Contributions to Discrete Mathematics

GROUP IRREGULAR LABELINGS OF DISCONNECTED GRAPHS

MARCIN ANHOLCER AND SYLWIA CICHACZ

ABSTRACT. We investigate the *group irregularity strength* ($s_g(G)$) of graphs, i.e. the smallest value of s such that taking any Abelian group $\mathcal{G}$ of order s, there exists a function $f : E(G) \to \mathcal{G}$ such that the sums of edge labels at every vertex are distinct. We give the exact values and bounds on $s_g(G)$ for chosen families of disconnected graphs. In addition we present some results for the *modular edge gracefulness* $k(G)$, i.e. the smallest value of s such that there exists a function $f : E(G) \to \mathbb{Z}_s$ such that the sums of edge labels at every vertex are distinct.

1. INTRODUCTION

It is an easy exercise to show that in any simple graph G there are at least two vertices of the same degree. The situation changes if we consider an edge labeling $f : E(G) \to \{1, \ldots, s\}$ and calculate the weighted degree (or weight) of each vertex x as the sum of labels of all the edges incident to x. The labeling f is called *irregular* if the weighted degrees of all the vertices are distinct. The smallest value of s that allows some irregular labeling is called the *irregularity strength of G* and denoted by $s(G)$.

The problem of finding $s(G)$ was introduced by Chartrand et al. in [4] and has been investigated by numerous authors. In particular, Jacobson conjectured (as mentioned in [12]) that there is an absolute constant c, such that for every d-regular graph G without isolated edges and with at most one isolated vertex, $s(G) \le n/d + c$. It is also believed that the generalized variant of this bound holds for an arbitrary graph, i.e. $s(G) \le n/\delta + c$, where δ is the minimum degree of G. The best published general result due to Kalkowski et al. (see [10]) is $s(G) \le 6n/\delta$. It was recently improved by Majerski and Przybyło ([14]) for dense graphs of sufficiently large order ($s(G) \le (4 + o(1))n/\delta + 4$ in this case). The exact value of $s(T)$ for a tree T was investigated, e.g., by Aigner and Triesch ([1]), Amar and Togni ([2]), Ferrara et al. ([5]), and Togni ([15]).

Lo in [13] defined *edge-graceful labelings*; a graph G of order n and size m is edge-graceful if there exists a bijective mapping $f : E(G) \to \{1, 2, \ldots, m\}$ such that the induced vertex labeling $f' : V(G) \to \{0, 1, 2, \ldots, n-1\}$ defined as

$$f'(u) = \sum_{v \in N(u)} f(uv) \mod n$$

is a bijection. In the above definition of an edge-graceful labeling of a connected graph G of order $n > 2$ and size m, the edge labeling f is required to be a bijection. However, since the induced vertex labels $f'(u)$ are obtained by summation in $\mathbb{Z}_n$, the mapping f is actually a mapping from $E(G)$ to $\mathbb{Z}_n$ and is in general not one-to-one. It is easy to see that the mapping $f : E(G) \to \mathbb{Z}_n$ is one-to-one if and only if $m = n - 1$ or $m = n$ ([8]). Jones combined the concepts of graceful labeling and modular edge coloring into a labeling called *modular edge-graceful labeling* ([8, 6, 9]). He defined the *modular edge-gracefulness of graphs* as the smallest integer $k(G) = k \geq n$ for which there exists an edge labeling $f : E(G) \to \mathbb{Z}_k$ such that the induced vertex labeling $f' : V(G) \to \mathbb{Z}_k$ defined by

$$f'(u) = \sum_{v \in N(u)} f(uv) \mod k$$

is one-to-one.

Assume $\mathcal{G}$ is an Abelian group of order $m \geq n$ with the operation denoted by $+$ and the identity element by 0. For convenience we will write ka to denote $a + a + \cdots + a$ (where element a appears k times), $-a$ to denote the inverse of a, and we will use $a - b$ instead of $a + (-b)$. Moreover, the notation $\sum_{a \in S} a$ will be used as a short form for $a_1 + a_2 + a_3 + \ldots$, where $a_1, a_2, a_3, \ldots$ are all the elements of the set S. The order of the element $a \neq 0$ is the smallest r such that $ra = 0$. It is well-known by Lagrange's Theorem that r divides $|\mathcal{G}|$ [7].

We consider an edge labeling $f : E(G) \to \mathcal{G}$ leading us to the weighted degrees defined as the sums (in $\mathcal{G}$):

$$w(v) = \sum_{v \in N(u)} f(uv).$$

The concept of a $\mathcal{G}$-irregular labeling is a generalization of a modular edge-graceful labeling. In both cases the labeling f is called $\mathcal{G}$-*irregular* if all the weighted degrees are distinct. However, the *group irregularity strength* of G, denoted $s_g(G)$, is the smallest integer s such that for every Abelian group $\mathcal{G}$ of order s there exists a $\mathcal{G}$-irregular labeling f of G. Thus the following observation is true.

Observation 1.1. *For every graph G with no component of order less than 3, $k(G) \leq s_g(G)$.*

The following theorem, determining the value of $s_g(G)$ for every connected graph G of order $n \geq 3$, was proved by Anholcer, Cichacz, and Milanič [3].

Theorem 1.2 ([3]). *Let G be an arbitrary connected graph of order $n \geq 3$. Then*

$$s_g(G) = \begin{cases} n+2, & \text{if } G \cong K_{1,3^{2q+1}-2} \text{ for some integer } q \geq 1 \\ n+1, & \text{if } n \equiv 2 \pmod 4 \wedge G \not\cong K_{1,3^{2q+1}-2} \text{ for any integer } q \geq 1 \\ n, & \text{otherwise.} \end{cases}$$

In [8] it was proved in turn that for every connected graph G of order $n \geq 3$

$$k(G) = \begin{cases} n, & \text{if } n \not\equiv 2 \pmod 4, \\ n+1, & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

In order to distinguish n vertices in an arbitrary (not necessarily connected) graph we need at least n distinct elements of $\mathcal{G}$. However, n elements are not always enough, as shown in the following lemma.

Lemma 1.3 ([3]). *Let G be a graph of order n. If $n \equiv 2 \pmod 4$, then $s_g(G) \geq n+1$.*

In fact, we will need a stronger version of the above lemma. The proof is similar to the one presented in [3], but we present it for sake of completeness.

Lemma 1.4. *Let G be a graph of order n. If $n \equiv 2 \pmod 4$, then there is no $\mathcal{G}$-irregular labeling of G for any Abelian group $\mathcal{G}$ of order n.*

Proof. Let $\mathcal{G}$ be an Abelian group of order $n = 2(1 + 2k)$. The fundamental theorem of finite Abelian groups states that the finite Abelian group $\mathcal{G}$ can be expressed as the direct sum of cyclic subgroups of prime-power order. In particular, $\mathcal{G} \cong \mathbb{Z}_2 \times A$ for some Abelian group A of order $1 + 2k$ and one can write $g \in \mathcal{G}$ as either $g = (0, a)$ or $g = (1, a)$, where $a \in A$. Notice that in the group $\mathcal{G}$ there are $1 + 2k$ elements with the first coordinate 0 and $1 + 2k$ with the first coordinate 1.

Assume there is a $\mathcal{G}$-irregular labeling f of G with resulting vertex weighting w. Now let $w(G) = \sum_{v \in V(G)} w(v) = \sum_{g \in \mathcal{G}} g$. Clearly $w(G)$ is a vector with the first coordinate 1 (since we are summing in $\mathbb{Z}_2$). On the other hand $w(G) = \sum_{v \in V(G)} (\sum_{e \ni v} f(e))$, so each label $f(e)$ for any $e \in E(G)$ appears in the sum twice. Therefore $w(G)$ is a vector with the first coordinate 0 (since we are summing in $\mathbb{Z}_2$), a contradiction. $\square$

In this paper we generalize the above results for some families of disconnected graphs. Our main results are three following theorems.

Theorem 1.5. *Let G be a graph of order n with no component of order less than 3 and with all the bipartite components having both color classes of even order. Then*

$$\begin{aligned} s_g(G) &= n, & \text{if } n \equiv 1 \pmod 2, \\ s_g(G) &= n+1, & \text{if } n \equiv 2 \pmod 4, \\ s_g(G) &\leq n+1, & \text{if } n \equiv 0 \pmod 4. \end{aligned}$$

Theorem 1.6. *Let G be a graph of order n with no component of order less than 3 and with all the bipartite components having both color classes of even order. Then*

$$k(G) = \begin{cases} n, & if \quad n \not\equiv 2 \pmod 4, \\ n+1, & if \quad n \equiv 2 \pmod 4. \end{cases}$$

Theorem 1.7. *Let G be a graph of order n having neither a component of order less than 3 nor a $K_{1,2u+1}$ component for any integer $u \geq 1$. Then*

$$\begin{array}{lll} k(G) = n, & if \quad n \equiv 1 \pmod 2, \\ k(G) = n+1, & if \quad n \equiv 2 \pmod 4, \\ k(G) \leq n+1, & if \quad n \equiv 0 \pmod 4. \end{array}$$

2. Proofs of the main results

We will start this section with two simple observations.

Observation 2.1. *If $\mathcal{G}$ is an Abelian group of odd order n, then for any element $a \in \mathcal{G}$ there exists unique element $b \in \mathcal{G}$ such that $2b = a$.*

Proof. If for some $g_1, g_2 \in \mathcal{G}$ we have $2g_1 = 2g_2$, then it follows that $2(g_1 - g_2) = 0$ and consequently $g_1 - g_2 = 0$, as there is no involution in $\mathcal{G}$. Thus $g_1 = g_2$. $\qquad\square$

For sake of simplicity, for any element $a \in \mathcal{G}$, we are going to use the notation $a/2$ for an arbitrarily chosen element $b \in \mathcal{G}$ satisfying $2b = a$. Let $\mu(\mathcal{G})$ be the number of elements $g \in \mathcal{G}$ such that there exists $g/2 \in \mathcal{G}$.

Observation 2.2. *Let $\mathcal{G}$ be an Abelian group of order $n = 2^\alpha(2\beta + 1)$, for some integers $\alpha > 0$ and $\beta \geq 0$ and let A be an Abelian group of order $2\beta + 1$. If $\mathcal{G} \cong \mathbb{Z}_{2^\alpha} \times A$, then $\mu(\mathcal{G}) = |\mathcal{G}|/2$. Moreover if $i \in \mathcal{G}$ is an involution, then there exists $i/2 \in \mathcal{G}$ if and only if $\alpha > 1$.*

Proof. Notice that for any $a \in \mathbb{Z}_{2^\alpha}$, $a = 2h + 1$ for some $h \in \mathbb{Z}_{2^\alpha}$, there is no such element $a' \in \mathbb{Z}_{2^\alpha}$ that $2a' = a$. Thus if $g = (2h + 1, a)$, then there does not exist $g/2 \in \mathcal{G}$. On the other hand, if $g = (2h, a)$ for $a \in A$, then by Observation 2.1 there exists $a/2 \in A$ and thus $g/2 = (h, a/2)$. As it can be easily seen, there are exactly $2^{\alpha-1}(2\beta + 1)$ elements $g \in \mathcal{G}$ such that there exists $g/2$. Notice that if $i \in \mathcal{G}$ is an involution, then $i = (2^{\alpha-1}, 0)$. Thus there exists $i/2 \in \mathcal{G}$ if and only if $\alpha > 1$. $\qquad\square$

Given any two vertices x_1 and x_2 belonging to the same connected component of G, there exist walks from x_1 to x_2. Some of them may consist of even number of vertices (some of them being repetitions). We are going to call them *even walks*. The walks with odd number of vertices will be called *odd walks*. We will always choose the shortest even or the shortest odd walk from x_1 to x_2.

We start with 0 on all the edges of G. Then, in every step we will choose x_1 and x_2 and add some labels to all the edges of chosen walk from x_1 to x_2. To be more specific, we will add some element a of the group to the

labels of all the edges having odd position on the walk (starting from x_1) and $-a$ to the labels of all the edges having even position. It is possible that some labels will be modified more than once, as the walk does not need to be a path. We will denote such situation with $\phi_e(x_1, x_2) = a$ if we label the shortest even walk and $\phi_o(x_1, x_2) = a$ if we label the shortest odd walk. Observe that putting $\phi_e(x_1, x_2) = a$ means adding a to the weighted degrees of both vertices, while $\phi_o(x_1, x_2) = a$ means adding a to the weighted degree of x_1 and $-a$ to the weighted degree of x_2. In both cases the operation does not change the weighted degree of any other vertex of the walk. Note that if some component G_1 of G is not bipartite, then for any vertices $x_1, x_2 \in G_1$ there exist both even and odd walks.

The next theorem gives us sufficient conditions for a group $\mathcal{G}$ to allow some $\mathcal{G}$-irregular labeling of given graph G.

Theorem 2.3. *Let G be a disconnected graph of order n with all the bipartite components having both color classes of even order and with no component of order less than 3. Let $s = n + 1$ if $n \equiv 2 \pmod 4$ and $s = n$ otherwise. Then for every integer $t \geq s$ there exists a $\mathcal{G}$-irregular labeling for every Abelian group $\mathcal{G}$ of order t with at most one involution i.*

Proof. Let us denote the connected components of G with $G_1, G_2, \ldots, G_r$. Let $G_1, G_2, \ldots, G_p$ be the components of odd order, $G_{p+1}, G_{p+2}, \ldots, G_q$ the nonbipartite components of even order and $G_{q+1}, G_{q+2}, \ldots, G_r$ the bipartite components.

As there is at most one involution in $\mathcal{G}$, we can be sure that $\mathcal{G} \cong \mathbb{Z}_{2^\alpha} \times A$ where A is an Abelian group of order $2\beta + 1$ for some integers $\alpha, \beta \geq 0$. Let the elements of $\mathcal{G}$ be $g_0 = 0, g_1, g_2, \ldots, g_{t-1}$. We can thus represent the elements of $\mathcal{G}$ by the ordered pairs (z, a), where $z \in \mathbb{Z}_{2^\alpha}$ and $a \in A$. We will use a instead of $(0, a)$ when $\alpha = 0$ (i.e. when $|\mathcal{G}| = 2\beta + 1$).

Notice that $t \geq n + 1$ for $n \equiv 2 \pmod 4$ and $t \geq n$ otherwise. Thus the number of elements $g \in \mathcal{G}$ for which $g/2$ exists is $\mu(\mathcal{G}) \geq |\mathcal{G}|/2 \geq n/2 > n/3 \geq r \geq p$. Notice that if there exists $g/2 \in \mathcal{G}$, then exists $-g/2 \in \mathcal{G}$ and $g/2 \neq -g/2$ if $g \neq 0$. Therefore we can pick $k = \lfloor p/2 \rfloor$ elements, say $g_1, g_2, \ldots, g_k$, such that $g_j \neq 0$, $g_j \neq -g_j$ and there exists $g_j/2 \in \mathcal{G}$ for $j = 1, 2, \ldots, k$. We choose any vertex $x_j \in G_j$ for $j = 1, \ldots, 2k - 2$ and we put

$$\phi_e(x_j, x_j) = \begin{cases} \frac{g_{(j+1)/2}}{2}, & \text{if } j \equiv 1 \pmod 2, \\ \frac{-g_{j/2}}{2}, & \text{if } j \equiv 0 \pmod 2. \end{cases}$$

Now we consider two cases, depending on the parity of n. If n is odd, then p is also odd. In such a situation we choose any vertex $x_j \in G_j$ for $j = 2k - 1, 2k = p - 1$, we put

$$\phi_e(x_{2k-1}, x_{2k-1}) = \frac{g_k}{2},$$
$$\phi_e(x_{2k}, x_{2k}) = \frac{-g_k}{2},$$

and leave the vertex x_p with weighted degree 0. After that, we are left with an even number $n - p$ of vertices that can be joined with even walks into pairs (x_j^1, x_j^2), $j = 1, \ldots, (n-p)/2$. On the other hand there are at least $(n-p)/2$ disjoint pairs $(g_j, -g_j)$, $g_j \neq 0$ of unused elements of $\mathcal{G}$: $(|\mathcal{G}| - p)/2$ when $|\mathcal{G}|$ is odd and $(|\mathcal{G}| - p - 1)/2$ otherwise. We put

$$\phi_o(x_j^1, x_j^2) = g_j$$

for every $j = 1, \ldots, (n-p)/2$ and we are done.

If n is even, then also p is even. Our next steps depend on the value of t. If $t \geq n + 1$, then we proceed as in the previous case, except for the fact that $p = 2k$ and none of the vertex weights in the odd components are equal to 0. Observe that also in this case, after weighting one vertex in every odd component, we still have at least $(n-p)/2$ disjoint pairs $(g_j, -g_j)$, $g_j \neq 0$ of unused elements of $\mathcal{G}$ and $n - p$ vertices to label. If $t = n$, this means that $n \equiv 0 \pmod 4$ and so there exists $i/2 \in \mathcal{G}$, where i is the only involution of $\mathcal{G}$. So we choose any vertex $x_{2k-1} \in G_{2k-1}$, we put

$$\phi_e(x_{2k-1}, x_{2k-1}) = \frac{i}{2}$$

and we leave the vertex x_p with weighted degree 0. After that we still have exactly $(n-p)/2$ disjoint pairs $(g_j, -g_j)$, $g_j \neq 0$ of unused elements of $\mathcal{G}$ and $n - p$ vertices to label, so we proceed as in the previous cases and we can complete the labeling. $\square$

In particular, the following corollary is true, as in every cyclic group $\mathbb{Z}_k$ there is exactly one involution $i = k/2$ if k is even and no involution if k is odd.

Corollary 2.4. *Let G be a disconnected graph of order n with all the bipartite components having both color classes of even order and with no component of order less than 3. Let $s = n + 1$ if $n \equiv 2 \pmod 4$ and $s = n$ otherwise. Then for every integer $t \geq s$ there exists a $\mathbb{Z}_t$-irregular labeling.*

The next lemma allows to find group irregular labelings for another class of graphs.

Lemma 2.5. *Let G be a graph of order n having neither a component of order less than 3 nor a $K_{1,2u+1}$ component for any integer $u \geq 1$. Then for every odd integer $t \geq n$ there exists a $\mathbb{Z}_t$-irregular labeling.*

Proof. We are going to use the following theorem, proved in [11].

Theorem 2.6 ([11]). *Let $s = r_1 + r_2 + \cdots + r_q$ be a partition of the positive integer s, where $r_i \geq 2$ for $i = 1, 2, \ldots, q$. Let $A = \{1, 2, \ldots, s\}$. Then the set A can be partitioned into pairwise disjoint subsets $A_1, A_2, \ldots, A_q$ such that for every $1 \leq i \leq q$, $|A_i| = r_i$ with $\sum_{a \in A_i} a \equiv 0 \pmod{s+1}$ if s is even and $\sum_{a \in A_i} a \equiv 0 \pmod s$ if s is odd.*

We are going to divide the vertices of G into triples and pairs. Let p_1 be the number of bipartite components of G with both color classes odd, p_2 with both classes even, and p_3 with one class odd and one even. Let p_4 be the number of remaining components of odd order and p_5 be the number of remaining components of even order. The number of triples equals $2p_1 + p_3 + p_4$. The remaining vertices form the pairs.

Now we are going to partition the elements of $\mathbb{Z}_t$. We will consider two cases. If $G \cong lK_{3,3}$ for some positive integer l and $t = n + 1$, then n is even and we use Theorem 2.6 with $s = n$. In such a case the authors of [11] present a construction that guarantees the existence of $2l$ triples ($6l = n = s$), each of sum 0 in $\mathbb{Z}_{s+1}$, such that in l triples there are elements $a_j = j, j = 1, 2, \ldots, l$, while in the remaining l triples there are elements $a_{l+j} = s + 1 - a_j$, i.e. the elements $a_{l+j} \equiv -a_j \pmod{t}$. Denoting the remaining elements of each triple with b_j and c_j, we can be sure that no element $-a_j$ can be equal to b_i or c_i for any i.

In the remaining case (i.e. when $G \not\cong lK_{3,3}$ or $t > n+1$), we are interested in the variant of the theorem where t is odd. The authors of [11] show a construction of $2l + 1$ triples $B_1, B_2, \ldots, B_{2l+1}$ and m pairs $C_1, C_2, \ldots, C_m$ where $l = \lfloor (2p_1 + p_3 + p_4)/2 \rfloor$ and $m = (t - 6l - 3)/2$ (observe that both l and m are nonnegative in this case). The construction provided by the authors guarantees this time the existence of l triples with the elements $a_j = j, j = 1, 2, \ldots, l$, and l triples with elements $a_{l+j} = t - a_j$, i.e. the elements $a_{l+j} \equiv -a_j \pmod{t}$. The last triple is $B_{2l+1} = \{3l + k + 1, 3l + k + 2, t\}$. Denoting the remaining elements of each triple with b_j and c_j, we can be sure that no element $-a_j$ can be equal to b_i or c_i for any i, while it still belongs to some triple. We will denote the elements of pairs with $C_j = (d_j, -d_j)$, $j = 1, \ldots, k$. Each of those triples and pairs sums up to 0 $\pmod{t}$. It is easy to observe that for a given element $g \in \mathbb{Z}_t$ not belonging to any triple, we have $(g, -g) = C_j$ for some j. If n is even, then we remove t (i.e. 0) from the triple B_{2l+1} to which it belongs in order to obtain additional pair C_{m+1} (observe that after removing 0, the number of remaining elements of $\mathbb{Z}_t$ is still enough to distinguish all the vertices of G as in this case $t \geq n + 1$).

Let us start the labeling. We number the pairs and triples consecutively, starting with 1 for both lists.

Given any bipartite component G with both color classes even, we divide the vertices of every color class into pairs (x_j^1, x_j^2), putting

$$\phi_o(x_j^1, x_j^2) = d_j$$

for every such pair. We proceed in similar way in the case of all the nonbipartite components of even order, pairing the vertices of every such component in any way.

If both color classes of a bipartite component are of odd order, then they both have at least three vertices. We choose three of them, denoted with x_j, y_j, and z_{j+1}, in one class and another three, x_{j+1}, y_{j+1}, and z_j, in another

one and we put

$$\phi_e(x_j, z_j) = b_j,$$
$$\phi_e(y_j, z_j) = c_j,$$
$$\phi_e(x_{j+1}, z_{j+1}) = b_{j+1},$$
$$\phi_e(y_{j+1}, z_{j+1}) = c_{j+1}.$$

We proceed with the remaining vertices of this kind of components as in the case when both color classes are even.

In the case that all the components are of odd order we choose three vertices x_j, y_j, and z_j (in the case the bipartite component z_j belongs to the odd color class and two other vertices to the even one). We put

$$\phi_e(x_j, z_j) = b_j,$$
$$\phi_e(y_j, z_j) = c_j.$$

Observe that the numbers of triples and pairs of elements of $\mathbb{Z}_t$ are at least equal to the numbers of triples and pairs of vertices of G. Thus the labeling defined above is $\mathbb{Z}_t$-irregular. Indeed, in the jth triple of vertices the weights are equal to $w(x_j) = b_j$, $w(y_j) = c_j$, and $w(z_j) = -a_j$ and in the jth pair we have $w(x_j^1) = d_j$ and $w(x_j^2) = -d_j$. □

The main results easily follow from the above theorems and observations. Let $\mathcal{G}$ be a group of order t, where $t = n$ if n is odd and $t = n+1$ otherwise. In both cases t is odd and there is no involution $i \in \mathcal{G}$. Thus Theorem 1.5 easily follows from Theorem 2.3 and Lemma 1.4. Theorem 1.6 follows from Corollary 2.4 and Lemma 1.4. Finally, Theorem 1.7 is a consequence of Lemma 2.5 and Lemma 1.4.

ACKNOWLEDGMENTS

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POZNAŃ UNIVERSITY OF ECONOMICS AND BUSINESS, FACULTY OF INFORMATICS AND ELECTRONIC ECONOMY, AL. NIEPODLEGŁOŚCI 10, 61-875 POZNAŃ, POLAND
E-mail address: `m.anholcer@ue.poznan.pl`

AGH UNIVERSITY OF SCIENCE AND TECHNOLOGY, FACULTY OF APPLIED MATHEMATICS, AL. MICKIEWICZA 30, 30-059 KRAKÓW, POLAND,
E-mail address: `cichacz@agh.edu.pl`

UNIVERSITY OF PRIMORSKA, FACULTY OF MATHEMATICS, NATURAL SCIENCES AND INFORMATION TECHNOLOGIES, GLAGOLJAŠKA 8, SI-6000 KOPER, SLOVENIA
E-mail address: `marcin.anholcer@famnit.upr.si`
`sylwia.cichacz-przenioslo@famnit.upr.si`