



A CHARACTERIZATION OF WELL-FOUNDED ALGEBRAIC LATTICES

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ABSTRACT. We characterize well-founded algebraic lattices by means of forbidden subsemilattices of the join-semilattice made of their compact elements. More specifically, we show that an algebraic lattice L is well-founded if and only if $K(L)$, the join-semilattice of compact elements of L , is well-founded and contains neither $[\omega]^{<\omega}$, nor $\underline{\Omega}(\omega^*)$ as a join-subsemilattice. As an immediate corollary, we get that an algebraic modular lattice L is well-founded if and only if $K(L)$ is well-founded and contains no infinite independent set. If $K(L)$ is a join-subsemilattice of $I_{<\omega}(Q)$, the set of finitely generated initial segments of a well-founded poset Q , then L is well-founded if and only if $K(L)$ is well-quasi-ordered.

1. INTRODUCTION AND SYNOPSIS OF RESULTS

Algebraic lattices and join-semilattices (with a 0) are two aspects of the same thing, as expressed in the following basic result ([13], see also [11]).

Theorem 1.1. *The collection $J(P)$ of ideals of a join-semilattice P , once ordered by inclusion, is an algebraic lattice and the subposet $K(J(P))$ of its compact elements is isomorphic to P . Conversely, the subposet $K(L)$ of compact elements of an algebraic lattice L is a join-semilattice with a 0 and $J(K(L))$ is isomorphic to L .*

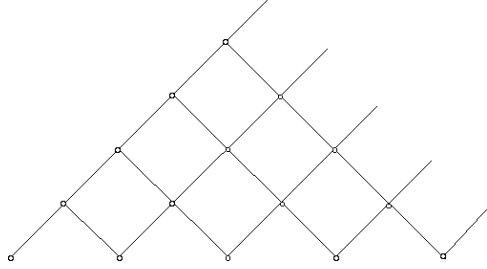
In this paper, we characterize well-founded algebraic lattices by means of forbidden join-subsemilattices of the join-semilattice made of their compact elements. In the sequel ω denotes the chain of non-negative integers, and when this causes no confusion, the first infinite cardinal as well as the first infinite ordinal. We denote ω^* the chain of negative integers. We recall that a poset P is *well-founded* provided that every nonempty subset of P has a minimal element. With the axiom of dependent choices, this amounts to the fact that P contains no subset isomorphic to ω^* . Let $\Omega(\omega^*)$ be the set

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FIGURE 1. $\Omega(\omega^*)$

$[\omega]^2$ of two-element subsets of ω , identified to pairs (i, j) , $i < j < \omega$, ordered so that $(i, j) \leq (i', j')$ if and only if $i' \leq i$ and $j \leq j'$ with respect to the natural order on ω . Let $\underline{\Omega}(\omega^*) := \Omega(\omega^*) \cup \{\emptyset\}$ be obtained by adding a least element. Note that $\underline{\Omega}(\omega^*)$ is isomorphic to the set of bounded intervals of ω (or ω^*) ordered by inclusion. Moreover $\underline{\Omega}(\omega^*)$ is a join-semilattice (the join of two elements is given by $(i, j) \vee (i', j') = (i \wedge i', j \vee j')$). The join-semilattice $\underline{\Omega}(\omega^*)$ embeds in $\Omega(\omega^*)$ as a join-semilattice; the advantage of $\underline{\Omega}(\omega^*)$ with respect to our discussion is to have a zero. Let κ be a cardinal number, e.g. $\kappa := \omega$; denote $[\kappa]^{<\omega}$ (resp. $\wp(\kappa)$) the set, ordered by inclusion, consisting of finite (resp. arbitrary) subsets of κ . The posets $\underline{\Omega}(\omega^*)$ and $[\kappa]^{<\omega}$ are well-founded lattices, whereas the algebraic lattices $J(\underline{\Omega}(\omega^*))$ and $J([\kappa]^{<\omega})$ (κ infinite) are not well-founded (and we may note that $J([\kappa]^{<\omega})$ is isomorphic to $\wp(\kappa)$). As a poset $\underline{\Omega}(\omega^*)$ is isomorphic to a subset of $[\omega]^{<\omega}$, but not as a join-subsemilattice. This is our first result.

Proposition 1.2. *As a join-subsemilattice, $\underline{\Omega}(\omega^*)$ does not embed in $[\omega]^{<\omega}$; more generally, if Q is a well-founded poset then $\underline{\Omega}(\omega^*)$ does not embed as a join-subsemilattice into $I_{<\omega}(Q)$, the join-semilattice made of finitely generated initial segments of Q .*

Our next result expresses that $\underline{\Omega}(\omega^*)$ and $[\omega]^{<\omega}$ are unavoidable examples of well-founded join-semilattices whose set of ideals is not well-founded.

Theorem 1.3. *An algebraic lattice L is well-founded if and only if $K(L)$ is well-founded and contains no join-subsemilattice isomorphic to $\underline{\Omega}(\omega^*)$ or to $[\omega]^{<\omega}$.*

The fact that a join-semilattice P contains a join-subsemilattice isomorphic to $[\omega]^{<\omega}$ amounts to the existence of an infinite independent set. Let us recall that a subset X of a join-semilattice P is *independent* if $x \not\leq \bigvee F$ for every $x \in X$ and every nonempty finite subset F of $X \setminus \{x\}$. Conditions which may ensure the existence of an infinite independent set or consequences of the inexistence of such sets have been considered within the framework of the structure of closure systems (cf. the research on the “free-subset problem” of Hajnal [22] or on the cofinality of posets [10, 17]). A basic result is the following.

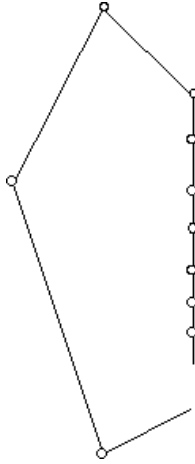


FIGURE 2. $L(\omega^*)$

Theorem 1.4 ([6, 15]). *Let κ be a cardinal number; for a join-semilattice P the following properties are equivalent:*

- (i) P contains an independent set of size κ ;
- (ii) P contains a join-subsemilattice isomorphic to $[\kappa]^{<\omega}$;
- (iii) P contains a subposet isomorphic to $[\kappa]^{<\omega}$;
- (iv) $J(P)$ contains a subposet isomorphic to $\wp(\kappa)$;
- (v) $\wp(\kappa)$ embeds in $J(P)$ via a map preserving arbitrary joins.

Let $L(\alpha) := 1 + (1 \oplus J(\alpha)) + 1$ be the lattice made of the direct sum of the one-element chain 1 and the chain $J(\alpha)$, (α finite or equal to ω^*), with the top and bottom added.

Clearly $J(\underline{\Omega}(\omega^*))$ contains a sublattice isomorphic to $L(\omega^*)$. Since a modular lattice contains no sublattice isomorphic to $L(2)$, we get as a corollary of Theorem 1.3:

Theorem 1.5. *An algebraic modular lattice L is well-founded if and only if $K(L)$ is well-founded and contains no infinite independent set.*

Another consequence is this:

Theorem 1.6. *For a join-semilattice P , the following properties are equivalent:*

- (i) P is well-founded with no infinite antichain ;
- (ii) P contains no infinite independent set and embeds as a join-semilattice into a join-semilattice of the form $I_{<\omega}(Q)$ where Q is some well-founded poset.

Posets which are well-founded and have no infinite antichain are called *well-partially-ordered* or *well-quasi-ordered*, wqo for short. They play an important role in several areas (see [9]). If P is a wqo join-semilattice

then $J(P)$, the poset of ideals of P , is well-founded and one may assign to every $J \in J(P)$ an ordinal, its *height*, denoted by $h(J, J(P))$. This ordinal is defined by induction, setting $h(J, J(P)) := \text{Sup}(\{h(J', J(P)) + 1 : J' \in J(P), J' \subset J\})$ and $h(J', J(P)) := 0$ if J' is minimal in $J(P)$. The ordinal $h(J(P)) := h(P, J(P)) + 1$ is the *height* of $J(P)$. If $P := I_{<\omega}(Q)$, with Q wqo, then $J(P)$ contains a chain of order type $h(J(P))$. This is an equivalent form of the famous result of de Jongh and Parikh [7] asserting that among the linear extensions of a wqo, one has a maximum order type.

Problem 1.7. *Let P be a wqo join-semilattice; does $J(P)$ contain a chain of order type $h(J(P))$?*

An immediate corollary of Theorem 1.6 is:

Corollary 1.8. *A join-semilattice P of $[\omega]^{<\omega}$ contains either $[\omega]^{<\omega}$ as a join-semilattice or is wqo.*

Let us compare join-subsemilattices of $[\omega]^{<\omega}$. Set $P \leq P'$ for two such join-subsemilattices if P embeds in P' as a join-semilattice. This gives a quasi-order and, according to Corollary 1.8, the poset corresponding to this quasi-order has a largest element (namely $[\omega]^{<\omega}$), and all other members come from wqo join-semilattices. Basic examples of join-subsemilattices of $[\omega]^{<\omega}$ are the $I_{<\omega}(Q)$'s where Q is a countable poset such that no element is above infinitely many elements. These posets Q are exactly those which are embeddable in the poset $[\omega]^{<\omega}$ ordered by inclusion. An interesting subclass is made of posets of the form $Q = (\mathbb{N}, \leq)$ where the order \leq is the intersection of the natural order \mathfrak{N} on \mathbb{N} and of a linear order \mathfrak{L} on \mathbb{N} , (that is $x \leq y$ if $x \leq y$ w.r.t. \mathfrak{N} and $x \leq y$ w.r.t. \mathfrak{L}). If α is the type of the linear order, a poset of this form is a *sierpinskiisation* of α . The corresponding join-semilattices are wqo provided that the posets Q have no infinite antichain; in the particular case of a sierpinskiisation of α this amounts to the fact that α is well-ordered.

As shown in [20], sierpinskiisations given by a bijective map $\psi : \omega\alpha \rightarrow \omega$ which is order-preserving on each component $\omega \cdot \{i\}$ of $\omega\alpha$ are all embeddable in each other, and for this reason denoted by the same symbol $\Omega(\alpha)$. Among the representatives of $\Omega(\alpha)$, some are join-semilattices, and among them, join-subsemilattices of the direct product $\omega \times \alpha$ (this is notably the case of the poset $\Omega(\omega^*)$ we previously defined). We extend the first part of Proposition 1.2, showing that except for $\alpha \leq \omega$, the representatives of $\Omega(\alpha)$ which are join-semilattices never embed in $[\omega]^{<\omega}$ as join-semilattices, whereas they embed as posets (see Corollary 4.11 and Example 4.15). From this result, it follows that the posets $\Omega(\alpha)$ and $I_{<\omega}(\Omega(\alpha))$ do not embed in each other as join-semilattices.

These two posets provide examples of a join-semilattice P such that P contains no chain of type α while $J(P)$ contains a chain of type $J(\alpha)$. However, if α is not well ordered then $I_{<\omega}(\Omega(\alpha))$ and $[\omega]^{<\omega}$ embed in each other as join-semilattices.

Problem 1.9. *Let α be a countable ordinal. Is there a minimum member among the join-subsemilattices P of $[\omega]^{<\omega}$ such that $J(P)$ contains a chain of type $\alpha + 1$? Is it true that this minimum is $I_{<\omega}(\Omega(\alpha))$ if α is indecomposable?*

Theorem 1.1 is a particular instance of the duality between compact zero-dimensional semilattices and discrete semilattices developed in [13] (e.g. Theorem 3.9, p. 19). The following was suggested to us:

Problem 1.10. *Is it possible to derive our results or simplify the proofs by making use of this duality?*

A version of this paper was posted on ArXiv (arXiv:0812.2300). The results are included in Chapter 1 of the Thèse d'État defended by the first author [4]. Part of these results have been presented at the International Conference on Discrete Mathematics and Computer Science (DIMACOS'11) organized by A. Boussaïri, M. Kabil, and A. Taik in Mohammedia (Morocco) May, 5–8, 2011. We thank the organizers for their help.

2. DEFINITIONS AND BASIC RESULTS

Our definitions and notations are standard and agree with [11] except on minor points that we will mention. We adopt the same terminology as in [6]. We recall only few things. Let P be a poset. A subset I of P is an *initial segment* of P if $x \in P$, $y \in I$, and $x \leq y$ imply $x \in I$. If A is a subset of P , then $\downarrow A = \{x \in P : x \leq y \text{ for some } y \in A\}$ denotes the least initial segment containing A . If $I = \downarrow A$ we say that I is *generated* by A or A is *cofinal* in I . If $A = \{a\}$ then I is a *principal initial segment* and we write $\downarrow a$ instead of $\downarrow \{a\}$. We denote $\text{down}(P)$ the set of principal initial segments of P . A *final segment* of P is any initial segment of P^* , the dual of P . We denote by $\uparrow A$ the final segment generated by A . If $A = \{a\}$ we write $\uparrow a$ instead of $\uparrow \{a\}$. A subset I of P is *directed* if every finite subset of I has an upper bound in I (that is I is nonempty and every pair of elements of I has an upper bound). An *ideal* is a directed initial segment of P (in particular it is nonempty; but note that in some other texts, the empty set is an ideal). We denote $I(P)$ (resp. $I_{<\omega}(P)$, $J(P)$) the set of initial segments (resp. finitely generated initial segments, ideals of P) ordered by inclusion and we set $J_*(P) := J(P) \cup \{\emptyset\}$, $I_0(P) := I_{<\omega}(P) \setminus \{\emptyset\}$. Others authors use *down set* for initial segment. Note that $\text{down}(P)$ is not to be confused with $I(P)$. If P is a join-semilattice with a 0, an element $x \in P$ is *join-irreducible* if it is distinct from 0, and if $x = a \vee b$ implies $x = a$ or $x = b$ (this is a slight variation from [11]). We denote $\mathbb{J}_{\text{irr}}(P)$ the set of join-irreducibles of P . An element a in a lattice L is *compact* if for every $A \subseteq L$, $a \leq \bigvee A$ implies $a \leq \bigvee A'$ for some finite subset A' of A . The lattice L is *compactly generated* if every element is a supremum of compact elements. A lattice is *algebraic* if it is complete and compactly generated.

We note that $I_{<\omega}(P)$ is the set of compact elements of $I(P)$, hence $J(I_{<\omega}(P)) \cong I(P)$. Moreover $I_{<\omega}(P)$ is a lattice, and in fact a distributive lattice, if and only if P is \downarrow -closed, that is, the intersection of two principal initial segments of P is a finite union, possibly empty, of principal initial segments. We also note that $J(P)$ is the set of join-irreducible elements of $I(P)$; moreover, $I_{<\omega}(J(P)) \cong I(P)$ whenever P has no infinite antichain.

Notably for the proof of Theorem 4.16, we will need the following results.

Theorem 2.1. *Let P be a poset.*

- (i) $I_{<\omega}(P)$ is well-founded if and only if P is well-founded (see [1]);
- (ii) $I_{<\omega}(P)$ is wqo if and only if P is wqo if and only if $I(P)$ is well-founded (see [12]);
- (iii) if P is a well-founded join-semilattice with a 0, then every member of P is a finite join of join-irreducible elements of P (see [1]);
- (iv) a join-semilattice P with a zero is wqo if and only if every member of P is a finite join of join-irreducible elements of P and the set $\mathbb{J}_{irr}(P)$ of these join-irreducible elements is wqo (follows from (ii) and (iii)).

A poset P is *scattered* if it does not contain a copy of η , the chain of rational numbers. A topological space T is *scattered* if every nonempty closed set contains some isolated point. The power set of a set, once equipped with the product topology, is a compact space. The set $J(P)$ of ideals of a join-semilattice P with a 0 is a closed subspace of $\wp(P)$, hence it is a compact space too. Consequently, an algebraic lattice L can be viewed as a poset and a topological space as well. It is easy to see that if L is topologically scattered then it is order scattered. It is a more significant fact, due to M. Mislove [18], that the converse holds if L is distributive.

3. SEPARATING CHAINS OF IDEALS AND PROOFS OF PROPOSITION 1.2 AND THEOREM 1.3

Let P be a join-semilattice. If $x \in P$ and $J \in J(P)$, then $\downarrow x$ and J have a join $\downarrow x \vee J$ in $J(P)$ and $\downarrow x \vee J = \downarrow \{x \vee y : y \in J\}$. Instead of $\downarrow x \vee J$ we also use the notation $\{x\} \vee J$. Note that $\{x\} \vee J$ is the least member of $J(P)$ containing $\{x\} \cup J$. We say that a nonempty chain \mathcal{I} of ideals of P is *separating* if for every $I \in \mathcal{I} \setminus \{\cup \mathcal{I}\}$ and every $x \in \cup \mathcal{I} \setminus I$, there is some $J \in \mathcal{I}$ such that $I \not\subseteq \{x\} \vee J$.

If \mathcal{I} is separating then \mathcal{I} has a least element implies it is a singleton set. In $P := [\omega]^{<\omega}$, the chain $\mathcal{I} := \{I_n : n < \omega\}$ where I_n consists of the finite subsets of $\{m : n \leq m\}$ is separating. In $P := \omega^*$, the chain $\mathcal{I} := \{\downarrow x : x \in P\}$ is nonseparating, as well as all of its infinite subchains. In $P := \Omega(\omega^*)$ the chain $\mathcal{I} := \{I_n : n < \omega\}$ where $I_n := \{(i, j) : n \leq i < j < \omega\}$ has the same property.

We may observe that a *join-preserving embedding from a join-semilattice P into a join-semilattice Q transforms every separating (resp. nonseparating) chain of ideals of P into a separating (resp. nonseparating) chain of*

ideals of Q (If \mathcal{I} is a separating chain of ideals of P , then $\mathcal{J} = \{\downarrow f(I) : I \in \mathcal{I}\}$ is a separating chain of ideals of Q). Hence the containment of $[\omega]^{<\omega}$ (resp. of ω^* or of $\Omega(\omega^*)$), as a join-subsemilattice, provides a chain of ideals which is separating (resp. nonseparating, as are all its infinite subchains, as well). We show in the next two lemmas that the converse holds.

Lemma 3.1. *A join-semilattice P contains an infinite independent set if and only if it contains an infinite separating chain of ideals.*

Proof. Let $X = \{x_n : n < \omega\}$ be an infinite independent set. Let I_n be the ideal generated by $X \setminus \{x_i : 0 \leq i \leq n\}$. The chain $\mathcal{I} = \{I_n : n < \omega\}$ is separating. Let \mathcal{I} be an infinite separating chain of ideals. Define inductively an infinite sequence $x_0, I_0, \dots, x_n, I_n, \dots$ such that $I_0 \in \mathcal{I} \setminus \{\cup \mathcal{I}\}$, $x_0 \in \cup \mathcal{I} \setminus I_0$ and such that:

- a_n) $I_n \in \mathcal{I}$;
- b_n) $I_n \subset I_{n-1}$;
- c_n) $x_n \in I_{n-1} \setminus (\{x_0 \vee \dots \vee x_{n-1}\} \vee I_n)$ for every $n \geq 1$.

The construction is immediate. Indeed, since \mathcal{I} is infinite then $\mathcal{I} \setminus \{\cup \mathcal{I}\} \neq \emptyset$. Choose arbitrary $I_0 \in \mathcal{I} \setminus \{\cup \mathcal{I}\}$ and $x_0 \in \cup \mathcal{I} \setminus I_0$. Let $n \geq 1$. Suppose x_k, I_k defined and satisfying $a_k), b_k), c_k)$ for all $k \leq n-1$. Set $I := I_{n-1}$ and $x := x_0 \vee \dots \vee x_{n-1}$. Since $I \in \mathcal{I}$ and $x \in \cup \mathcal{I} \setminus I$, there is some $J \in \mathcal{I}$ such that $I \not\subseteq \{x\} \vee J$. Let $z \in I \setminus (\{x\} \vee J)$. Set $x_n := z$, $I_n := J$. The set $X := \{x_n : n < \omega\}$ is independent. Indeed if $x \in X$ then since $x = x_n$ for some n , $n < \omega$, condition c_n) asserts that there is some ideal containing $X \setminus \{x\}$ and excluding x . \square

Lemma 3.2. *A join-semilattice P contains either ω^* or $\Omega(\omega^*)$ as a join-subsemilattice if and only if it contains an ω^* -chain \mathcal{I} of ideals such that all infinite subchains are nonseparating.*

Proof. Let \mathcal{I} be an ω^* -chain of ideals and let A be its largest element (that is $A = \cup \mathcal{I}$). Let E denote the set $\{x : x \in A \text{ and } I \sqsubset x \text{ for some } I \in \mathcal{I}\}$.

Case i: For every $I \in \mathcal{I}$, $I \cap E \neq \emptyset$.

We can build an infinite strictly decreasing sequence x_0, \dots, x_n, \dots of elements of P . Indeed, let us choose $x_0 \in E \cap (\cup \mathcal{I})$ and I_0 such that $I_0 \sqsubset x_0$. Suppose x_0, \dots, x_n and I_0, \dots, I_n defined such that $I_i \sqsubset x_i$ for all $i = 0, \dots, n$. As $E \cap I_n \neq \emptyset$ we can select $x_n \in E \cap I_n$ and by definition of E , we can select some $I_{n+1} \in \mathcal{I}$ such that $I_{n+1} \sqsubset x_{n+1}$. Thus $\omega^* \leq P$.

Case ii: There is some $I \in \mathcal{I}$ such that $I \cap E = \emptyset$.

In particular all members of \mathcal{I} included in I are unbounded in I . Since all infinite subchains of \mathcal{I} are nonseparating then, with no loss of generality, we may suppose that $I = A$ (hence $E = \emptyset$). We set $I_{-1} := A$ and define a sequence $x_0, I_0, \dots, x_n, I_n, \dots$ such that $I_n \in \mathcal{I}$, $x_n \in I_{n-1} \setminus I_n$ and $I_n \subseteq \{x_n\} \vee I$ for all $I \in \mathcal{I}$, all $n < \omega$. Members of this sequence being defined for all $n', n' < n$, observe that the set $\mathcal{I}_n := \{I \in \mathcal{I} : I \subseteq I_{n-1}\}$ being infinite is nonseparating, hence there are $I \in \mathcal{I}_n$ and $x \in I_{n-1} \setminus I$

such that $I \subseteq \{x\} \vee J$ for all $J \in \mathcal{I}_n$. Set $I_n := I$ and $x_n := x$. Next, we define a sequence $y_0 := x_0, \dots, y_n, \dots$ such that for every $n \geq 1$:

- a_n) $x_n \leq y_n \in I_{n-1}$;
- b_n) $y_n \not\leq y_0 \vee y_{n-1}$;
- c_n) $y_j \leq y_i \vee y_n$ for every $i \leq j \leq n$.

Suppose y_0, \dots, y_{n-1} defined for some $n, n \geq 1$. Since I_{n-1} is unbounded, we may select $z \in I_{n-1}$ such that $z \not\leq y_0 \vee \dots \vee y_{n-1}$. If $n = 1$, we set $y_1 := x_1 \vee z$. Suppose $n \geq 2$. Let $0 \leq j \leq n-2$. Since $y_{j+1} \vee \dots \vee y_{n-1} \in I_j \subseteq \{x_j\} \vee I_{n-1}$ we may select $t_j \in I_{n-1}$ such that $y_{j+1} \vee \dots \vee y_{n-1} \leq x_j \vee t_j$.

Set $t := t_0 \vee \dots \vee t_{n-2}$ and $y_n := x_n \vee z \vee t$. Let $f : \Omega(\omega^*) \rightarrow P$ be defined by $f(i, j) := y_i \vee y_j$ for all $(i, j), i < j < \omega$. Condition c_n) ensures that f is join-preserving. Indeed, let $(i, j), (i', j') \in \Omega(\omega^*)$. We have $(i, j) \vee (i', j') = (i \wedge i', j \vee j')$ hence $f((i, j) \vee (i', j')) = f(i \wedge i', j \vee j') = y_{i \wedge i'} \vee y_{j \vee j'}$. If F is a finite subset of ω with minimum a and maximum b then condition c_n) forces $\bigvee \{y_n : n \in F\} = y_a \vee y_b$. If $F := \{i, j, i', j'\}$ then, taking account of $i < j$ and $i' < j'$, we have $f(i, j) \vee f(i', j') = y_i \vee y_j \vee y_{i'} \vee y_{j'} = y_{i \wedge i'} \vee y_{j \vee j'}$. Hence $f((i, j) \vee (i', j')) = f(i, j) \vee f(i', j')$, proving our claim.

Next, f is one-to-one. Let $(i, j), (i', j') \in \Omega(\omega^*)$ such that $f(i, j) = f(i', j')$, that is $y_i \vee y_j = y_{i'} \vee y_{j'}$ (1). Suppose $j < j'$. Since $0 \leq i < j$, condition c_j) implies $y_i \leq y_0 \vee y_j$. On the other hand, since $0 \leq j \leq j' - 1$, condition $c_{j'-1}$) implies $y_j \leq y_0 \vee y_{j'-1}$. Hence $y_i \vee y_j \leq y_0 \vee y_{j'-1}$. From (1) we get $y_{j'} \leq y_0 \vee y_{j'-1}$, contradicting condition $b_{j'}$). Hence $j' \leq j$. Exchanging the roles of j, j' gives $j' \leq j$ thus $j = j'$. If $i < i'$ then, conditions $a_{i'}$) and $a_{j'}$) assure $y_{i'} \in I_{i'-1}$ and $y_{j'} \in I_{j'-1}$. Since $I_{j'-1} \subseteq I_{i'-1}$ we have $y_{i'} \vee y_{j'} \in I_{i'-1}$. On the other hand $x_i \notin I_i$ and $x_i \leq y_i \vee y_j$ thus $y_i \vee y_j \notin I_i$. From $I_{i'-1} \subseteq I_i$, we have $y_i \vee y_j \notin I_{i'-1}$, hence $y_i \vee y_j \neq y_{i'} \vee y_{j'}$ and $i' \leq i$. Similarly we get $i \leq i'$. Consequently $i = i'$. □

3.1. Proof of Proposition 1.2. If $\underline{\Omega}(\omega^*)$ embeds in $[\omega]^{<\omega}$ then $[\omega]^{<\omega}$ contains a nonseparating ω^* -chain of ideals. This is impossible: a nonseparating chain of ideals of $[\omega]^{<\omega}$ has necessarily a least element. Indeed, if the pair x, I ($x \in [\omega]^{<\omega}, I \in \mathcal{I}$) witnesses the fact that the chain \mathcal{I} is nonseparating then there are at most $|x| + 1$ ideals belonging to \mathcal{I} which are included in I (note that the set $\{\cup I \setminus \cup J : J \subseteq I, J \in \mathcal{I}\}$ is a chain of subsets of x). The proof of the general case requires more care. If $\underline{\Omega}(\omega^*)$ embeds in $I_{<\omega}(Q)$ as a join-semilattice then we may find a sequence $x_0, I_0, \dots, x_n, I_n, \dots$ such that $I_n \subset I_{n-1} \in J(I_{<\omega}(Q))$, $x_n \in I_{n-1} \setminus I_n$ and $I_n \subseteq \{x_n\} \vee I_m$ for every $n < \omega$ and every $m < \omega$. Set $I_\omega := \bigcap \{I_n : n < \omega\}$, $\bar{I}_n := \cup I_n$ for every $n \leq \omega$, $Q' := Q \setminus \bar{I}_\omega$ and $y_n := x_n \setminus \bar{I}_\omega$ for every $n < \omega$. We claim that y_0, \dots, y_n, \dots form a strictly descending sequence in $I_{<\omega}(Q')$. According to property (i) stated in Theorem 2.1, Q' , thus Q , is not well-founded.

First, $y_n \in I_{<\omega}(Q')$. Indeed, if $a_n \in [Q]^{<\omega}$ generates $x_n \in I_{<\omega}(Q)$ then, since $\bar{I}_\omega \in I(Q)$, $a_n \setminus \bar{I}_\omega$ generates $x_n \setminus \bar{I}_\omega \in I(Q')$. Next, $y_{n+1} \subset y_n$. It

suffices to prove that the following inclusions hold:

$$x_{n+1} \cup \bar{I}_\omega \subseteq \bar{I}_n \subset x_n \cup \bar{I}_\omega.$$

Indeed, subtracting \bar{I}_ω , from the sets figuring above, we get:

$$y_{n+1} = (x_{n+1} \cup \bar{I}_\omega) \setminus \bar{I}_\omega \subset (x_n \cup \bar{I}_\omega) \setminus \bar{I}_\omega = y_n.$$

The first inclusion is obvious. For the second note that, since $J(I_{<\omega}(Q))$ is isomorphic to $I(Q)$, complete distributivity holds, hence with the hypotheses on the sequence $x_0, I_0, \dots, x_n, I_n, \dots$ we have $I_n \subseteq \bigcap \{ \{x_n\} \vee I_m : m < \omega \} = \{x_n\} \vee \bigcap \{ I_m : m < \omega \} = \{x_n\} \vee I_\omega$, thus $\bar{I}_n \subset x_n \cup \bar{I}_\omega$. \square

Remark. One can deduce the fact that $\Omega(\omega^*)$ does not embed as a join-semilattice in $[\omega]^{<\omega}$ from the fact that it contains a strictly descending chain of completely meet-irreducible ideals (namely the chain $\mathcal{I} := \{I_n : n < \omega\}$ where $I_n := \{(i, j) : n \leq i < j < \omega\}$) (see Proposition 4.10) but this fact by itself does not prevent the existence of some well-founded poset Q such that $\Omega(\omega^*)$ embeds as a join semilattice in $I_{<\omega}(Q)$.

3.2. Proof of Theorem 1.3. In terms of join-semilattices and ideals, the result becomes this: let P be a join-semilattice, then $J(P)$ is well-founded if and only if P is well-founded and contains no join-subsemilattice isomorphic to $\Omega(\omega^*)$ or to $[\omega]^{<\omega}$.

The proof goes as follows. Suppose that $J(P)$ is not well-founded. If some ω^* -chain in $J(P)$ is separating then, according to Lemma 3.1, P contains an infinite independent set. From Theorem 1.4, it contains a join-subsemilattice isomorphic to $[\omega]^{<\omega}$. If no ω^* -chain in $J(P)$ is separating, then all the infinite subchains of an arbitrary ω^* -chain are nonseparating. From Lemma 3.2, either ω^* or $\Omega(\omega^*)$ embed in P as a join-semilattice. The converse is obvious. \square

4. JOIN-SUBSEMILATTICES OF $I_{<\omega}(Q)$ AND PROOF OF THEOREM 1.6

In this section, we consider join-semilattices which embed in join-semilattices of the form $I_{<\omega}(Q)$. These are easy to characterize internally (see Proposition 4.4). This is also the case if the posets Q are antichains (see Proposition 4.10) but does not go so well if the posets Q are well-founded (see Lemma 4.8).

Let us recall that if P is a join-semilattice, an element $x \in P$ is *join-prime* (or prime if there is no confusion), if it is distinct from the least element 0, if any, and if $x \leq a \vee b$ implies $x \leq a$ or $x \leq b$. This amounts to the fact that $P \setminus \uparrow x$ is an ideal. We denote $\mathbb{J}_{\text{pri}}(P)$, the set of join-prime members of P . We recall that $\mathbb{J}_{\text{pri}}(P) \subseteq \mathbb{J}_{\text{irr}}(P)$; the equality holds provided that P is a distributive lattice. It also holds if $P = I_{<\omega}(Q)$. Indeed:

Fact 4.1. *For an arbitrary poset Q , we have:*

$$(4.1) \quad \mathbb{J}_{\text{irr}}(I_{<\omega}(Q)) = \mathbb{J}_{\text{pri}}(I_{<\omega}(Q)) = \text{down}(Q)$$

Fact 4.2. *For a poset P , the following properties are equivalent:*

- P is isomorphic to $I_{<\omega}(Q)$ for some poset Q ;
- P is a join-semilattice with a least element in which every element is a finite join of primes.

Proof. Observe that the primes in $I_{<\omega}(Q)$ are the $\downarrow x$, $x \in Q$. Let $I \in I_{<\omega}(Q)$ and $F \in [Q]^{<\omega}$ generating I , we have $I = \cup\{\downarrow x : x \in F\}$. Conversely, let P be a join-semilattice with a 0. If every element in P is a finite join of primes, then $P \cong I_{<\omega}(Q)$ where $Q := \mathbb{J}_{\text{pri}}(P)$. \square

Let L be a complete lattice. For $x \in L$, set $x^+ := \bigwedge\{y \in L : x < y\}$. We recall that $x \in L$ is *completely meet-irreducible* if $x = \bigwedge X$ implies $x \in X$, or equivalently $x \neq x^+$. We denote $\Delta(L)$ the set of completely meet-irreducible members of L . We recall the following Lemma.

Lemma 4.3. *Let P be a join-semilattice, $I \in J(P)$ and $x \in P$. Then $x \in I^+ \setminus I$ if and only if I is a maximal ideal of $P \setminus \uparrow x$.*

Proposition 4.4. *Let P be a join-semilattice. The following properties are equivalent:*

- (i) P embeds in $I_{<\omega}(Q)$, as a join-semilattice, for some poset Q ;
- (ii) P embeds in $I_{<\omega}(J(P))$ as a join-semilattice;
- (iii) P embeds in $I_{<\omega}(\Delta(J(P)))$ as a join-semilattice;
- (iv) For every $x \in P$, $P \setminus \uparrow x$ is a finite union of ideals.

Proof. (i) \Rightarrow (iv). Let φ be an embedding from P in $P' := I_{<\omega}(Q)$. We may suppose that P has a least element 0 and that $\varphi(0) = \emptyset$ (if P has no least element, add one, say 0, and set $\varphi(0) := \emptyset$; if P has a least element, say a , and $\varphi(a) \neq \emptyset$, add to P an element 0 below a and set $\varphi(0) := \emptyset$). For $J' \in \wp(P')$, let $\varphi^{-1}(J') := \{x \in P : \varphi(x) \in J'\}$. Since φ is order-preserving, $\varphi^{-1}(J') \in I(P)$ whenever $J' \in I(P')$; moreover, since φ is join-preserving, $\varphi^{-1}(J') \in J(P)$ whenever $J' \in J(P')$. Now, let $x \in P$. We have $\varphi^{-1}(P' \setminus \varphi(x)) := P \setminus \uparrow x$. Since $\varphi(x)$ is a finite join of primes, $P' \setminus \uparrow \varphi(x)$ is a finite union of ideals. Since their inverse images are ideals, $P \setminus \uparrow x$ is a finite union of ideals too.

(iv) \Rightarrow (iii). We use the well-known method for representing a poset by a family of sets.

Fact 4.5. *Let P be a poset and $Q \subseteq I(P)$. For $x \in P$ set $\varphi_Q(x) := \{J \in Q : x \notin J\}$. Then:*

- (i) $\varphi_Q(x) \in I(Q)$;
- (ii) $\varphi_Q : P \rightarrow I(Q)$ is an order-preserving map;
- (iii) φ_Q is an order-embedding if and only if for every $x, y \in P$ such that $x \not\leq y$ there is some $J \in Q$ such that $x \notin J$ and $y \in J$.

Applying this to $Q := \Delta(J(P))$ we get immediately that φ_Q is join-preserving. Moreover, $\varphi_Q(x) \in I_{<\omega}(Q)$ if and only if $P \setminus \uparrow x$ is a finite union of ideals. Indeed, we have $P \setminus \uparrow x = \cup \varphi_Q(x)$, proving that $P \setminus \uparrow x$ is

a finite union of ideals provided that $\varphi_Q(x) \in I_{<\omega}(Q)$. Conversely, if $P \searrow \uparrow x$ is a finite union of ideals, say I_0, \dots, I_n , then since ideals are prime members of $I(P)$, every ideal included in I is included in some I_i , proving that $\varphi_Q(x) \in I_{<\omega}(Q)$. To conclude, note that if P is a join-semilattice then φ_Q is join-preserving.

(iii) \Rightarrow (ii). Trivial.

(ii) \Rightarrow (i). Trivial. □

Corollary 4.6. *If a join-semilattice P has no infinite antichain, it embeds in $I_{<\omega}(J(P))$ as a join-subsemilattice.*

Proof. As is well known, if a poset has no infinite antichain then every initial segment is a finite union of ideals (cf. [8], see also [9] 4.7.3 p. 125). Thus Proposition 4.4 applies. □

Another corollary of Proposition 4.4 is the following.

Corollary 4.7. *Let P be a join-semilattice. If for every $x \in P$, $P \searrow \uparrow x$ is a finite union of ideals and $\Delta(J(P))$ is well-founded then P embeds as a join-subsemilattice in $I_{<\omega}(Q)$, for some well-founded poset Q .*

The converse does not hold:

Example 4.8. *There is a bipartite poset Q such that $I_{<\omega}(Q)$ contains a join-semilattice P for which $\Delta(J(P))$ is not well-founded.*

Proof. Let $\underline{2} := \{0, 1\}$ and $Q := \mathbb{N} \times \underline{2}$. Order Q in such a way that $(m, i) < (n, j)$ if $m > n$ in \mathbb{N} and $i < j$ in $\underline{2}$.

Let P be the set of subsets X of Q of the form $X := F \times \{0\} \cup G \times \{1\}$ such that F is a nonempty final segment of \mathbb{N} , G is a nonempty finite subset of \mathbb{N} and

$$(4.2) \quad \min(F) - 1 \leq \min(G) \leq \min(F)$$

where $\min(F)$ and $\min(G)$ denote the least element of F and G with respect to the natural order on \mathbb{N} . For each $n \in \mathbb{N}$, let $I_n := \{X \in P : (n, 0) \notin X\}$.

Claim.

1. Q is bipartite and P is a join-subsemilattice of $I_{<\omega}(Q)$.
2. The I_n 's form a strictly descending sequence of members of $\Delta(J(P))$.

Proof of the Claim. 1. The poset Q is decomposed into two antichains, namely $\mathbb{N} \times \{0\}$ and $\mathbb{N} \times \{1\}$ and for this reason it is called *bipartite*. Next, P is a subset of $I_{<\omega}(Q)$. Indeed, Let $X \in P$. Let F, G such that $X = F \times \{0\} \cup G \times \{1\}$. Set $G' := G \times \{1\}$. If $\min(G) = \min(F) - 1$, then $X = \downarrow G'$ whereas if $\min(G) = \min(F)$ then $X = \downarrow G' \cup \{(\min(F), 0)\}$. In both cases $X \in I_{<\omega}(Q)$. Finally, P is a join-semilattice. Indeed, let $X, X' \in P$ with $X := F \times \{0\} \cup G \times \{1\}$ and $X' := F' \times \{0\} \cup G' \times \{1\}$. Obviously $X \cup X' = (F \cup F') \times \{0\} \cup (G \cup G') \times \{1\}$. Since $X, X' \in P$, $F \cup F'$ is a nonempty final segment of \mathbb{N} and $G \cup G'$ is a nonempty finite subset of \mathbb{N} . We have $\min(G \cup G') = \min(\{\min(G), \min(G')\}) \leq \min(\{\min(F), \min(F')\}) =$

$\min(F \cup F')$ and similarly $\min(F \cup F') - 1 = \min\{\min(F), \min(F')\} - 1 = \min\{\min(F) - 1, \min(F') - 1\} \leq \min\{\min(G), \min(G')\} = \min(G \cup G')$, proving that inequalities as in (4.2) hold. Thus $X \cup X' \in I_{<\omega}(Q)$.

2. Due to its definition, I_n is a nonempty initial segment of P which is closed under finite unions, hence $I_n \in J(P)$. Let $X_n := \{(n, 1), (m, 0) : m \geq n + 1\}$ and $Y_n := X_n \cup \{(n, 0)\}$. Clearly, $X_n \in I_n$ and $Y_n \in P$. We claim that $I_n^+ = I_n \vee \{Y_n\}$. Indeed, let J be an ideal containing strictly I_n . Let $Y := \{m \in \mathbb{N} : m \geq p\} \times \{0\} \cup G \times \{1\} \in J \setminus I_n$. Since $Y \notin I_n$, we have $p \leq n$ hence $Y_n \subseteq Y \cup X_n \in J$. It follows that $Y_n \in J$, thus $I_n^+ \subseteq J$, proving our claim.

Since $I_n^+ \neq I_n$, $I_n \in \Delta(J(P))$. Since, trivially, $I_n^+ \subseteq I_{n-1}$ we have $I_n \subset I_{n-1}$, proving that the I_n 's form a strictly descending sequence. \square

Let E be a set and \mathcal{F} be a subset of $\mathcal{P}(E)$, the power set of E . For $x \in E$, set $\mathcal{F}_{-x} := \{F \in \mathcal{F} : x \notin F\}$ and for $X \subset \mathcal{F}$, set $\overline{X} := \bigcup X$. Let $\mathcal{F}^{<\omega}$ (resp. \mathcal{F}^\cup) be the collection of finite (resp. arbitrary) unions of members of \mathcal{F} . Ordered by inclusion, \mathcal{F}^\cup is a complete lattice, the least element and the largest element being the empty set and $\bigcup \mathcal{F}$, respectively.

Lemma 4.9. *Let Q be a poset, \mathcal{F} be a subset of $I_{<\omega}(Q)$ and $P := \mathcal{F}^{<\omega}$ ordered by inclusion.*

- (i) *The map $X \rightarrow \overline{X}$ is an isomorphism from $J(P)$ onto \mathcal{F}^\cup ordered by inclusion.*
- (ii) *If $I \in \Delta(J(P))$ then there is some $x \in Q$ such that $I = P_{-x}$.*
- (iii) *If $\downarrow q$ is finite for every $q \in Q$ then $\overline{I^+} \setminus \overline{I}$ is finite for every $I \in J(P)$ and the set $\varphi_\Delta(X) := \{I \in \Delta(J(P)) : X \notin I\}$ is finite for every $X \in P$.*

Proof. (i) Let I and J be two ideals of P . Then J contains I if and only if \overline{J} contains \overline{I} . Indeed, if $I \subseteq J$ then, clearly $\overline{I} \subseteq \overline{J}$. Conversely, suppose $\overline{I} \subseteq \overline{J}$. If $X \in I$, then $X \subseteq \overline{I}$, thus $X \subseteq \overline{J}$. Since $X \in I_{<\omega}(Q)$, and $X \subseteq \overline{J}$, there are $X_1, \dots, X_n \in J$ such that $X \subseteq Y = X_1 \cup \dots \cup X_n$. Since J is an ideal $Y \in J$, it follows that $X \in J$.

(ii) Let $I \in \Delta(J(P))$. From (a), we have $\overline{I} \subset \overline{I^+}$. Let $x \in \overline{I^+} \setminus \overline{I}$. Clearly P_{-x} is an ideal containing I . Since $x \notin \overline{P_{-x}}$, P_{-x} is distinct from I^+ . Hence $P_{-x} = I$. Note that the converse of assertion (ii) does not hold in general.

(iii) Let $I \in \Delta(J(P))$ and $X \in I^+ \setminus I$. We have $\{X\} \vee I = I^+$, hence from (i) $\overline{\{X\} \vee I} = \overline{I^+}$. Since $\overline{\{X\} \vee I} = X \cup \overline{I}$ we have $\overline{I^+} \setminus \overline{I} \subseteq X$. From our hypothesis on P , X is finite, hence $\overline{I^+} \setminus \overline{I}$ is finite. Let $X \in P$. If $I \in \varphi_\Delta(X)$ then according to (ii) there is some $x \in Q$ such that $I = P_{-x}$. Necessarily $x \in X$. Since X is finite, the number of these I s is finite. \square

Proposition 4.10. *Let P be a join-semilattice. The following properties are equivalent:*

- (i) *P embeds in $[E]^{<\omega}$ as a join-subsemilattice for some set E ;*
- (ii) *for every $x \in P$, $\varphi_\Delta(x)$ is finite.*

Proof. (i) \Rightarrow (ii). Let φ be an embedding from P in $[E]^{<\omega}$ which preserves joins. Set $\mathcal{F} := \varphi(P)$. Apply part (iii) of Lemma 4.9.

(ii) \Rightarrow (i). Set $E := \Delta(J(P))$. We have $\varphi_\Delta(x) \in [E]^{<\omega}$. According to Fact 4.5 and Lemma 4.3, the map $\varphi_\Delta : P \rightarrow [E]^{<\omega}$ is an embedding preserving joins. \square

Corollary 4.11. *Let β be a countable order type. If a proper initial segment contains infinitely many nonprincipal initial segments then no sierpinskiisation P of β with ω can embed in $[\omega]^{<\omega}$ as a join-semilattice (whereas it embeds as a poset).*

Proof. According to Proposition 4.10 it suffices to prove that P contains some x for which $\varphi_\Delta(x)$ is infinite.

Let P be a sierpinskiisation of β and ω . It is obtained as the intersection of two linear orders L, L' on the same set and having respectively order type β and ω . We may suppose that the ground set is \mathbb{N} and L' the natural order.

Claim 4.12. *A nonempty subset I is a nonprincipal ideal of P if and only if this is a nonprincipal initial segment of L .*

Proof of Claim 4.12. Suppose that I is a nonprincipal initial segment of L . Then, clearly, I is an initial segment of P . Let us check that I is up-directed. Let $x, y \in I$; since I is nonprincipal in L , the set $A := I \cap \uparrow_L x \cap \uparrow_L y$ of upper-bounds of x and y with respect to L which belong to I is infinite; since $B := \downarrow_{L'} x \cup \downarrow_{L'} y$ is finite, $A \setminus B$ is nonempty. An arbitrary element $z \in A \setminus B$ is an upper bound of x, y in I with respect to the poset P proving that I is up-directed. Since I is infinite, I cannot have a largest element in P , hence I is a nonprincipal ideal of P . Conversely, suppose that I is a nonprincipal ideal of P . Let us check that I is an initial segment of L . Let $x \leq_L y$ with $y \in I$. Since I nonprincipal in P , $A := \uparrow_P y \cap I$ is infinite; since $B := \downarrow_{L'} x \cup \downarrow_{L'} y$ is finite, $A \setminus B$ is nonempty. An arbitrary element of $A \setminus B$ is an upper bound of x and y in I with respect to P . It follows that $x \in I$. If I has a largest element with respect to L then such an element must be maximal in I with respect to P , and since I is an ideal, I is a principal ideal, a contradiction.

Claim 4.13. *Let $x \in \mathbb{N}$. If there is a nonprincipal ideal of L which does not contain x , there is a maximal one, say I_x . If P is a join-semilattice, $I_x \in \Delta(P)$.*

Proof of Claim 4.13. The first part follows from Zorn's Lemma. The second part follows from Claim 4.12 and Lemma 4.3.

Claim 4.14. *If an initial segment I of β contains infinitely many nonprincipal initial segments then there is an infinite sequence $(x_n)_{n < \omega}$ of elements of I such that the I_{x_n} 's are all distinct.*

Proof of Claim 4.14. With Ramsey's theorem, obtain a sequence $(I_n)_{n < \omega}$ of nonprincipal initial segments which is either strictly increasing or strictly

decreasing. Separate two successive members by some element x_n and apply the first part of Claim 4.13.

If we pick $x \in \mathbb{N} \setminus I$ then it follows from Claim 4.14 and the second part of Claim 4.12 that $\varphi_\Delta(x)$ is infinite. \square

Example 4.15. *If α is a countably infinite order type distinct from ω , $\Omega(\alpha)$ is not embeddable in $[\omega]^{<\omega}$ as a join-semilattice.*

Indeed, $\Omega(\alpha)$ is a sierpinskiisation of $\omega\alpha$ and ω . If α is distinct from ω , α contains some element which majorizes infinitely many others. Thus $\beta := \omega\alpha$ satisfies the hypothesis of Corollary 4.11.

Note that on an other hand, for every ordinal $\alpha \leq \omega$, there are representatives of $\Omega(\alpha)$ which are embeddable in $[\omega]^{<\omega}$ as join-semilattices.

Theorem 4.16. *Let Q be a well-founded poset and let $\mathcal{F} \subseteq I_{<\omega}(Q)$. The following properties are equivalent:*

- 1) \mathcal{F} has no infinite antichain;
- 2) $\mathcal{F}^{<\omega}$ is wqo;
- 3) $J(\mathcal{F}^{<\omega})$ is topologically scattered;
- 4) \mathcal{F}^\cup is order-scattered;
- 5) $\wp(\omega)$ does not embed in \mathcal{F}^\cup ;
- 6) $[\omega]^{<\omega}$ does not embed in $\mathcal{F}^{<\omega}$;
- 7) \mathcal{F}^\cup is well-founded.

Proof. We prove the following chain of implications:

$$1) \implies 2) \implies 3) \implies 4) \implies 5) \implies 6) \implies 7) \implies 1).$$

1) \implies 2). Since Q is well-founded then, as mentioned in (i) of Theorem 2.1, $I_{<\omega}(Q)$ is well-founded. It follows first that $\mathcal{F}^{<\omega}$ is well-founded, hence from property (iii) of Theorem 2.1, every member of $\mathcal{F}^{<\omega}$ is a finite join of join-irreducibles. Next, as a subset of $\mathcal{F}^{<\omega}$, \mathcal{F} is well-founded, hence wqo according to our hypothesis. The set of join-irreducible members of $\mathcal{F}^{<\omega}$ is wqo as a subset of \mathcal{F} . From property (iv) of Theorem 2.1, $\mathcal{F}^{<\omega}$ is wqo.

2) \implies 3). If $\mathcal{F}^{<\omega}$ is wqo then $I(\mathcal{F}^{<\omega})$ is well-founded (cf. property (ii) of Theorem 2.1). It follows that $I(\mathcal{F}^{<\omega})$ is topologically scattered (cf. [18]); hence all its subsets are topologically scattered, in particular $J(\mathcal{F}^{<\omega})$.

3) \implies 4). Suppose that \mathcal{F}^\cup is not ordered scattered. Let $f : \eta \rightarrow \mathcal{F}^\cup$ be an embedding. For $r \in \eta$ set $\check{f}(r) = \cup\{f(r') : r' < r\}$. Let $X := \{\check{f}(r) : r < \eta\}$. Clearly $X \subseteq \mathcal{F}^\cup$. Furthermore X contains no isolated point (Indeed, since $\check{f}(r) = \cup\{\check{f}(r') : r' < r\}$, $\check{f}(r)$ belongs to the topological closure of $\{\check{f}(r') : r' < r\}$). Hence \mathcal{F}^\cup is not topologically scattered.

4) \implies 5). Suppose that $\wp(\omega)$ embeds in \mathcal{F}^\cup . Since $\eta \leq \wp(\omega)$, we have $\eta \leq \mathcal{F}^\cup$.

5) \implies 6). Suppose that $[\omega]^{<\omega}$ embeds in $\mathcal{F}^{<\omega}$, then $J([\omega]^{<\omega})$ embeds in $J(\mathcal{F}^{<\omega})$. Lemma 4.9 ensures that $J(\mathcal{F}^{<\omega})$ is isomorphic to \mathcal{F}^\cup . On the other hand $J([\omega]^{<\omega})$ is isomorphic to $\wp(\omega)$. Hence $\wp(\omega)$ embeds in \mathcal{F}^\cup .

6) \implies 7). Suppose \mathcal{F}^\cup not well-founded. Since Q is well-founded, (i) of

Theorem 2.1 assures $I_{<\omega}(Q)$ well-founded, but $\mathcal{F}^{<\omega} \subseteq I_{<\omega}(Q)$, hence $\mathcal{F}^{<\omega}$ is well-founded. Furthermore, since $I_{<\omega}(Q)$ is closed under finite unions, we have $\mathcal{F}^{<\omega} \subseteq I_{<\omega}(Q)$, Proposition 1.2 implies that $\underline{\Omega}(\omega^*)$ does not embed in $\mathcal{F}^{<\omega}$. From Theorem 1.3, we have $\mathcal{F}^{<\omega}$ not well-founded.

7) \implies 1). Clearly, \mathcal{F} is well-founded. If F_0, \dots, F_n, \dots is an infinite antichain of members of \mathcal{F} , define $f(i, j) : [\omega]^2 \rightarrow Q$, choosing $f(i, j)$ arbitrary in $\max(F_i) \setminus F_j$. Divide $[\omega]^3$ into $R_1 := \{(i, j, k) \in [\omega]^3 : f(i, j) = f(i, k)\}$ and $R_2 := [\omega]^3 \setminus R_1$. From Ramsey's theorem, cf. [21], there is some infinite subset X of ω such that $[X]^3$ is included in R_1 or in R_2 . The inclusion in R_2 is impossible since $\{f(i, j) : j < \omega\}$, being included in $\max(F_i)$, is finite for every i . For each $i \in X$, set $G_i := \cup\{F_j : i \leq j \in X\}$. This defines an ω^* -chain in \mathcal{F}^\cup . \square

Remark. If $\mathcal{F}^{<\omega}$ is closed under finite intersections then equivalence between 3) and 4) follows from Mislove's Theorem mentioned in [18].

Theorem 4.16 above was obtained by the second author and M. Sobrani in the special case where Q is an antichain [19, 23].

Corollary 4.17. *If P is a join-subsemilattice of a join-semilattice of the form $[\omega]^{<\omega}$, or more generally of the form $I_{<\omega}(Q)$ where Q is some well-founded poset, then $J(P)$ is well-founded if and only if P has no infinite antichain.*

Remark. If, in Theorem 4.16 above, we suppose that \mathcal{F} is well-founded instead of Q , all implications in the above chain hold, except 6) \implies 7). A counterexample is provided by $Q := \omega \oplus \omega^*$, the direct sum of the chains ω and ω^* , and \mathcal{F} , the image of $\underline{\Omega}(\omega^*)$ via a natural embedding.

4.1. Proof of Theorem 1.6. (i) \implies (ii). Suppose that (i) holds. Set $Q := J(P)$. Since P contains no infinite antichain, P embeds as a join-subsemilattice in $I_{<\omega}(Q)$ (Corollary 4.6). From (ii) of Theorem 2.1 Q is well-founded. Since P has no infinite antichain, it has no infinite independent set.

(ii) \implies (i). Suppose that (ii) holds. Since Q is well-founded, then from (i) of Theorem 2.1, $I_{<\omega}(Q)$ is well-founded. Since P embeds in $I_{<\omega}(Q)$, P is well-founded. From our hypothesis, P contains no infinite independent set. According to implication (iii) \implies (i) of Theorem 1.4, it does not embed $[\omega]^{<\omega}$. From implication 6) \implies 1) of Theorem 4.16, it has no infinite antichain.

REFERENCES

1. G. Birkhoff, *Lattice Theory*, 3rd ed., A.M.S. Coll. Pub., vol. 25, American Mathematical Society, 1967.
2. I. Chakir, *Chaînes d'idéaux et dimension algébrique des treillis distributifs*, Ph.D. thesis, Université Claude-Bernard(Lyon1), 1992.
3. ———, *The length of chains in modular lattices*, Order **24** (2007), 227–247.
4. ———, *Chains conditions in algebraic lattices*, Ph.D. thesis, University Mohamed V, Faculty of Sciences, Rabat, May 2009, arXiv:1609.07167.

5. I. Chakir and M. Pouzet, *The length of chains in distributive lattices*, Abstracts of papers presented to the A.M.S. **92**, 502–503, T-06-118.
6. ———, *Infinite independent sets in distributive lattices*, Algebra Universalis **53** (2005), no. 2, 211–225.
7. D. H. J. de Jongh and R. Parikh., *Well-partial orderings and hierarchies*, Indag. Math. (Proc) **80** (1977), no. 3, 195–207.
8. P. Erdős and A. Tarski, *On families of mutually exclusive sets*, Annals of Math. **44** (1943), 315–329.
9. R. Fraïssé, *Theory of Relations*, North-Holland Publishing Co., Amsterdam, 2000.
10. F. Galvin, E. C. Milner, and M. Pouzet, *Cardinal representations for closures and preclosures*, Trans. Amer. Math. Soc. **328** (1991), 667–693.
11. G. Grätzer, *General Lattice Theory*, Birkhäuser, Basel, 1998.
12. G. Higman, *Ordering by divisibility in abstract algebras*, Proc. London. Math. Soc. **2** (1952), no. 3, 326–336.
13. K. H. Hofmann, M. Mislove, and A. R. Stralka, *The Pontryagin duality of compact 0-dimensional semilattices and its applications*, Lecture Note in Mathematics, vol. 396, Springer-Verlag., 1974.
14. J. D. Lawson, M. Mislove, and H. A. Priestley, *Infinite antichains in semilattices*, Order **2** (1985), 275–290.
15. ———, *Ordered sets with no infinite antichains*, Discrete Math. **63** (1987), 225–230.
16. ———, *Infinite antichains and duality theories*, Houston Journal of Mathematics **14** (1988), no. 3, 423–441.
17. E. C. Milner and M. Pouzet, *Combinatorics, Paul Erdős is Eighty*, Report, 1993, pp. 277–299.
18. M. Mislove, *When are order scattered and topologically scattered the same?*, Annals of Discrete Math. **23** (1984), 61–80.
19. M. Pouzet and M. Sobrani, *Ordinal invariants of an age*, Tech. report, Université Claude-Bernard (Lyon 1), August 2002.
20. M. Pouzet and N. Zaguia, *Ordered sets with no chains of ideals of a given type*, Order **1** (1984), 159–172.
21. F. P. Ramsey, *On a problem of formal logic*, Proc. London Math. Soc. **30** (1930), 264–286.
22. S. Shelah, *Independence of strong partition relation for small cardinals, and the free subset problem*, The Journal of Symbolic Logic **45** (1980), 505–509.
23. M. Sobrani, *Sur les âges de relations et quelques aspects homologiques des constructions $D+M$* , Ph.D. thesis, Université S. M. Ben Abdallah-Fez, Fez, Morocco, 2002.

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