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BOUNDS FOR THE m-ETERNAL DOMINATION NUMBER OF A GRAPH

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Abstract. Mobile guards on the vertices of a graph are used to defend the graph against an infinite sequence of attacks on vertices. A guard must move from a neighboring vertex to an attacked vertex (we assume attacks happen only at vertices containing no guard and that each vertex contains at most one guard). More than one guard is allowed to move in response to an attack. The m-eternal domination number, $\gamma_{\mathrm{m}}^{\infty}(G)$, of a graph G is the minimum number of guards needed to defend G against any such sequence. We show that if G is a connected graph with minimum degree at least 2 and of order $n \geq 5$, then $\gamma_{\mathrm{m}}^{\infty}(G) \leq \lfloor (n-1)/2 \rfloor$, and this bound is tight. We also prove that if G is a cubic bipartite graph of order n, then $\gamma_{\mathrm{m}}^{\infty}(G) \leq 7n/16$.

1. Introduction

Let G be an undirected graph with vertex set $V(G)$ and edge set $E(G)$. The order of G is given by $n(G) = |V(G)|$ and its size by $m(G) = |E(G)|$. If the graph G is clear from the context, we simply write V, E, n and m rather than $V(G)$, $E(G)$, $n(G)$, and $m(G)$, respectively. Several recent papers have considered problems associated with using mobile guards to defend G against an infinite sequence of attacks; see the survey by Klostermeyer and Mynhardt [12]. We will be interested in a particular version of this problem known as m-eternal domination, defined below.

A *dominating set* of graph G is a set $D \subseteq V$ with the property that for each $u \in V \setminus D$, there exists $x \in D$ adjacent to u. The minimum cardinality amongst all dominating sets of G is the *domination number* $\gamma(G)$. Further background on domination can be found in [6]. Let $D_i \subseteq V$, $1 \leq i$, be a set of vertices with one guard located on each vertex of D_i. In this paper, we allow

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at most one guard to be located on a vertex at any time. Eternal domination problems can be modeled as a two-player game between a *defender* and an *attacker*: the defender chooses D_1 as well as each D_i, $i > 1$, while the attacker chooses the locations of the attacks $r_1, r_2, \ldots$. Note that the location of an attack can be chosen by the attacker depending on the location of the guards. Each attack is handled by the defender by choosing the next D_i subject to some constraints that depend on the particular game. The defender wins the game if they can successfully defend any series of attacks, subject to the constraints of the game; the attacker wins otherwise.

In the *eternal dominating set problem*, each D_i, $i \geq 1$, is required to be a dominating set, $r_i \in V$ (assume, without loss of generality, that $r_i \notin D_i$), and D_{i+1} is obtained from D_i by moving one guard to r_i from a vertex $v \in D_i$, where $v \in N(r_i)$. The smallest size of an eternal dominating set for G is denoted $\gamma^\infty(G)$. This problem was first studied in [1].

In the *m-eternal dominating set problem*, each D_i, $i \geq 1$, is required to be a dominating set, $r_i \in V$ (assume, without loss of generality, that $r_i \notin D_i$), and D_{i+1} is obtained from D_i by moving guards to neighboring vertices. That is, each guard in D_i may move to an adjacent vertex. It is required that $r_i \in D_{i+1}$. The smallest size of an m-eternal dominating set for G, denoted $\gamma^\infty_{\rm m}(G)$, is the *m-eternal domination number* of G. This "all-guards move" version of the problem was introduced in [4] and has been subsequently studied in a number of papers such as [2, 3, 5, 8, 9, 10, 11]. It is clear that $\gamma^\infty(G) \geq \gamma^\infty_{\rm m}(G) \geq \gamma(G)$ for all graphs G. An example that will be important to us is $\gamma^\infty_{\rm m}(C_n) = \lceil n/3 \rceil$. We say that a vertex is *protected* if there is a guard on the vertex or on an adjacent vertex. We say that an attack at v is *defended* if we send a guard to v. More generally, we *defend* a graph by defending all the attacks in an attack sequence.

Our aim in this paper is twofold. Our first is to establish a tight upper bound on the m-eternal domination number of a connected graph in terms of its order. Our second is to prove that the m-eternal domination number of a cubic bipartite graph with n vertices is at most $7n/16$.

1.1. **Notation.** For notation and graph theory terminology, we in general follow [6]. The *open neighborhood* of a vertex $v \in V(G)$ is $N_G(v) = \{u \in V \mid uv \in E(G)\}$ and its *closed neighborhood* is the set $N_G[v] = N_G(v) \cup \{v\}$. Further, for $S \subseteq V$, let $N_G(S) = \bigcup_{x \in S} N_G(x)$. The degree of v is $d_G(v) = |N_G(v)|$. The minimum degree among all the vertices of G is denoted by $\delta(G)$. A *leaf* is a vertex of degree 1, while its neighbor is a *support vertex*. If the graph G is clear from the context, we simply write V, E, $N(v)$, $N[v]$, and $d(v)$ rather than $V(G)$, $E(G)$, $N_G(v)$, $N_G[v]$ and $d_G(v)$, respectively.

For a set S of vertices, $G-S$ is the graph obtained from G by removing all vertices of S and removing all edges incident to vertices of S. The subgraph induced by S is denoted by $G[S]$. For a set F of edges of G, $G-F$ is the graph obtained from G by removing all edges of F from G. A *cycle* on n vertices is denoted by C_n and a *path* on n vertices (of length $n-1$) by P_n.

An n-cycle is a cycle C_n. An *odd-length cycle* (*odd-length path*) is a cycle (respectively, path) of odd length. An *even-length cycle* is a cycle of even length. A *dumb-bell* is a connected graph on $n = n_1 + n_2$ vertices that can be constructed by joining a vertex of a cycle C_{n_1} to a vertex of a cycle C_{n_2} by an edge. We denote the resulting dumb-bell by $D_b(n_1, n_2)$. A *non-trivial graph* is a graph on at least two vertices.

An *independent set* of vertices in G is a set $I \subseteq V$ with the property that no two vertices in I are adjacent. The maximum cardinality amongst all independent sets is the *independence number*, which we denote as $\alpha(G)$. A set of pairwise independent edges of G is called a *matching* in G. If M is a matching in G, then a vertex incident with an edge of M is called M-*matched*. A *perfect matching* M in G is a matching such that every vertex of G is incident to an edge of M. If X and Y are vertex disjoint sets in G, we let $G[X, Y]$ denote the set of edges of G with one end in X and the other end in Y. We use the standard notation $[k] = \{1, 2, \ldots, k\}$.

1.2. **Known Results on Domination and m-Eternal Domination.** Ore [16] established the following classical upper bound on the domination number of a graph in terms of its order.

Theorem 1.1 ([16]). *If G is a graph of order n with no isolated vertex, then $\gamma(G) \le n/2$.*

Chambers et al. [2] showed that the 1/2-bound due to Ore on the domination number almost holds for the m-eternal domination. More precisely, they prove the following result, which also appears with a different proof in the survey [12].

Theorem 1.2 ([2, 12]). *If G is a connected graph of order n, then $\gamma_{\rm m}^{\infty}(G) \le \lceil n/2 \rceil$.*

McCuaig and Shepherd [15] proved the following result.

Theorem 1.3 ([15]). *If G is a connected graph of order $n \ge 8$ with $\delta(G) \ge 2$, then $\gamma(G) \le 2n/5$.*

If we restrict the minimum degree to be at least three, then Reed [18] showed that the upper bound in Theorem 1.3 can be improved from two-fifths the order to three-eighths the order.

Theorem 1.4 ([18]). *If G is a graph of order n with $\delta(G) \ge 3$, then $\gamma(G) \le 3n/8$.*

As a special case of Theorem 1.4, we have the following result.

Theorem 1.5 ([18]). *If G is a cubic graph of order n, then $\gamma(G) \le 3n/8$.*

The two non-planar cubic graphs of order $n = 8$ (shown in Figure 1(a) and 1(b)) both have domination number 3 and achieve the upper bound in Theorem 1.5.

Kostochka and Stocker [13] improved Reed's bound as follows.

FIGURE 1. The two non-planar cubic graphs of order eight.

Theorem 1.6 ([13]). *If G is a connected cubic graph of order $n \geq 10$, then $\gamma(G) \leq 5n/14$.*

2. Graphs with Minimum Degree Two

Recall that if we restrict the minimum degree to be at least two, then the upper bound on the (ordinary) domination number improves from one-half the order to two-fifths the order (for connected graphs of order at least 8), as shown by Theorem 1.1 and Theorem 1.3. However, this is not the case for the m-eternal domination number. We show in this section that the upper bound of Theorem 1.2 can be improved ever-so-slightly if the minimum degree is at least 2 and the order at least 5. More precisely, we shall prove the following result.

Theorem 2.1. *If G is a connected graph with $\delta(G) \geq 2$ of order $n \neq 4$, then*

$$\gamma_{\rm m}^{\infty}(G) \leq \left\lfloor \frac{n-1}{2} \right\rfloor,$$

and this bound is tight.

By Theorem 1.2, if G is a connected graph with $\delta(G) \geq 2$ of order $n > 4$, then $\gamma_{\rm m}^{\infty}(G) \leq \lceil n/2 \rceil = \lfloor (n-1)/2 \rfloor + 1$. Thus, Theorem 2.1 states that the upper bound in Theorem 1.2 can be decreased by 1 if we restrict the connected graph G to have minimum degree at least 2 and order at least 5.

That the bound of Theorem 2.1 is tight may be seen as follows. Let G_k be the graph obtained from two vertex disjoint 4-cycles by adding an edge joining a vertex from one copy of a 4-cycle to a vertex from the other copy, and then subdividing the added edge $2k+1$ times for some integer $k \geq 0$. The resulting connected graph G_k has order $n = 2k+9$ and satisfies $\gamma_{\rm m}^{\infty}(G_k) = k+4 = (n-1)/2 = \lfloor (n-1)/2 \rfloor$.

2.1. **A Proof of Theorem 2.1.** Let G be a graph with $\delta(G) \geq 2$. We define a vertex v of G to be *large* if $d_G(v) \geq 3$ and *small* if $d_G(v) = 2$. In order to prove Theorem 2.1, define a graph G to be an *edge-minimal graph* if G is edge-minimal among all graphs G satisfying (i) $\delta(G) \geq 2$ and (ii) G is connected. In this section, we prove the following result about edge-minimal graphs.

Theorem 2.2. *If G is an edge-minimal graph of order n, then $G = C_4$ or $\gamma_{\rm m}^{\infty}(G) < n/2$.*

Proof. Suppose, to the contrary, that the theorem is false and that G is a counterexample with minimum value of $n(G) + m(G)$, where $n = n(G)$ and $m = m(G)$. Thus, G is an edge-minimal graph with $\gamma_{\mathrm{m}}^{\infty}(G) \geq n/2$ and $G \neq C_4$, but if G' is an edge-minimal graph with $n(G') + m(G') < n(G) + m(G)$, then $G' = C_4$ or $\gamma_{\mathrm{m}}^{\infty}(G') < n(G')/2$. If $n = 3$, then $G = C_3$ and $\gamma_{\mathrm{m}}^{\infty}(G) = 1 < n/2$. If $n = 4$, then $G = C_4$. Hence, $n \geq 5$. By the minimality of G, we have the following observation.

Observation 2.3. *If $e \in E(G)$, then either e is a bridge of G or $\delta(G-e) = 1$.*

Let F be a connected graph with $n(F) + m(F) < n(G) + m(G)$ and with $\delta(F) \geq 2$. If F is an edge-minimal graph, let $F' = F$. Otherwise, let F' be an edge-minimal graph obtained from F by removing edges from F until we produce an edge-minimal graph. Since the m-eternal domination number cannot decrease if edges are removed, we note that $\gamma_{\mathrm{m}}^{\infty}(F) \leq \gamma_{\mathrm{m}}^{\infty}(F')$. By the minimality of G, $F' = C_4$ or $\gamma_{\mathrm{m}}^{\infty}(F') < n(F')/2$. If $F' = C_4$, then either $F = K_4$, in which case $\gamma_{\mathrm{m}}^{\infty}(F) = 1$, or $F \in \{C_4, K_4 - e\}$. We state this formally as follows.

Observation 2.4. *If F is a connected graph with $n(F) + m(F) < n(G) + m(G)$ and with $\delta(F) \geq 2$, then $F \in \{C_4, K_4 - e\}$ or $\gamma_{\mathrm{m}}^{\infty}(F) < n(F)/2$.*

If $G = C_n$ (and still $n \geq 5$), then $\gamma_{\mathrm{m}}^{\infty}(G) = \lceil n/3 \rceil < n/2$, a contradiction. Hence, G is not a cycle. Let $\mathcal{L}$ be the set of all large vertices of G and let $\mathcal{S}$ be the set of small vertices in G, i.e., $\mathcal{L} = \{v \in V(G) \mid d_G(v) \geq 3\}$ and $\mathcal{S} = \{v \in V(G) \mid d_G(v) = 2\}$. Since G is not a cycle, $|\mathcal{L}| \geq 1$. We will now prove a number of claims.

Claim I. *The set $\mathcal{L}$ is an independent set.*

Proof of Claim I. Suppose, to the contrary, that $\mathcal{L}$ is not an independent set. Let u and v be two adjacent vertices in $\mathcal{L}$ and let $e = uv$. By Observation 2.3, e is a bridge. Let G_1 and G_2 be the two components of $G - e$, where $u \in V(G_1)$. For $i = 1, 2$, let $|V(G_i)| = n_i$, and so $n = n_1 + n_2$. Since $u, v \in \mathcal{L}$ in G, we note that $\delta(G_1) \geq 2$ and $\delta(G_2) \geq 2$. Since G is edge-minimal, so too are both G_1 and G_2. By the minimality of G, for $i \in \{1, 2\}$, either $\gamma_{\mathrm{m}}^{\infty}(G_i) < n_i/2$ or $G_i = C_4$. If $\gamma_{\mathrm{m}}^{\infty}(G_1) < n_1/2$ or $\gamma_{\mathrm{m}}^{\infty}(G_2) < n_2/2$, then $\gamma_{\mathrm{m}}^{\infty}(G) \leq \gamma_{\mathrm{m}}^{\infty}(G_1) + \gamma_{\mathrm{m}}^{\infty}(G_2) < n_1/2 + n_2/2 = n/2$, a contradiction. Hence, both $G_1 = C_4$ and $G_2 = C_4$, and so $n = 8$ and $G = D_b(4, 4)$. Thus, $\gamma_{\mathrm{m}}^{\infty}(G) = 3 < n/2$, once again producing a contradiction. (□)

By Claim I, the set $\mathcal{L}$ is an independent set. Let C be any component of $G - \mathcal{L}$; it must be a path. If C has only one vertex, or has at least two vertices but the ends of C are adjacent in G to different large vertices, then we say that C is a 2-*path*. Otherwise we say that C is a 2-*handle*.

Claim II. *Every 2-path has order 1 and every 2-handle has order 2 or 3.*

Proof of Claim II. Suppose, to the contrary, that there is a 2-path of order 2 or more or a 2-handle of order 4 or more. Then, G contains a path uw_1w_2v on four vertices with both internal vertices, w_1 and w_2, having degree 2 in G and such that u and v are not adjacent. Let G' be the graph obtained from G by removing the vertices w_1 and w_2, and adding the edge uv. Then, G' is a connected graph of order $n' = n - 2$ with $\delta(G') \geq 2$. Let S' be a minimum m-eternal dominating set of G'. If $u \in S'$, let $S = S' \cup \{w_2\}$. If $u \notin S'$, let $S = S' \cup \{w_1\}$. In both cases, the set S is an m-eternal dominating set of G, implying that $\gamma_{\rm m}^{\infty}(G) \leq \gamma_{\rm m}^{\infty}(G') + 1$. By Observation 2.4, $G' \in \{C_4, K_4 - e\}$ or $\gamma_{\rm m}^{\infty}(G') < n'/2$. If $G' = K_4 - e$, then the graph G is determined and the set $\mathcal{L}$ is not independent, a contradiction. If $G' = C_4$, then $G = C_6$ contradicting the fact that $|\mathcal{L}| \geq 1$. Therefore, $\gamma_{\rm m}^{\infty}(G') < n'/2 = n/2 - 1$, and so $\gamma_{\rm m}^{\infty}(G) \leq \gamma_{\rm m}^{\infty}(G') + 1 < n/2$, once again producing a contradiction. (□)

Claim III. *The graph G contains no 2-handle of order 2.*

Proof of Claim III. Suppose, to the contrary, that G contains no 2-handle, v_1v_2, of order 2. Let v be the common neighbor of v_1 and v_2. Suppose that $d_G(v) \geq 4$. In this case, let $G' = G - \{v_1, v_2\}$. Since G is edge-minimal, so too is G'. Let G' have order n', and so $n' = n - 2$. Every minimum m-eternal dominating set of G' can be extended to an m-eternal dominating set of G by adding to it either v_1 or v_2, implying that $\gamma_{\rm m}^{\infty}(G) \leq \gamma_{\rm m}^{\infty}(G') + 1$. By the minimality of G, either $G' = C_4$ or $\gamma_{\rm m}^{\infty}(G') < n'/2 = n/2 - 1$. If $G' = C_4$, then the graph G is determined. In this case, $n = 6$ and $\gamma_{\rm m}^{\infty}(G) = 2 < n/2$, a contradiction. Hence, $\gamma_{\rm m}^{\infty}(G') < n'/2$, implying that $\gamma_{\rm m}^{\infty}(G) < n/2$, a contradiction. Therefore, $d_G(v) = 3$.

Let v_3 be the third neighbor of v. By Claim I, v_3 is a small vertex. By Claim II, v_3 belongs to a 2-path of order 1. Let w be the neighbor of v_3 different from v. Then, $w \in \mathcal{L}$. Let $G' = G - \{v, v_1, v_2, v_3\}$. Since G is edge-minimal, so too is G'. Let G' have order n', and so $n' = n - 4$. Every minimum m-eternal dominating set of G' can be extended to an m-eternal dominating set of G by adding to it the set $\{v, v_3\}$, implying that $\gamma_{\rm m}^{\infty}(G) \leq \gamma_{\rm m}^{\infty}(G') + 2$. By the minimality of G, either $G' = C_4$ or $\gamma_{\rm m}^{\infty}(G') < n'/2 = n/2 - 1$. If $G' = C_4$, then the graph G is determined. In this case, $n = 8$ and $\gamma_{\rm m}^{\infty}(G) = 3 < n/2$, a contradiction. Hence, $\gamma_{\rm m}^{\infty}(G') < n'/2 = n/2 - 2$, implying that $\gamma_{\rm m}^{\infty}(G) < n/2$, a contradiction. (□)

Claim IV. *The graph G contains no 2-handle.*

Proof of Claim IV. Suppose, to the contrary, that G contains a 2-handle. By Claim II and Claim III, such a 2-handle has order 3. Let $v_1v_2v_3$ be a 2-handle in G, and let v be the common neighbor of v_1 and v_3. Suppose that $d_G(v) \geq 4$. In this case, let $G' = G - \{v_1, v_2, v_3\}$. Since G is edge-minimal, so too is G'. Let G' have order n', and so $n' = n - 3$. Let S_v be a minimum m-eternal dominating set of G' that contains v, and let $\overline{S_v}$ be a minimum m-eternal dominating set of G' that does not contain v. Then, $S_v \cup \{v_1\}$ and $\overline{S_v} \cup \{v_2\}$ are both m-eternal dominating sets of G', implying that $\gamma_{\rm m}^{\infty}(G) \le \gamma_{\rm m}^{\infty}(G') + 1 \le n'/2 + 1 < n/2$, a contradiction. Therefore, $d_G(v) = 3$. Let v_4 be the third neighbor of v. By Claim I, v_4 is a small vertex. By Claim II, v_4 belongs to a 2-path of order 1. Let w be the neighbor of v_4 different from v. Then, $w \in \mathcal{L}$. Let $G' = G - \{v, v_1, v_2, v_3, v_4\}$. Since G is edge-minimal, so too is G'. Let G' have order n', and so $n' = n - 5$. Every minimum m-eternal dominating set of G' can be extended to an m-eternal dominating set of G by adding to it, for example, the set $\{v_2, v_4\}$ implying that $\gamma_{\rm m}^{\infty}(G) \le \gamma_{\rm m}^{\infty}(G') + 2 \le n'/2 + 2 < n/2$, a contradiction. (□)

By Claim I, $\mathcal{L}$ is an independent set. By Claim II, every 2-path has order 1. By Claim IV, the graph G contains no 2-handle. Therefore, G is a bipartite graph with partite sets $\mathcal{L}$ and $\mathcal{S}$. Counting the edges of G, we note that $3|\mathcal{L}| \le |E(G)| = 2|\mathcal{S}|$. Further, $n = |\mathcal{L}| + |\mathcal{S}|$, implying that $|\mathcal{L}| \le 2n/5$. For any vertex $v \in \mathcal{L}$ and an arbitrary neighbor v' of v, we note that $(\mathcal{L} \setminus \{v\}) \cup \{v'\}$ is a dominating set. We note that set $\mathcal{L}$ is an m-eternal dominating set of G, since for any vertex $v \in \mathcal{L}$ and an arbitrary neighbor v' of v, the set $(\mathcal{L} \setminus \{v\}) \cup \{v'\}$ is a dominating set. Therefore, $\gamma_{\rm m}^{\infty}(G) \le |\mathcal{L}| \le 2n/5$, a contradiction. This completes the proof of Theorem 2.2. □

Since the m-eternal domination number of a graph cannot decrease if edges are removed, Theorem 2.1 is an immediate consequence of Theorem 2.2.

3. Cubic Bipartite Graphs

By Theorem 1.5, if G is a cubic graph of order n, then $\gamma(G) \le 3n/8$. The two non-planar cubic graphs of order $n = 8$ shown in Figure 1 both satisfy $\gamma(G) = \gamma_{\rm m}^{\infty}(G) = 3n/8$. However, there are cubic graphs G with $\gamma_{\rm m}^{\infty}(G) > 3n/8$. For example, if G is the Petersen graph (of order $n = 10$), then as first observed in [8], $\gamma_{\rm m}^{\infty}(G) = 4 = 2n/5$. In this section, we focus our attention on cubic bipartite graphs. Our aim is to establish an upper bound on the m-eternal domination number of a cubic bipartite graph in terms of its order. More precisely, we shall prove the following result. A proof of Theorem 3.1 is presented in Section 3.2.

Theorem 3.1. *If G is a cubic bipartite graph of order n, then $\gamma_{\rm m}^{\infty}(G) \le 7n/16$.*

3.1. Preliminary Observations and Lemmas.

We begin with some preliminary lemmas and observations that will aid us when proving our main results. Recall that for $n \ge 4$, $D_b(4, n)$ denotes the dumb-bell on $n + 4$ vertices constructed by joining a vertex of a cycle C_4 to a vertex of a cycle C_n by an edge.

Lemma 3.2. *For $n \ge 4$ even, $\gamma_{\rm m}^{\infty}(D_b(4, n)) = \lceil n/3 \rceil + 1$.*

Proof. We maintain one of two configurations of guards: (a) $\lceil n/3 \rceil$ guards in C_n with one guard on the unique vertex of the C_n adjacent to the C_4

plus one guard in the C_4, the latter guard on the unique vertex of the C_4 at maximum distance from the C_n, or (b) $\lfloor n/3 \rfloor$ guards in C_n plus two guards in the C_4, one of the latter being on the unique vertex of the C_4 adjacent to the C_n. It is easy to see that we can maintain and switch between these two configurations eternally by rotating guards around the C_n and moving guards on and off vertices of the C_4. □

Let C_4^* denote the graph of order 5 obtained from a C_4 by adding a pendant edge to one vertex of the C_4.

Lemma 3.3. *$\gamma_{\rm m}^{\infty}(C_4^*) = 2$.*

Proof. This follows from inspection, noting that one must always keep a guard on either the leaf or the neighbor of the leaf. □

As it is well-known that cubic bipartite graphs are bridgeless, the next lemma follows from Petersen's theorem [17] that every cubic, bridgeless graph contains a perfect matching.

Lemma 3.4 ([17])**.** *Every cubic bipartite graph contains a perfect matching.*

3.2. **Proof of Theorem 3.1.** Before we present a proof of Theorem 3.1, we introduce some new terminology for notational convenience. A cycle of (even) length at least 6 we call a *large cycle*, while a cycle of length 4 we call a *small cycle*. By a *weak partition* of a set we mean a partition of the set in which some of the subsets may be empty. We are now in a position to present a proof of our main result. Recall its statement.

Theorem 3.1. *If G is a cubic bipartite graph of order n, then $\gamma_{\rm m}^{\infty}(G) \leq 7n/16$.*

Proof. By linearity, we may assume that G is connected, for otherwise we apply the result to each component of G. Let M be a perfect matching of G. Let G' be the graph formed by removing from G the edges in M, and so $G' = G - M$. Note that each component of G' is an even-length cycle. Since adding edges to a graph cannot increase the m-eternal domination number, $\gamma_{\rm m}^{\infty}(G) \leq \gamma_{\rm m}^{\infty}(G')$. Hence it suffices for us to show that $\gamma_{\rm m}^{\infty}(G') \leq 7n/16$. Form an auxiliary graph H as follows: each component of G' is a vertex of H and two vertices in H are adjacent if the cycles in G' corresponding to the vertices in H are joined by an edge in G.

We call a vertex in H a *large vertex* (respectively, *small vertex*) if it corresponds to a large cycle (respectively, small cycle) in G'. We let $\mathcal{L}$ and $\mathcal{S}$ denote the set of large and small vertices, respectively, in H. Thus, $(\mathcal{L}, \mathcal{S})$ is a weak partition of $V(H)$. A neighbor of a vertex, v in H that is a large vertex is called a *large neighbor* of v, while a neighbor of v that is a small vertex is called a *small neighbor* of v.

Let $\mathcal{S}_1$ be the subset of vertices of $\mathcal{S}$ that have at least one (small) neighbor in H that belongs to $\mathcal{S}$, and let $\mathcal{S}_2$ be the remaining vertices in $\mathcal{S}$. Thus, $\mathcal{S}_2$ is the subset of vertices of $\mathcal{S}$ all of whose neighbors are large and therefore belong to $\mathcal{L}$. We note that $(\mathcal{S}_1, \mathcal{S}_2)$ is a weak partition of $\mathcal{S}$. We first consider the subgraph $H_1 = H[\mathcal{S}_1]$ of H induced by the set $\mathcal{S}_1$ of small vertices. Let G_1 be the subgraph of G corresponding to H_1, and let G_1 have order n_1.

Claim A $\gamma_{\rm m}^{\infty}(G_1) \le 7n_1/16$.

Proof of Claim A. By linearity, we may assume that G_1 is connected. We note that, by definition of the subgraph H_1, every component of H_1 has order at least 2. We show that if H_1' is an arbitrary induced connected subgraph of H_1 on at least two vertices and if G_1' denotes the subgraph of G corresponding to H_1', then $\gamma_{\rm m}^{\infty}(G_1') \le 7n_1'/16$ where n_1' denotes the order of G_1'. In particular, this would imply that taking $H_1' = H_1$, we have $G_1 = G_1'$ and $\gamma_{\rm m}^{\infty}(G_1) \le 7n_1/16$. We proceed by induction on the order $k \ge 2$ of H_1'.

The subgraph G_1' contains as a spanning subgraph a 2-regular subgraph each component of which is a 4-cycle (corresponding to vertices of H_1'). Recall that H_1' has order $k \ge 2$ and that G_1' has order $n_1' = 4k$. Let T be a spanning tree of H_1'. Since each vertex of T corresponds to a C_4 and since G is cubic, the maximum degree of T is at most 4. If $k = 2$, then $n_1 = 8$ and G_1' consists of two copies of C_4 joined by at least one edge. Thus, G_1' has $D_b(4,4)$ as a spanning subgraph. By Lemma 3.2, $\gamma_{\rm m}^{\infty}(G_1') \le \lceil 4/3 \rceil + 1 = 3 = 3n_1'/8 < 7n_1'/16$. This establishes the base case. Suppose that $k \ge 3$ and that the desired result holds for all induced connected subgraphs of H_1 of order at least 2 and order less than k. Let T be a spanning tree of H_1' of order k. Since $k \ge 3$, we note that $\mathrm{diam}(T) \ge 2$.

We now root the tree T at a vertex r on a longest path in T. Necessarily, r is a leaf. Let u be a vertex at maximum distance from r. Necessarily, u is a leaf. Let v be the parent of u, and let w be the parent of v. Since u is a vertex at maximum distance from the root r, every child of v is a leaf. We proceed further with the following subclaim.

Claim A.1 *If* $\mathrm{diam}(T) = 2$*, then* $\gamma_{\rm m}^{\infty}(G_1') \le 7n_1'/16$.

Proof of Claim A.1. Suppose that $\mathrm{diam}(T) = 2$. In this case, T is a star $K_{1,r}$ for some $r \in \{2,3,4\}$, noting that $\Delta(T) \le 4$.

If $T = K_{1,2}$, then $n_1' = 12$ and G_1' consists of three copies of C_4, with one copy of C_4 joined by at least one edge to each of the other two copies of C_4. In this case, G_1' has $D_b(4,4) \cup C_4$ as a spanning subgraph, implying that $\gamma_{\rm m}^{\infty}(G_1') \le \gamma_{\rm m}^{\infty}(D_b(4,4)) + \gamma_{\rm m}^{\infty}(C_4) = 3 + 2 = 5 = 5n_1'/12 < 7n_1'/16$.

If $T = K_{1,3}$, then $n_1' = 16$ and G_1' consists of four copies of C_4, with one copy of C_4 joined by at least one edge to each of the other three copies of C_4. In this case, G_1' has $3C_4^* \cup K_1$ as a spanning subgraph, implying that $\gamma_{\rm m}^{\infty}(G_1') \le 3\gamma_{\rm m}^{\infty}(C_4^*) + \gamma_{\rm m}^{\infty}(K_1) = 3 \times 2 + 1 = 7 = 7n_1'/16$.

If $T = K_{1,4}$, then $n_1' = 20$ and G_1' consists of five copies of C_4, with one copy of C_4 joined by at least one edge to each of the other four copies of C_4. In this case, G_1' has $4C_4^*$ as a spanning subgraph, implying that $\gamma_{\rm m}^{\infty}(G_1') \le 4\gamma_{\rm m}^{\infty}(C_4^*) = 8 = 2n_1'/5 < 7n_1'/16$. (□)

By Claim A.1, we may assume that $\mathrm{diam}(T) \geq 3$, for otherwise the desired result follows. This implies that $d_T(w) \geq 2$ and that w is not the root of the tree T. We now consider the tree T' obtained from T by deleting the vertex v and its children. Let T' have order k'. Then, $2 \leq k' < k$. Recall that T_v denotes the maximal subtree of T induced by v and its descendants. Thus, T is obtained from the disjoint union of T' and T_v by adding the edge vw. Let G' be the subgraph of G corresponding to T' and let G'_v be the subgraph of G corresponding to T_v. Let ℓ_1 and ℓ_2 denotes the order of G' and G'_v, respectively, and note that $n'_1 = \ell_1 + \ell_2$. Applying the inductive hypothesis to the tree T' and the tree T_v, we note that $\gamma_{\mathrm{m}}^{\infty}(G') \leq 7\ell_1/16$ and $\gamma_{\mathrm{m}}^{\infty}(G'_v) \leq 7\ell_2/16$, implying that $\gamma_{\mathrm{m}}^{\infty}(G'_1) \leq \gamma_{\mathrm{m}}^{\infty}(G') + \gamma_{\mathrm{m}}^{\infty}(G'_v) \leq 7(\ell_1+\ell_2)/16 = 7n'_1/16$. This completes the proof of Claim A. ($\Box$)

We next consider the set of large vertices, $\mathcal{L}$, in H. Let $\mathcal{L}_{10}$ be the subset of (large) vertices in $\mathcal{L}$ that correspond to copies of C_{10} in G' and that have at least one (small) neighbor in H that belongs to $\mathcal{S}_2$. Thus, each vertex in $\mathcal{L}_{10}$ corresponds to a cycle C_{10} of G' that is joined with at least one edge in G to a cycle C_4 of G'. Further, such a cycle C_4 is only joined in G to large cycles of G' since it corresponds to a vertex of H that belongs to $\mathcal{S}_2$. Let F be the bipartite graph with partite sets $\mathcal{L}_{10}$ and $\mathcal{S}_2$, where a vertex in $\mathcal{L}_{10}$ is joined to a vertex $\mathcal{S}_2$ in F if they are adjacent in H. Let M_F be a maximum matching in F, and let H_2 be the subgraph of H induced by the set of M_F-matched vertices. Let $\mathcal{L}_M$ be the subset of vertices of $\mathcal{L}_{10}$ that are M_F-matched in F, and let $\mathcal{S}_M$ be the subset of vertices of $\mathcal{S}_2$ that are M_F-matched in F. We note that $V(H_2) = \mathcal{L}_M \cup \mathcal{S}_M$. Let G_2 be the subgraph of G corresponding to H_2, and let G_2 have order n_2.

Claim B $\gamma_{\mathrm{m}}^{\infty}(G_2) \leq 5n_2/14$.

Proof of Claim B. The subgraph G_2 contains as a spanning subgraph a subgraph each component of which is a dumb-bell $B_b(4,10)$ (that corresponds to an edge of the maximum matching M_F). In particular, we note that G_2 has order $n_2 = 14|M_F|$. By Lemma 3.2, $\gamma_{\mathrm{m}}^{\infty}(D_b(4,10)) = 5$, implying that $\gamma_{\mathrm{m}}^{\infty}(G_2) \leq 5|M_F| = 5n_2/14$. ($\Box$)

Let $\mathcal{L}_1$ be the subset of vertices of $\mathcal{L} \setminus \mathcal{L}_M$ that have at least one (small) neighbor in H that belongs to $\mathcal{S}_2 \setminus \mathcal{S}_M$, and let $\mathcal{L}_2$ be the remaining vertices in $\mathcal{L} \setminus \mathcal{L}_M$. Thus, $\mathcal{L}_2$ is the subset of vertices of $\mathcal{L} \setminus \mathcal{L}_M$ all of whose neighbors belong to $\mathcal{L} \cup \mathcal{S}_1 \cup \mathcal{S}_M$. We note that $(\mathcal{L}_1, \mathcal{L}_2)$ is weak partition of $\mathcal{L} \setminus \mathcal{L}_M$. By the maximality of the matching M_F, we note that no vertex in $\mathcal{L}_1$ corresponds to a cycle C_{10} of G'. Thus, each vertex in $\mathcal{L}_1$ corresponds to a large cycle of G' that is not a 10-cycle and is joined with at least one edge in G to a cycle C_4 of G'. Further, such a cycle C_4 is only joined in G to large cycles of G' since it corresponds to a vertex of H that belongs to $\mathcal{S}_2$. Let $H_3 = H[\mathcal{L}_2]$, and let G_3 be the subgraph of G corresponding to H_3. Further, let G_3 have order n_3.

Claim C $\gamma_{\mathrm{m}}^{\infty}(G_3) \leq 2n_2/5$.

Proof of Claim C. The subgraph G_3 contains as a spanning subgraph a 2-regular subgraph each component of which is a large cycle (corresponding to the vertices of H_2). Since $\gamma_{\rm m}^{\infty}(C_k) = \lceil k/3 \rceil$, for $k \ge 6$ an even integer we note that $\gamma_{\rm m}^{\infty}(C_k) \le 2k/5$, with equality if and only if $k = 10$. Applying this to each large cycle in the 2-regular spanning subgraph of G_3, we deduce that $\gamma_{\rm m}^{\infty}(G_3) \le 2n_3/5$. $(\Box)$

Let $H_4 = H[\mathcal{L}_1 \cup (\mathcal{S}_2 \setminus \mathcal{S}_M)]$. We note that the set $\mathcal{S}_2 \setminus \mathcal{S}_M$ is an independent set in H_4. Let G_4 be the subgraph of G corresponding to H_4, and let G_4 have order n_4.

Claim D $\gamma_{\rm m}^{\infty}(G_4) \le 7n_4/16$.

Proof of Claim D. Let (A, B) be a partition of $V(G_4)$, where the vertices in A belong to large cycles in G_4 associated with the large vertices that belong to $\mathcal{L}_1$ and where the vertices in B belong to small cycles in G_4 associated with the small vertices that belong to $\mathcal{S}_2 \setminus \mathcal{S}_M$. Let $|A| = a$ and $|B| = b$. Thus, $n_4 = |V(G_4)| = a + b$. For notation simplicity, we write $[A, B]$, rather than $G[A, B]$, to denote the set of edges of G with one end in A and the other end in B. Since the set B can be partitioned into sets each of which induce a 4-cycle, we note that $b = 4k$ for some $k \ge 1$. We proceed further with the following subclaim.

Claim D.1 $b \le n_4/2$.

Proof of Claim D.1 We count the number of edges, $|[A, B]|$, with one end in A and the other end in B. Since G is a cubic graph and since the subgraph, $G[B]$, of G induced by B is the disjoint union of k 4-cycles, each vertex in B is adjacent to exactly one vertex in A. Thus, $|[A, B]| = 4k = b$. Each vertex in A is adjacent to two other vertices in A (namely, its two neighbors on the large cycle in G' to which it belongs) and therefore to at most one vertex in B, and so $|[A, B]| \le a$. Consequently, $b \le a = n_3 - b$, or, equivalently, $b \le n_3/2$. $(\Box)$

Let G_A be the subgraph of G_4 induced by the set of vertices in A, and let G_B be the subgraph of G_4 induced by the set of vertices in B. The subgraph G_A contains as a spanning subgraph a 2-regular subgraph each component of which is a large cycle different from C_{10}. Since $\gamma_{\rm m}^{\infty}(C_r) = \lceil r/3 \rceil$, for $r \ge 6$ an even integer and $r \ne 10$ we note that $\gamma_{\rm m}^{\infty}(C_r) \le 3r/8$ (with equality if and only if $r \in \{8, 16\}$). Applying this to each large cycle in the 2-regular spanning subgraph of G_A, we deduce that $\gamma_{\rm m}^{\infty}(G_A) \le 3a/8$. As observed earlier, $b = 4k$ and G_B is a disjoint union of k 4-cycles, implying that

$\gamma_{\rm m}^\infty(G_B) = k\gamma_{\rm m}^\infty(G_4) = 2k = b/2$. Therefore, by Claim D.1,

$$\begin{array}{rcl} \gamma_{\rm m}^\infty(G_4) & \le & \gamma_{\rm m}^\infty(G_A) + \gamma_{\rm m}^\infty(G_B) \\ & \le & \frac{3}{8}a + \frac{1}{2}b \\ & = & \frac{3}{8}(n_4 - b) + \frac{1}{2}b \\ & = & \frac{3}{8}n_4 + \frac{1}{8}b \\ & \le & \frac{3}{8}n_4 + \frac{1}{16}n_4 \\ & = & \frac{7}{16}n_4. \end{array}$$

This completes the proof of Claim D. (□)

We now return to the proof of Theorem 3.1. By construction, every vertex of H belongs to exactly one of the subgraphs H_1, H_2, H_3, and H_4. Equivalently, every vertex of G belongs to exactly one of the subgraphs G_1, G_2, G_3, and G_4. Hence, by Claims A, B, C and D, we have that

$$\begin{array}{rcl} \gamma_{\rm m}^\infty(G) & \le & \gamma_{\rm m}^\infty(G_1) + \gamma_{\rm m}^\infty(G_2) + \gamma_{\rm m}^\infty(G_3) + \gamma_{\rm m}^\infty(G_4) \\ & \le & \frac{7}{16}n_1 + \frac{5}{12}n_2 + \frac{2}{5}n_3 + \frac{7}{16}n_4 \\ & \le & \frac{7}{16}(n_1 + n_2 + n_3 + n_4) \\ & = & \frac{7}{16}n. \end{array}$$

This completes the proof of Theorem 3.1. □

We remark that if G is a cubic bipartite graph of girth at least 6, then the proof of Theorem 3.1 simplifies considerably since in this case, adopting the notation introduced in the proof of Theorem 3.1, we have $G = G_3$ and by Claim C, $\gamma_{\rm m}^\infty(G) \le 2n/5$. Further, if $\gamma_{\rm m}^\infty(G) = 2n/5$, then the graph G' is a disjoint union of copies of C_{10}. We state this formally as follows.

Corollary 3.5. *If G is a cubic bipartite graph of order n and girth at least 6, then $\gamma_{\rm m}^\infty(G) \le 2n/5$.*

We close with the following conjectures.

Conjecture. *If G is a cubic bipartite graph of order n, then $\gamma_{\rm m}^\infty(G) \le 3n/8$.*

Conjecture. *If G is a cubic graph of order n, then $\gamma_{\rm m}^\infty(G) \le 2n/5$.*

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