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CONJECTURES ON UNIQUELY 3-EDGE COLORABLE GRAPHS

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Abstract. A graph G is *uniquely k-edge colorable* if the chromatic index of G is k and every two k-edge colorings of G produce the same partition of $E(G)$ into k independent subsets. For any $k \neq 3$, a uniquely k-edge colorable graph G is completely characterized: $G \cong K_2$ if $k = 1$, G is a path or an even cycle if $k = 2$, and G is a star $K_{1,k}$ if $k \geq 4$. On the other hand, there are infinitely many uniquely 3-edge colorable graphs, and hence, there are many conjectures for the characterization of uniquely 3-edge colorable graphs. In this paper, we introduce a new conjecture which connects conjectures of uniquely 3-edge colorable planar graphs with those of uniquely 3-edge colorable non-planar graphs.

1. Introduction

In this paper, we only deal with finite undirected simple graphs. A *k-edge coloring* of a graph G is a map $c : E(G) \to \{1, 2, \ldots, k\}$ such that for every two edges $e, e' \in E(G)$, $c(e) \neq c(e')$ if e and e' are adjacent (that is, they share the same vertex). A graph G is *k-edge colorable* if there exists a k-edge coloring of G, and the *chromatic index* of G, denoted by $\chi'(G)$, is the minimum number k such that G is k-edge colorable. Moreover, a graph G with $\chi'(G) = k$ is called a *k-edge chromatic* graph. A k-edge chromatic graph G is *uniquely k-edge colorable* if every two k-edge colorings of G produce the same partition of $E(G)$ into k independent subsets (color classes). In other words, G has only one k-edge coloring up to permutation of the colors, where the edge coloring is called a *unique k-edge coloring*. We denote the set of uniquely k-edge colorable graphs by UEC_k. Note that for any k, a *star* $K_{1,k}$ is in UEC_k. Moreover, for two distinct colors $i, j \in \{1, 2, \ldots, k\}$ in a k-edge coloring c of a graph G, define $G_{i,j}$ to be the subgraph of G induced by $c^{-1}(i) \cup c^{-1}(j)$ (see Figure 1, for example).

Unique edge colorability was introduced by Greenwell and Kronk [7] in 1973. A few years later, Fiorini and Wilson [4] found several necessary conditions for uniquely edge colorable graphs as follows. A *k-regular* graph G is a graph such that each vertex of G has degree exactly k, and K_n denotes

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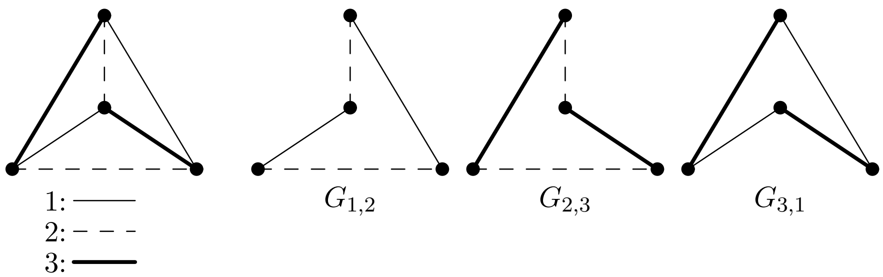

FIGURE 1. Example of a 3-edge coloring of $G(= K_4)$ and $G_{i,j}$'s

a complete graph with n vertices. In particular, a 3-regular graph is *cubic* and K_3 is called a *triangle*.

Proposition 1.1 (Fiorini and Wilson [4]). *Let G be a uniquely k-edge colorable graph with n vertices and m edges, and suppose that G is k-edge colored. Then the following holds:*

1. *Each edge of G is adjacent to edges colored by every other color.*
2. *The subgraph $G_{i,j}$ of G is either a path or a cycle for each $i, j \in \{1, 2, \ldots, k\}$.*
3. *$nk/2 - \binom{k}{2} \leq m \leq nk/2$ and both these bounds are sharp.*

Greenwell and Kronk [7], and Fiorini [3] independently showed that for every uniquely k-edge colorable graph $G \not\cong K_3$, $\chi'(G) = \Delta(G) = k$, where $\Delta(G)$ denotes the maximum degree in G. (Such a class of graphs G with $\chi'(G) = \Delta(G)$ is called a *class one*.) Observe that any uniquely 1-edge colorable graph is an edge, and any uniquely 2-edge colorable graph is a path or a cycle of even length. Moreover, for any positive integer $k \geq 4$, uniquely k-edge colorable graphs are completely characterized as follows.

Theorem 1.2 (Thomason [11]). *For any positive integer $k \geq 4$, every uniquely k-edge colorable graph is isomorphic to a star $K_{1,k}$.*

Thus, in what follows, we only consider uniquely 3-edge colorable graphs.

Uniquely 3-edge colorable graphs have been studied in many papers (for a survey of known results and related topics, see the thesis [5]). In particular, it is shown that the local structure of a minimum counterexample to the cycle double cover conjecture (cf. [9]) is closely related to uniquely 3-edge colorable graphs [12]. The structure requirements of uniquely 3-edge colorable graphs are in some way similar to those on snarks, where a *snark* is a cubic graph with no 3-edge coloring. Moreover, any uniquely 3-edge colorable graph has exactly three *Hamiltonian cycles* which are cycles through all vertices in the graph. So uniquely 3-edge colorable graphs are also dealt with in several papers about Hamiltonian cycles (for example, see [2, 11, 12]). On the other hand, uniquely 3-edge colorable graphs are related to uniquely vertex colorable graphs. A graph G is *planar* (resp., *plane*) if G can be drawn (resp., is already drawn) on the plane so that no two edges of G

cross. Tait [10] proved that if a graph G embedded in the plane is 3-edge colorable, then the *dual* G^* of G is 4-*vertex colorable*, i.e., G^* has a 4-vertex coloring $c : V(G) \to \{1, 2, 3, 4\}$ with $c(u) \neq c(v)$ for any edge $uv \in E(G)$. (The *dual* G^* of a plane graph G is $V(G^*) = F(G)$, which is a face set of G, and $e = uv \in E(G^*)$ if two faces corresponding to u, v share an edge in G.) So, the dual G^* of every uniquely 3-edge colorable planar graph G is *uniquely* 4-*vertex colorable*, i.e., G^* has only one 4-vertex coloring up to permutation of the colors. As above, since there are many applications of uniquely 3-edge colorable graphs, we are interested in the structure of them. In fact, there are many conjectures on the characterization of uniquely 3-edge colorable graphs, and in this paper, we consider the relationship between those conjectures using local transformations.

Let G be a graph with a vertex v of degree 3, and let $N(v) = \{a, b, c\}$, where $N(u)$ denotes the set of the neighbors of a vertex u. The transformation Y-Δ on v removes v, adds a triangle xyz and joins x, y, z to a, b, c, respectively (see Figure 2). Moreover, the inverse operation is called a Δ-Y, and if those operations break the simplicity of graphs, then we do not apply them.

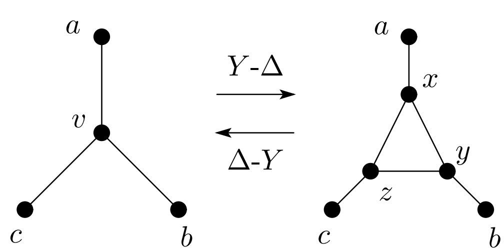

FIGURE 2. Transformations Y-Δ and Δ-Y

It is not difficult to see that Y-Δ and Δ-Y preserve the uniquely 3-edge colorability of graphs [6]. Hence, by applying Y-Δ to a vertex of degree 3 in a graph $G \in UEC_3$ repeatedly, we can obtain infinitely many uniquely 3-edge colorable graphs. Conversely, if a uniquely 3-edge colorable graph G has a triangle T and Δ-Y can be applied to T, then we can obtain another uniquely 3-edge colorable graph with fewer vertices than G.

First, Greenwell and Kronk [7] proposed a conjecture that if G is a uniquely 3-edge colorable cubic graph, then G is a planar graph with a triangle. However, Fiorini and Wilson [4] have observed that the non-planar graph $P(9, 2)$ shown in Figure 3 is uniquely 3-edge colorable and has no triangle, that is, the above conjecture is not true. (The graph $P(9, 2)$ is well known as one of *generalized Petersen graphs.*) Thus, Fiorini and Wilson [4] conjectured the following.

Conjecture 1.3 (Fiorini and Wilson [4])**.** *Every uniquely* 3*-edge colorable cubic non-planar graph can be obtained from* $P(9, 2)$ *by a sequence of* Y*-*Δ*.*

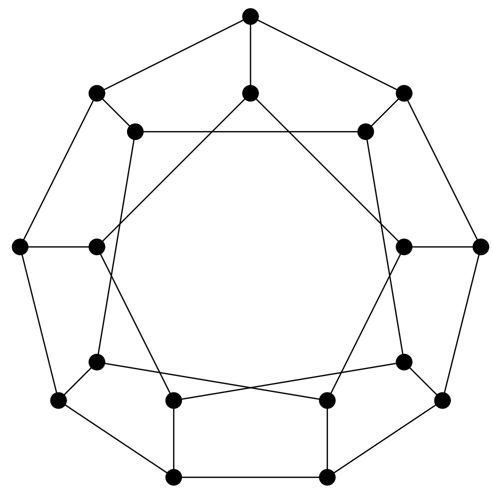

FIGURE 3. The graph $P(9,2)$

However, this conjecture is also not true since we can construct non-planar uniquely 3-edge colorable graphs with no triangle as follows:

Prepare two $P(9,2)$'s, denoted by G_1 and G_2. Let v_1 and v_2 be vertices of degree 3 in G_1 and G_2, and let $N(v_1) = \{a_1, a_2, a_3\}$ and $N(v_2) = \{b_1, b_2, b_3\}$. Moreover, $G_1 - v_1$ and $G_2 - v_2$ denote the graphs obtained from G_1 and G_2 by removing v_1 and v_2, respectively. Then we obtain the graph H from $G_1 - v_1$ and $G_2 - v_2$ by joining a_i and b_i for each $i \in \{1, 2, 3\}$. By a famous theorem called the *Parity Lemma* (cf. [8]), it is not difficult to check that H is a uniquely 3-edge colorable non-planar cubic graph and it has no triangle.

Therefore, by the above construction, we have the following which implies that Conjecture 1.3 is false in general.

Proposition 1.4 (Belcastro and Haas [1]). *There are infinitely many uniquely* 3*-edge colorable cubic non-planar graphs with no triangle.*

Now, we shall introduce another conjecture for uniquely 3-edge colorable cubic graphs. For two graphs G and H, H is a *minor* of G if H can be obtained from G by a sequence of contracting edges, removing edges and deleting isolated vertices. In this case, if H is the Petersen graph, then H is called a *Petersen minor* of G. For the relation between Petersen minors and uniquely 3-edge colorable graphs, Zhang [12], and Goldwasser and Zhang [6] conjectured the following. Note that if G has a Petersen minor, then clearly, G contains a snark as a minor.

Conjecture 1.5 (Zhang [12]). *Every uniquely* 3*-edge colorable non-planar cubic graph can be obtained from a uniquely* 3*-edge colorable cubic graph with a Petersen minor and no triangle by a sequence of* Y*-*Δ*.*

Conjecture 1.6 (Goldwasser and Zhang [6]). *Every uniquely* 3*-edge colorable cubic graph contains either a snark as a minor or a triangle.*

On the other hand, for uniquely 3-edge colorable planar graphs, there is the following conjecture.

Conjecture 1.7 (Fiorini [3], Fiorini and Wilson [4]). *Every uniquely* 3*-edge colorable planar graph G with $G \not\cong K_{1,3}$ contains a triangle.*

In [6], Goldwasser and Zhang observed the following relation between the above three conjectures, and they also gave a partial result for Conjectures 1.5 and 1.6.

Conjecture 1.5 $\Rightarrow$ Conjecture 1.6 $\Rightarrow$ Conjecture 1.7.

In 1998, Fowler [5] gave a positive solution to the following conjecture for uniquely 3-edge colorable planar *cubic* graphs. However, a computer is used in his proof and the corresponding paper has not yet been submitted. So, the following conjecture is still open. (Note that his proof is detailed and closely parallels the proof of the Four Color Theorem, so in fact, the following is considered true rather than open.)

Conjecture 1.8 (Fiorini and Wilson [4]). *Every uniquely* 3*-edge colorable planar cubic graph G can be obtained from K_4 by a sequence of Y-Δ.*

Observe that Conjecture 1.8 is not completely contained in Conjecture 1.7 since if Conjecture 1.8 is true, then every uniquely 3-edge colorable planar cubic graph contains at least two triangles, but Conjecture 1.7 does not guarantee it. On the other hand, a uniquely 3-edge colorable planar cubic graph with at most one triangle has not yet been found. Then we naturally conjecture the following which is equivalent Conjecture 1.8.

Conjecture 1.9. *Every uniquely* 3*-edge colorable planar cubic graph G has at least two triangles.*

In this paper, we consider the sub-cubic case, where a graph G is *sub-cubic* if $\deg(v) \leq 3$ for every vertex v in G and G has at least one vertex of degree at most 2. The transformation V_2-Δ (the inverse operation is called a Δ-V_2) on v with $\deg(v) = 2$ and $N(v) = \{u, w\}$ removes v, adds a triangle xyz and joins y, z to u, w, respectively (see Figure 4). Removing a vertex of degree 1 as shown in Figure 5 is called a V_1*-removing* (the inverse operation is called a V_1*-addition*). Similarly to Y-Δ (or Δ-Y), each of the above transformations preserves the uniquely 3-edge colorability of graphs.

A graph $G \in UEC_3$ is said to be *minimal* if we cannot apply any Δ-Y, Δ-V_2 and V_1-removing to G. Moreover, for each $i \in \{0, 1, 2, 3\}$, let H_i be the graph shown in Figure 6, and let $\mathcal{H} = \{H_0, H_1, H_2, H_3\}$. Then we propose the following new conjecture.

Conjecture 1.10. *Every uniquely* 3*-edge colorable planar graph G can be obtained from one of $\mathcal{H}$ by a sequence of Y-Δ, V_2-Δ and V_1-additions.*

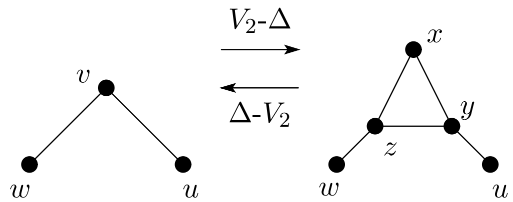

FIGURE 4. Transformations V_2-Δ and Δ-V_2

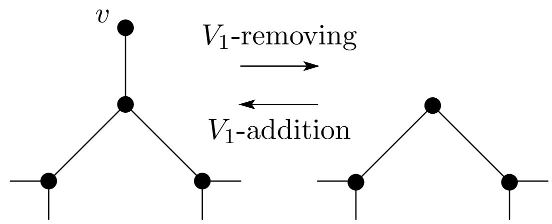

FIGURE 5. Transformations V_1-removing and V_1-addition

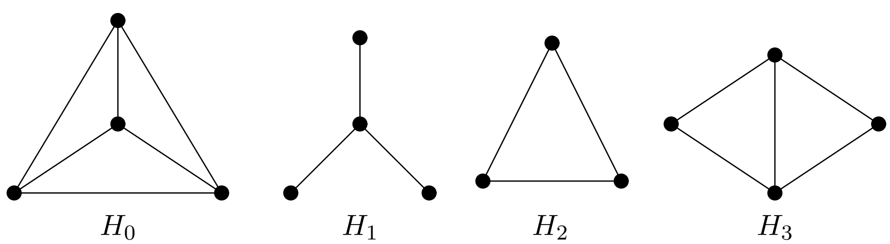

FIGURE 6. The graphs H_0, H_1, H_2, and H_3

Note that Conjecture 1.10 implies Conjecture 1.7, and it is equivalent to Conjecture 1.8 when G is cubic. Unfortunately, we have not yet directly proved Conjecture 1.10. However, we can prove the conjecture by assuming the truth of Conjectures 1.5 and 1.9 as follows.

Theorem 1.11. *The combination of Conjectures* 1.5 *and* 1.9 *implies Conjecture* 1.10*.*

By the above theorem, we have a relationship of the above conjectures as follows:

Conjectures 1.5 and 1.9 $\Rightarrow$ Conjecture 1.10 $\Rightarrow$ Conjectures 1.7 and 1.8.

Therefore, to solve Conjecture 1.6 and all conjectures for uniquely 3-edge colorable planar graphs, it is sufficient to prove Conjectures 1.5 and 1.9.

2. Minimal uniquely 3-edge colorable graphs

In this section, we characterize minimal uniquely 3-edge colorable graphs with a triangle. For two integers $d_1, d_2 \geq 0$, let $N(d_1, d_2)$ be a set of uniquely

3-edge colorable graphs with d_1 vertices of degree 1 and d_2 vertices of degree 2. Note that $H_0 \in N(0,0)$, $H_1 \in N(3,0)$, $H_2 \in N(0,3)$ and $H_3 \in N(0,2)$. It is not difficult to see that the number of kinds of (d_1, d_2) is exactly 16 by Proposition 1.1. However, we can see that almost of them cannot exist as follows.

Proposition 2.1. *$N(d_1, d_2) = \emptyset$ if $(d_1, d_2) = (1,0), (0,1), (2,2), (1,3), (1,4), (0,4), (0,5),$ or $(0,6)$.*

Proof. Let $G \in UEC_3$ be a planar graph in $N(d_1, d_2)$, and let $c : E(G) \to \{1,2,3\}$ be a unique 3-edge coloring of G. First observe that for every vertex v of degree 3 in G, the degree of v in $G_{i,j}$ is exactly 2 for each $i, j \in \{1,2,3\}$ by Proposition 1.1.

For any vertex v of degree 2 in G, there exist two distinct pairs of i, j such that v has a degree 1 in $G_{i,j}$. Since there are three kinds of $G_{i,j}$, if G has at least four vertices of degree 2, then G has a subgraph $G_{i,j}$ which is neither a path nor a cycle, which contradicts Proposition 1.1. Therefore, we have $N(1,4) = N(0,4) = N(0,5) = N(0,6) = \emptyset$. Similarly, we have $N(1,0) = N(0,1) = \emptyset$ since if G has only one vertex of degree 1 (or 2), then there exists a subgraph $G_{i,j}$ which is neither a path nor a cycle.

Now we suppose $(d_1, d_2) = (1,3)$. By the above argument, if at least two of three vertices of degree 2 in G receive the same pair of colors, then there exists a subgraph $G_{i,j}$ which is neither a path nor a cycle. Thus, we may suppose that the three vertices receive exactly three pairs of colors $\{1,2\}$, $\{2,3\}$ and $\{3,1\}$. In this case, since a unique vertex of degree 1 in G must appear in some $G_{i,j}$, the subgraph $G_{i,j}$ is neither a path nor a cycle, a contradiction.

Finally, we let $(d_1, d_2) = (2,2)$. By the similar argument to the above, two vertices of degree 2 receive two different colorings, say $\{1,2\}$ and $\{2,3\}$, by symmetry. Let e and e' be edges incident to two vertices v and v' of degree 1, respectively. Now, if e receives a color 1 or 3, then $G_{1,3}$ has at least three vertices of degree 1, a contradiction. Hence, both e and e' receive the color 2. However, $G_{1,2}$ is clearly neither a path nor a cycle in this case, a contradiction. Therefore, the proposition holds. □

Let $G \not\cong K_{1,3}$ be a uniquely 3-edge colorable graph in $N(d_1, d_2)$. By applying V_1-removings to G, we can easily convert the above pair (d_1, d_2) into another as follows (note that a vertex of degree 1 and one of degree 2 are not adjacent in G, otherwise this contradicts $G \in UEC_3$):

$$(3,0) \to (0,3), (2,0) \to (0,2), (2,1) \to (0,3), (1,1) \to (0,2), \text{ or } (1,2) \to (0,3).$$

Therefore, since it suffices to consider only three sets, $N(0,0), N(0,2)$, and $N(0,3)$, we shall prove the following.

Proposition 2.2. *Let G be a minimal uniquely 3-colorable planar graph with a triangle. Then, $G \cong H_0 \in N(0,0)$, $G \cong H_2 \in N(0,3)$, or $G \cong H_3 \in N(0,2)$.*

Proof. If G is in $N(3,0), N(2,0), N(2,1), N(1,1)$, or $N(1,2)$, then we can obtain a graph in $N(0,0), N(0,3)$, or $N(0,2)$ from G by applying V_1-removings. Note that any V_1-removing does not decrease the number of triangles. Suppose that G is in $N(0,0), N(0,2)$, or $N(0,3)$, and let T be a triangle in G with $V(T) = \{a,b,c\}$. If at least two of $V(T)$ have degree exactly 2, then we immediately have $G \cong H_2$ since otherwise, this contradicts $G \in UEC_3$. Then, we consider the following two cases by symmetry.

Case 1: $\deg(a) = \deg(b) = \deg(c) = 3$.

Let u, v and w be neighbors of a, b and c, respectively, where $\{u,v,w\} \cap \{a,b,c\} = \emptyset$. If u, v and w are distinct, then it is easy to see that we can apply Δ-Y to T, contradicts the minimally of G. Moreover, if $u = v = w$, then G is clearly isomorphic to H_0. Thus, we may suppose that $u = v$ by symmetry. Now suppose that $\deg(u) = 3$ and let $z \neq a, b$ be a neighbor of u $(= v)$. In this case, since the 4-cycle $aubc$ is also a cycle in $G_{i,j}$ for some pair of i, j, we have $\deg(z) = \deg(w) = 1$ (otherwise, contradicts that $G \in UEC_3$ since $G_{i,j}$ is not connected). However, this is a contradiction since G has no vertex of degree 1. So, we have $\deg(u) = 2$ and also have $\deg(c) = 2$, that is, $G \cong H_3$.

Case 2: $\deg(a) = 2$ and $\deg(b) = \deg(c) = 3$.

Let v and w be neighbors of b and c, respectively, where $\{v,w\} \cap \{a,b,c\} = \emptyset$. Similarly to Case 1, since if $v \neq w$, then can apply Δ-V_2 to T, we may suppose that $v = w$. In this case, if $\deg(v) = 3$, then $G_{i,j}$ is not connected for some pair of i, j similarly to Case 1, a contradiction. Hence, we have $\deg(v) = 2$, that is, $G \cong H_3$.

□

3. Proof of Theorem 1.11

In this section, we shall prove Theorem 1.11.

Proof of Theorem 1.11. Let G be a minimal uniquely 3-edge colorable planar graph and suppose that Conjectures 1.5 and 1.9 hold. By way of contradiction, we suppose that $G \ncong H_i$ for any $i \in \{0,1,2,3\}$.

By Conjecture 1.9, if G is cubic, then it is easy to see that we can transform G into H_0 by applying Δ-Y (without the truth of Conjecture 1.5). Hence, we may suppose that G is not cubic, that is, G is sub-cubic. Moreover, if G has a vertex of degree 1, then we can transform G into G' by applying V_1-removings at most three times, where $G' \in N(0,2)$ or $G' \in N(0,3)$. In this case, if a V_1-removing is not applicable, that is, this operation makes G be not uniquely 3-edge colorable, then $G \cong H_1$, a contradiction. Therefore, we may also suppose that G has no vertex of degree 1.

If G has a triangle T, then $G \cong H_2$ or H_3 by Proposition 2.2. So we may suppose that G has no triangle. Then we consider the following two cases depending on embeddings of G on the plane.

Case 1: There exists an embedding of G on the plane such that all vertices of degree 2 lie on the boundary of a face of G.

We fix the embedding of G such that all vertices of degree 2 lie on the boundary of a face of G. In this case, if $G \in N(0,3)$, then we add a single vertex x and join x and $a, b, c \in V(G)$, where $\deg(a) = \deg(b) = \deg(c) = 2$, and let G' be the resulting graph. Clearly, G' is cubic and planar, and it is easy to check $G' \in EUC_3$ (by the proof of Proposition 2.1). Now, if G' has at least two triangles, then b and a, c are adjacent since G has no triangle, that is, $N(b) = \{a, c\}$. However, in this case, it is not difficult to see that $G \cong H_2$ or G is not uniquely 3-edge colorable, a contradiction. Hence, G' has at most one triangle, however, this contradicts Conjecture 1.9.

We next suppose $G \in N(0,2)$ and fix the embedding of G satisfying the claimed condition. Then we add an edge uv to G, where $u, v \in V(G)$ and $\deg(u) = \deg(v) = 2$ (note that $uv \notin E(G)$ since otherwise, $G_{i,j}$ is neither a path nor a cycle for some pair of i, j), and G' is the resulting graph. Similarly to the previous case, G' is a uniquely 3-edge colorable cubic planar graph since two pairs of two edges incident to vertices of degree 2 are colored by the same two colors. If G' has at least two triangles, then $N(u) = N(v) = \{y, y'\}$. However, since the 4-cycle $uyvy'$ is also a cycle in $G_{i,j}$ for some pair of i, j, G must be H_3, a contradiction. By Conjecture 1.9, we complete the proof of the case.

Case 2: Otherwise.

In this case, similarly to Case 1, we add a single vertex x (resp., an edge uv) to G if $G \in N(0,3)$ (resp., $G \in N(0,2)$) which has been embedded in the plane and join x to a, b, c in G, where $a, b, c, u, v \in V(G)$ and these vertices are of degree 2, and let G' be the resulting graph. (Each label is the same as one that is used in the previous case.) By the assumption, G' is not planar but it is a uniquely 3-edge colorable cubic graph.

Now, by Conjecture 1.5, G' is obtained from a uniquely 3-edge colorable cubic non-planar graph which has a Petersen minor by applying Y-Δ repeatedly. Thus, G' also has a Petersen minor. Then it is well known that $G' - x$ (or $G' - uv$) has a $K_{3,3}$-minor, where $G' - x$ (resp., $G' - uv$) is obtained from G' by removing a vertex x (resp., an edge uv). However, this contradicts the planarity of G by Kuratowski's theorem.

□

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