triangles $(1,1,1)$ and $(2,2,2)$ are Ramsey objects in $\mathcal{R}$, but triangles $(1,2,3)$, $(3,3,1)$, and $(2,2,1)$ are not Ramsey objects in $\mathcal{R}$.

Without going into the details, we mention that we can verify corresponding result for the unique ergodicity. This can be done in the same way as in Theorem 9.1 in [8]. In the following, we sketch only the main combinatorial construction that can be done. Let $\mathbb{A} = (A, R_1^A, R_2^A, R_3^A)$ be a given structure in $\mathcal{R}$ and let $k = |A|$. Then there is a constant C such that for every k-uniform hypergraph on $n \geq k$ vertices with girth at least 4 and with at least $Cn^{\frac{4}{3}}$ edges. In particular, every two edges in this hypergraph can intersect in at most one point and there are no three edges E_1, E_2, E_3 such that $E_i \cap E_j \neq \emptyset$ for all $i, j \in [3]$. Let G be one such set with n vertices and with edges $E_1, ..., E_s$. Let $\phi_i : A \to E_i$ be a random bijection for each $i \in [s]$. On each E_i we introduce relations $R_t^{E_i}$ such that ϕ_i is a bijection between $\mathbb{A}$ and $(E_i, R_1^{E_i}, R_2^{E_2}, R_3^{E_3})$. We consider the structure $\mathbb{G} = (G, R_1^G, R_2^G, R_3^G)$ such that $(E_i, R_1^{E_i}, R_2^{E_2}, R_3^{E_3})$ for all $i \in [s]$. If $x \in E_i$, $y \in E_i$, $i \neq j$, and $E_i \cap E_j = \emptyset$, then $R_2^G(x,y)$. If $x \in E_i$, $y \in E_i$, $i \neq j$, $E_i \cap E_j = \{z\}$, and $R_3^G(z,x)$ and $R_3^G(z,y)$ then $R_1^G(x,y)$; otherwise $R_2^G(x,y)$. If $x, y \in G$

are such that $\{x, y\} \nsubseteq \bigcup_{i=1}^{s} E_i$ then we take $R_2^G(x, y)$. Now it is a simple verification that $\mathbb{G} \in \mathcal{R}$.

6. Conclusion

For $i \in [3]$, let $\mathbb{S}_i = (S_i, R_1^{S_i}, R_2^{S_2}, R_3^{S_3})$ be structures in $\mathcal{K}$ given by:

$$
\begin{gathered}
S_1 = S_2 = S_3 = \{v_1, v_2, v_3\},\\
R_1^{S_1}(v_1, v_2), R_1^{S_1}(v_2, v_3), R_1^{S_1}(v_3, v_1),\\
R_2^{S_2}(v_1, v_2), R_2^{S_2}(v_1, v_3), R_1^{S_3}(v_3, v_2),\\
R_3^{S_3}(v_1, v_2), R_1^{S_3}(v_2, v_3), R_1^{S_3}(v_3, v_1).
\end{gathered}
$$

We define $\mathcal{S}$ as the class of structures $\mathbb{A} =(A, R_1^A, R_2^A, R_3^A)$ in $\mathcal{F}$ with the property that

$$
\begin{gathered}
(\forall i \in [3])(\mathbb{S}_i \not\hookrightarrow \mathbb{A}),\\
(\forall x \neq y \in A)(R_1^A(x, y) \vee R_2^A(x, y) \vee R_3^A(x, y)).
\end{gathered}
$$

We may consider a functor

$$
\begin{gathered}
\nabla : \mathcal{R} \to \mathcal{S},\\
\nabla((A, R_1^A, R_2^A, R_3^A)) = (S, R_1^S, R_2^S, R_3^S),
\end{gathered}
$$

such that $\nabla(A) = S$ and for all $x, y \in A$, we have $R_1^S(x, y) \Leftrightarrow R_3^A(x, y)$, $R_2^S(x, y) \Leftrightarrow R_1^A(x, y)$ and $R_3^S(x, y) \Leftrightarrow R_2^A(x, y)$. We see that $\nabla(\mathbb{T}_i) = \mathbb{S}_i$ and that ∇ is a functor that gives equivalence between classes $\mathcal{R}$, and $\mathcal{S}$, as well as between $\mathcal{OR}$ and $\mathcal{OS}$. Then, according to [5] we have the following.

Corollary 6.1. *$\mathcal{OS}$ is a Ramsey class and $\mathcal{OS}$ satisfies the OP with respect to $\mathcal{S}$.*

References

1. D. Amato, G. Cherlin, D. Macpherson, *Metrically Homogeneous Graphs in Diameter 3*, `https://www1.maths.leeds.ac.uk/~pmthdm/diameter3_final.pdf`
2. O. Angel, A. S. Kechris, R. Lyons, *Random orderings and unique ergodicity of automorphism groups*, J. Eur. Math. Soc. **16** (2014), no. 10, 2059–2095.
3. A. W. Hales, R. I. Jewett, *Regularity and positional games*, Trans. Amer. Math. Soc. **106** (1963), 222–229.
4. A. S. Kechris, V. G. Pestov, S. Todorčević, *Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups*, Geom. Funct. Anal. **15** (2005), no. 1, 106–189.
5. D. Mašulović, L. Scow, *Categorical equivalence and the Ramsey Property for finite powers of a primal algebra*, Algebra Universalis **78** (2017), 159–179.
6. J. Nešetřil, *Metric spaces are Ramsey*, European J. Combin. **28** (2007), no. 1, 457–468.
7. J. Nešetřil, V. Rödl, *The partite construction and Ramsey set systems*, Discrete Math. **75** (1989), no. 1–3, 327-334.

8. M. Pawliuk, M. Sokić, *Amenability and unique ergodicity of automorphism groups of countable homogeneous directed graphs*, Ergodic Theory Dynam. Systems, 1–51, doi:10.1017/etds.2018.84.

Toronto, ON, Canada
E-mail address: miosokic@gmail.com