Contributions to Discrete Mathematics

Volume 16, Number 1, Pages 20–27
ISSN 1715-0868

COMBINATORIAL INTERPRETATIONS OF TWO IDENTITIES OF GUO AND YANG

MIRCEA MERCA

ABSTRACT. The restricted partitions in which the largest part is less than or equal to N and the number of parts is less than or equal to k were investigated by Andrews. These partitions were extended recently by the author to partitions into parts of two kinds. In this paper, we use a new class of restricted partitions into parts of two kinds to provide new combinatorial interpretations for two identities of Guo and Yang.

1. INTRODUCTION

A partition of n into at most k parts, each part less than or equal to N, is an unordered sum of n that uses at most k positive integers less than or equal to N. These partitions were investigated by Andrews in [2]. Following the notation in [2], the number of such partitions will be denoted in this paper by $p(N,k,n)$. According to [2, Theorem 3.1], the generating function of $p(N,k,n)$ is given by

$$\sum_{n=0}^{Nk} p(N,k,n)q^n = \begin{bmatrix} N+k \\ N \end{bmatrix}_q, \tag{1.1}$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} 0, & \text{if } k<0 \text{ or } k>n, \\ \dfrac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, & \text{otherwise} \end{cases}$$

is the Gaussian polynomial or the q-binomial coefficient. Recall that

$$(a;q)_n = \begin{cases} 1, & \text{for } n=0, \\ (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}), & \text{for } n>0 \end{cases}$$

is the q-shifted factorial and

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n.$$

Received by the editors April 30, 2017, and in revised form May 31, 2020.
2010 *Mathematics Subject Classification.* 05A17, 11P81, 11P82.
Key words and phrases. integer partitions, restricted partitions.

Because the infinite product $(a;q)_\infty$ diverges when $a \neq 0$ and $|q| \geqslant 1$, whenever $(a;q)_\infty$ appears in a formula, we shall assume that $|q| < 1$.

Assume there are positive integers of two kinds: λ and $\overline{\lambda}$. We denote by $\overline{p}_r(N_1, N_2, k_1, k_2, n)$ the number of partitions of n into parts of two kinds with at most k_1 parts of the first kind, each part divisible by r and less than or equal to $N_1 r$, and at most k_2 parts of the second kind, each part less than or equal to N_2. For example, $\overline{p}_2(2,3,2,2,4) = 6$ because the six partitions in question are:

$$4, \quad 2+2, \quad 2+\overline{2}, \quad 2+\overline{1}+\overline{1}, \quad \overline{3}+\overline{1}, \quad \text{and} \quad \overline{2}+\overline{2}.$$

Recently, Merca [6] considered some properties of the Gaussian polynomials and obtained few properties of $\overline{p}_1(N_1, N_2, k_1, k_2, n)$ (see for instance [6, Theorems 3.1, 3.3, 3.4, 3.5, 4.1, 4.5, 5.1, 5.2, 6.1]). In this paper, motivated by these results, we provide some similar results for $\overline{p}_r(N_1, N_2, k_1, k_2, n)$. Two partition formulas involving $\overline{p}_2(N_1, N_2, k_1, k_2, n)$ and $\overline{p}_4(N_1, N_2, k_1, k_2, n)$ are derived in the last section of this paper as corollaries of two identities of Guo and Yang [4]:

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \begin{bmatrix} m+k \\ k \end{bmatrix}_{q^2} \begin{bmatrix} m+1 \\ n-2k \end{bmatrix}_q q^{\binom{n-2k}{2}} = \begin{bmatrix} m+n \\ n \end{bmatrix}_q, \tag{1.2}$$

$$\sum_{k=0}^{\lfloor n/4 \rfloor} \begin{bmatrix} m+k \\ k \end{bmatrix}_{q^4} \begin{bmatrix} m+1 \\ n-4k \end{bmatrix}_q q^{\binom{n-4k}{2}} = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \begin{bmatrix} m+k \\ k \end{bmatrix}_{q^2} \begin{bmatrix} m+n-2k \\ n-2k \end{bmatrix}_q. \tag{1.3}$$

As we can see in [5, Theorems 2.1 and 2.5], these identities are specializations of two convolutions for the complete and elementary symmetric functions.

2. Some general results

The following convolution is a connection between the partition functions $p(N,k,n)$ and $\overline{p}_r(N_1, N_2, k_1, k_2, n)$.

Theorem 2.1. *For $N_1, N_2, k_1, k_2, n \geqslant 0$, $r \geqslant 1$,*

$$\overline{p}_r(N_1, N_2, k_1, k_2, n) = \sum_{j=0}^{\lfloor n/r \rfloor} p(N_1, k_1, j) p(N_2, k_2, n - rj).$$

Proof. Let

$$\lambda_1 + \cdots + \lambda_a + \overline{\lambda}_1 + \cdots + \overline{\lambda}_b = n$$

be a partition into parts of two kinds with $\lambda_i \leqslant N_1 r$, $\overline{\lambda}_i \leqslant N_2$, λ_i divisible by r, and $0 \leqslant a \leqslant k_1$, $0 \leqslant b \leqslant k_2$. This partition can be rewritten as

$$r \sum_{i=1}^{a} \lambda'_i + \sum_{i=1}^{b} \overline{\lambda}_i = n,$$

where $\lambda'_i = \lambda/r$ and $\lambda'_i \leqslant N_1$. The identity follows easily from this relation. □

This convolution allows us to give the following generating function for $\overline{p}_r(N_1, N_2, k_1, k_2, n)$.

Theorem 2.2. *For* $N_1, N_2, k_1, k_2 \geqslant 0$, $r \geqslant 1$,

$$\sum_{n=0}^{N_1 k_1 r + N_2 k_2} \overline{p}_r(N_1, N_2, k_1, k_2, n) q^n = \begin{bmatrix} N_1 + k_1 \\ N_1 \end{bmatrix}_{q^r} \begin{bmatrix} N_2 + k_2 \\ N_2 \end{bmatrix}_q .$$

Proof. Taking into account (1.1), we can write

$$\begin{aligned} &\begin{bmatrix} N_1 + k_1 \\ N_1 \end{bmatrix}_{q^r} \begin{bmatrix} N_2 + k_2 \\ N_2 \end{bmatrix}_q \\ &\quad = \left(\sum_{n=0}^{N_1 k_1} p(N_1, k_1, n) q^{nr} \right) \left(\sum_{n=0}^{N_2 k_2} p(N_2, k_2, n) q^n \right) \\ &\quad = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{\lfloor n/r \rfloor} p(N_1, k_1, j) p(N_2, k_2, n - rj) \right) q^n, \end{aligned}$$

where we have invoked the Cauchy multiplication of two power series. The proof follows considering Theorem 2.1. □

The recurrence relations for the Gaussian polynomials

$$\begin{bmatrix} N \\ k \end{bmatrix}_{q^r} = q^{kr} \begin{bmatrix} N-1 \\ k \end{bmatrix}_{q^r} + \begin{bmatrix} N-1 \\ k-1 \end{bmatrix}_{q^r} . \tag{2.1}$$

and

$$\begin{bmatrix} N \\ k \end{bmatrix}_{q^r} = \begin{bmatrix} N-1 \\ k \end{bmatrix}_{q^r} + q^{(N-k)r} \begin{bmatrix} N-1 \\ k-1 \end{bmatrix}_{q^r} . \tag{2.2}$$

can be used to derive the following generalization of [6, Theorem 3.3].

Theorem 2.3. *For* $N_1, N_2, k_1, k_2, n \geqslant 0$, $r \geqslant 1$,

(1) $\overline{p}_r(N_1, N_2, k_1, k_2, n) - \overline{p}_r(N_1 - 1, N_2 - 1, k_1, k_2, n - k_1 r - k_2)$
$-\overline{p}_r(N_1 - 1, N_2, k_1, k_2 - 1, n - k_1 r) - \overline{p}_r(N_1, N_2 - 1, k_1 - 1, k_2, n - k_2)$
$-\overline{p}_r(N_1, N_2, k_1 - 1, k_2 - 1, n) = 0$;

(2) $\overline{p}_r(N_1, N_2, k_1, k_2, n) - \overline{p}_r(N_1 - 1, N_2 - 1, k_1, k_2, n)$
$-\overline{p}_r(N_1 - 1, N_2, k_1, k_2 - 1, n - N_2) - \overline{p}_r(N_1, N_2 - 1, k_1 - 1, k_2, n - N_1 r)$
$-\overline{p}_r(N_1, N_2, k_1 - 1, k_2 - 1, n - N_1 r - N_2) = 0$;

(3) $\overline{p}_r(N_1, N_2, k_1, k_2, n) - \overline{p}_r(N_1 - 1, N_2 - 1, k_1, k_2, n - k_1 r)$
$-\overline{p}_r(N_1 - 1, N_2, k_1, k_2 - 1, n - k_1 r - N_2) - \overline{p}_r(N_1, N_2 - 1, k_1 - 1, k_2, n)$
$-\overline{p}_r(N_1, N_2, k_1 - 1, k_2 - 1, n - N_2) = 0.$

Proof. By (2.1), we have

$$\begin{aligned}
&\begin{bmatrix} N_1 + k_1 \\ k_1 \end{bmatrix}_{q^r} \begin{bmatrix} N_2 + k_2 \\ k_2 \end{bmatrix}_q \\
&\quad = \left(q^{k_1 r} \begin{bmatrix} N_1 - 1 + k_1 \\ k_1 \end{bmatrix}_{q^r} + \begin{bmatrix} N_1 + k_1 - 1 \\ k_1 - 1 \end{bmatrix}_{q^r} \right) \\
&\qquad \times \left(q^{k_2} \begin{bmatrix} N_2 - 1 + k_2 \\ k_2 \end{bmatrix}_q + \begin{bmatrix} N_2 + k_2 - 1 \\ k_2 - 1 \end{bmatrix}_q \right) \\
&\quad = q^{k_1 r + k_2} \begin{bmatrix} N_1 - 1 + k_1 \\ k_1 \end{bmatrix}_{q^r} \begin{bmatrix} N_2 - 1 + k_2 \\ k_2 \end{bmatrix}_q \\
&\qquad + q^{k_1 r} \begin{bmatrix} N_1 - 1 + k_1 \\ k_1 \end{bmatrix}_{q^r} \begin{bmatrix} N_2 + k_2 - 1 \\ k_2 - 1 \end{bmatrix}_q \\
&\qquad + q^{k_2} \begin{bmatrix} N_1 + k_1 - 1 \\ k_1 - 1 \end{bmatrix}_{q^r} \begin{bmatrix} N_2 - 1 + k_2 \\ k_2 \end{bmatrix}_q \\
&\qquad + \begin{bmatrix} N_1 + k_1 - 1 \\ k_1 - 1 \end{bmatrix}_{q^r} \begin{bmatrix} N_2 + k_2 - 1 \\ k_2 - 1 \end{bmatrix}_q .
\end{aligned}$$

This allows us to write

$$\begin{aligned}
&\sum_{n=0}^{N_1 k_1 r + N_2 k_2} \overline{p}_r(N_1, N_2, k_1, k_2, n) q^n \\
&\quad = \sum_{n=0}^{(N_1-1)k_1 r + (N_2-1)k_2} \overline{p}_r(N_1 - 1, N_2 - 1, k_1, k_2, n) q^{n + k_1 r + k_2} \\
&\qquad + \sum_{n=0}^{(N_1-1)k_1 r + N_2(k_2-1)} \overline{p}_r(N_1 - 1, N_2, k_1, k_2 - 1, n) q^{n + k_1 r} \\
&\qquad + \sum_{n=0}^{N_1(k_1-1)r + (N_2-1)k_2} \overline{p}_r(N_1, N_2 - 1, k_1 - 1, k_2, n) q^{n + k_2} \\
&\qquad + \sum_{n=0}^{N_1(k_1-1)r + N_2(k_2-1)} \overline{p}_r(N_1, N_2, k_1 - 1, k_2 - 1, n) q^n.
\end{aligned}$$

The proof of the first relation follows equating the coefficient of q^n in this identity. Similarly, considering (2.2) we obtain the second recurrence relation. The last relation follows combining (2.1) and (2.2). □

For $r \geqslant 1$, it is clear that the Gaussian polynomial

$$\begin{bmatrix} N+k \\ N \end{bmatrix}_{q^r}$$

is symmetric in N and k. In addition, this polynomial is self-reciprocal. These properties allow us to derive the following relations for the partition function $\overline{p}_r(N_1, N_2, k_1, k_2, n)$.

Theorem 2.4. *For all $N_1, N_2, k, n \geqslant 0$, $r \geqslant 1$,*

$$\begin{aligned} (1) \quad \overline{p}_r(N_1, N_2, k_1, k_2, n) &= \overline{p}_r(k_1, N_2, N_1, k_2, n) \\ &= \overline{p}_r(N_1, k_2, k_1, N_2, n) \\ &= \overline{p}_r(k_1, k_2, N_1, N_2, n); \\ (2) \quad \overline{p}_r(N_1, N_2, k_1, k_2, n) &= \overline{p}_r(N_1, N_2, k_1, k_2, N_1 k_1 r + N_2 k_2 - n). \end{aligned}$$

Proof. The poof is similar to [6, Theorem 3.4]. □

For $r > 1$, we remark that the Gaussian polynomial

$$\begin{bmatrix} N+k \\ N \end{bmatrix}_{q^r}$$

is not unimodal. Thus the third relation of [6, Theorem 3.4] cannot be generalized in this way.

We denote by $\overline{Q}_r(N_1, N_2, k_1, k_2, n)$ the number of partitions of n into parts of two kinds with exactly k_1 distinct parts of the first kind, each part divisible by r and less than or equal to N_1, and exactly k_2 distinct parts of the second kind, each part less than or equal to N_2. We have the following bijection between restricted partitions into parts of two kinds.

Theorem 2.5. *For $N_1, N_2, k_1, k_2, n \geqslant 0$, $r \geqslant 1$,*

$$\begin{aligned} &\overline{Q}_r(N_1, N_2, k_1, k_2, n) \\ &\qquad = \overline{p}_r\left(N_1 - k_1, N_2 - k_2, k_1, k_2, n - r\binom{k_1+1}{2} - \binom{k_2+1}{2}\right). \end{aligned}$$

Proof. The poof is similar to [6, Theorem 6.1]. □

The generating function for $\overline{Q}_r(N_1, N_2, k_1, k_2, n)$ can be easily derived from Theorems 2.2 and 2.5.

Theorem 2.6. *For $N_1, N_2, k_1, k_2 \geqslant 0$, $r \geqslant 1$,*

$$\sum_{n=0}^{\infty} \overline{Q}_r(N_1, N_2, k_1, k_2, n) q^n = q^{r\binom{k_1+1}{2} + \binom{k_2+1}{2}} \begin{bmatrix} N_1 \\ k_1 \end{bmatrix}_{q^r} \begin{bmatrix} N_2 \\ k_2 \end{bmatrix}_q .$$

A similar result to Theorem 2.1 is also possible for $\overline{Q}_r(N_1, N_2, k_1, k_2, n)$ if we consider the number of partitions of n into exactly k distinct parts, each part less than or equal to N, which is denoted in [6] by $Q(N, k, n)$. In fact, all results obtained for $\overline{p}_r(N_1, N_2, k_1, k_2, n)$ can be rewritten in terms of the partition functions $\overline{Q}_r(N_1, N_2, k_1, k_2, n)$.

3. Two partition formulas

As we can see in [6, Theorems 4.5 and 5.1], the partition function $p(N,k,n)$ can be expressed in terms of the partition function $\overline{p}_1(N_1,N_2,k_1,k_2,n)$. For instance, the identity

$$p(N,k,n)=\sum_{j=0}^{k}\overline{p}_1(N-j,k-j,j,j,n-j^2)$$

is a specialization of [6, Theorem 5.1]. The following result shows that the partition function $p(N,k,n)$ can be expressed in terms of the partition function $\overline{p}_2(N_1,N_2,k_1,k_2,n)$.

Theorem 3.1. *For $N,k,n \geqslant 0$,*

$$p(N,k,n)=\sum_{j=0}^{\lfloor k/2\rfloor}\overline{p}_2\left(N,N+1-k+2j,j,k-2j,n-\binom{k-2j}{2}\right).$$

Proof. Taking into account the first identity of Guo and Yang (1.2), we can write

$$\begin{aligned}
&\sum_{n=0}^{Nk}p(N,k,n)q^n\\
&\quad=\begin{bmatrix}N+k\\k\end{bmatrix}_q\\
&\quad=\sum_{j=0}^{\lfloor k/2\rfloor}\begin{bmatrix}N+j\\j\end{bmatrix}_{q^2}\begin{bmatrix}N+1\\k-2j\end{bmatrix}_q q^{\binom{k-2j}{2}}\\
&\quad=\sum_{j=0}^{\lfloor k/2\rfloor}\sum_{n=0}^{2Nj+(N+1-k+2j)(k-2j)}\overline{p}_2(N,N+1-k+2j,j,k-2j,n)q^{n+\binom{k-2j}{2}}\\
&\quad=\sum_{j=0}^{\lfloor k/2\rfloor}\sum_{n=\binom{k-2j}{2}}^{Nk-\binom{k-2j}{2}}\overline{p}_2\left(N,N+1-k+2j,j,k-2j,n-\binom{k-2j}{2}\right)q^n\\
&\quad=\sum_{n=0}^{Nk}\sum_{j=0}^{\lfloor k/2\rfloor}\overline{p}_2\left(N,N+1-k+2j,j,k-2j,n-\binom{k-2j}{2}\right)q^n.
\end{aligned}$$

The identity follows equating the coefficient of q^n in this relation. □

As a consequence of this theorem, we remark the following formula for the partition function $p(n)$.

Corollary 3.2. *For $n \geqslant 0$,*

$$p(n) = \sum_{j=\left\lceil \frac{n}{2}-\frac{1}{4}-\sqrt{\frac{n}{2}+\frac{1}{16}} \right\rceil}^{\lfloor n/2 \rfloor} \overline{p}_2\left(n, n-2j, j, 2j+1, n-\binom{n-2j}{2}\right).$$

It is clear that the expansion of $p(n)$ by this corollary requires about $1+\sqrt{n/2}$ terms. For example, the case $n=6$ of this corollary is read as

$$p(6) = \overline{p}_2(6,4,1,3,0) + \overline{p}_2(6,2,2,5,5) + \overline{p}_2(6,0,3,7,6) = 1+7+3 = 11.$$

The partition functions $\overline{p}_2(N_1,N_2,k_1,k_2,n)$ and $\overline{p}_4(N_1,N_2,k_1,k_2,n)$ are related by the following identity.

Theorem 3.3. *For $N,k,n \geqslant 0$,*

$$\sum_{j=0}^{\lfloor k/4 \rfloor} \overline{p}_4\left(N, N+1-k+4j, j, k-4j, n-\binom{k-4j}{2}\right)$$
$$= \sum_{j=0}^{\lfloor k/2 \rfloor} (-1)^j \overline{p}_2\left(N,N,j,k-2j,n\right).$$

Proof. We have

$$\sum_{j=0}^{\lfloor k/4 \rfloor} \begin{bmatrix} N+j \\ j \end{bmatrix}_{q^4} \begin{bmatrix} N+1 \\ k-4j \end{bmatrix}_q q^{\binom{k-4j}{2}}$$
$$= \sum_{j=0}^{\lfloor k/4 \rfloor} \sum_{n=0}^{4Nj+(N+1-k+4j)(k-4j)} \overline{p}_4(N, N+1-k+4j, j, k-4j, n) q^{n+\binom{k-4j}{2}}$$
$$= \sum_{j=0}^{\lfloor k/4 \rfloor} \sum_{n=\binom{k-4j}{2}}^{Nk-\binom{k-4j}{2}} \overline{p}_4\left(N, N+1-k+4j, j, k-4j, n-\binom{k-4j}{2}\right) q^n$$
$$= \sum_{n=0}^{Nk} \sum_{j=0}^{\lfloor k/4 \rfloor} \overline{p}_4\left(N, N+1-k+4j, j, k-4j, n-\binom{k-4j}{2}\right) q^n$$

and

$$\sum_{j=0}^{\lfloor k/2 \rfloor} (-1)^j \begin{bmatrix} N+j \\ j \end{bmatrix}_{q^2} \begin{bmatrix} N+k-2j \\ k-2j \end{bmatrix}_q$$
$$= \sum_{j=0}^{\lfloor k/2 \rfloor} \sum_{n=0}^{2Nj+N(k-2j)} (-1)^j \overline{p}_2(N,N,j,k-2j,n) q^n$$
$$= \sum_{n=0}^{Nk} \sum_{j=0}^{\lfloor k/2 \rfloor} \overline{p}_2\left(N,N,j,k-2j,n\right) q^n.$$

The identity follows easily considering identity (1.3). □

4. Concluding remarks

Theorems 3.1 and 3.3 provide new combinatorial interpretations for the identities (1.2) and (1.3) of Guo and Yang. According to Andrews [1, Theorem 5.3], there are two identities quite similar to (1.2):

$$\sum_{k\geqslant 0} \begin{bmatrix} m+k \\ k \end{bmatrix}_{q^2} \begin{bmatrix} 2n+1 \\ 2m+2k+1 \end{bmatrix}_q q^{k(2m+2k+1)} = \begin{bmatrix} 2n-m \\ m \end{bmatrix}_{q^2} (-q;q)_{2n-2m}$$

and

$$\sum_{k\geqslant 0} \begin{bmatrix} m+k \\ k \end{bmatrix}_{q^2} \begin{bmatrix} 2n \\ 2m+2k \end{bmatrix}_q q^{k(2m+2k-1)}$$
$$= \begin{bmatrix} 2n-m \\ m \end{bmatrix}_{q^2} (-q;q)_{2n-2m} \frac{1-q^{2n}}{1-q^{4n-2m}}.$$

Following the directions in Andrews's paper [1], it would be very appealing to see that the identity (1.2) is an instance of the q-Pfaff-Saalschütz summation [3, p. 68]:

$$\sum_{k=0}^{n} \frac{(a;q)_k (b;q)_k (q^{-n};q)_k}{(c;q)_k (abq^{1-n}/c;q)_k (q;q)_k} q^k = \frac{(c/a;q)_n (c/b;q)_n}{(c;q)_n (c/ab;q)_n}.$$

Acknowledgements

The author likes to thank the referees for their helpful comments.

References

1. G. E. Andrews, Applications of basic hypergeometric functions, *SIAM Rev.* **16**(4) (1974), 441–484.
2. ______, *The Theory of Partitions*, Addison–Wesley Publishing Co., Reading, M.A., 1976.
3. W. N. Bailey, *Generalized Hypergeometric Series*, Cambridge University Press, London, 1935.
4. V. J. W. Guo, D.-M. Yang, A q-analogue of some binomial coefficient identities of Y. Sun, *Electron. J. Combin.* **18** (2011), #P78.
5. M. Merca, Generalizations of two identities of Guo and Yang, *Quaest. Math.* **41**(5) (2018), 643–652.
6. ______, Combinatorial interpretations of q-Vandermonde's identities, *Ann. Acad. Rom. Sci. Ser. Math. Appl.* **11**(1) (2019), 98–114.

Department of Mathematics, University of Craiova
200585 Craiova, Romania

Academy of Romanian Scientists, Ilfov 3, Sector 5, Bucuresti, Romania
E-mail address: mircea.merca@profinfo.edu.ro