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THE 2-TUPLE DOMINATING INDEPENDENT NUMBER OF A RANDOM GRAPH

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Abstract. In this note, we show that 2-tuple dominating independent number of the Erdős–Rényi graph $G(n, p)$ a.a.s. has a two-point concentration when p is a constant.

1. Introduction and main result

In a simple graph $G = (V, E)$, a vertex is said to dominate itself and its neighbors. The k-tuple domination set of G is a subset D of V such that any vertex in $V \setminus D$ is dominated by at least k vertices in D. Furthermore, if D is also an independent set (i.e. it does not induce any edge), then D is called a k-tuple dominating independent set. The k-tuple dominating independent number of G, $i_k(G)$, is the smallest integer ℓ such that there exists a k-tuple dominating independent set of cardinality ℓ, see [5] and [6] for more information about (k-tuple) independent domination in graphs.

The Erdős–Rényi random graph $G(n, p)$ is the set of graphs on n vertices and every two vertices are connected by an edge independently with probability p. Wieland and Godbole [8] proved the domination number of $G(n, p)$ asymptotically almost surely[1] (a.a.s.) is concentrated at two points for the constant p and for p tends to 0 with suitable rate. Later, Wang and Xiang [7] considered the k-tuple domination number of $G(n, p)$ and got the two-point concentration when p is a constant. Clark and Johnson [3] showed the independent domination (i.e. 1-tuple dominating independent) number of $G(n, p)$ for $p^2 \ln n \leq 64 \ln((\ln n)/p)$ a.a.s. also has the same property. Recently, Włoch [9] introduced 2-tuple dominating independent sets (called the 2-domination kernels in [9]), and characterized some classes of graphs having a 2-dominating kernel. In general, computing the independent domination

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[1]Here for a given graph property A, we say A occurs asymptotically almost surely if the probability that G_n has property A tends to 1 as $n \to \infty$.

number is NP-complete (see [4]), so is the k-tuple dominating independent number. Hence, it is interesting to decide $i_k(G)$ for a given graph G. In this note, we show that the 2-tuple dominating independent number of $G(n,p)$ a.a.s. also has a two-point concentration when p is constant. Our main results can be stated as follows.

Theorem 1.1. *Let $p \in (0,1)$ is a constant which is independent of n and $b = 1/(1-p)$. Then in $G(n,p)$ a.a.s.*

$$\lfloor \log_b n - \log_b \ln n + \log_b 2p \rfloor + 2 \le i_2(G(n,p)) \le \lfloor \log_b n - \log_b \ln n + \log_b 2p \rfloor + 3.$$

Here $\lfloor x \rfloor$ is the largest integer which is no more that x for any $x \in \mathbb{R}$.

The following notation will be used. Write $\mathbf{P}(\cdot)$, $\mathbf{E}(\cdot)$, and $\mathbf{Var}(\cdot)$ for the probability, expected value, and variance of a random variable or event, respectively. For any two positive functions $f(n)$ and $g(n)$ of a natural-valued parameter n, denote $f(n) = O(g(n))$ if there is a positive constant C such that $f(n) \le Cg(n)$ when n is large enough; $f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ and $g(n) = O(f(n))$; and $f(n) = o(g(n))$ if $\lim_{n\to\infty} f(n)/g(n) = 0$.

2. Proof of Theorem 1.1

In this section, we appeal to the probabilistic method (see [1]) to prove Theorem 1.1. The lower bound is proved in Section 2.1 by Markov's inequality, and the upper bound is shown in Section 2.2 by Chebyshev's inequality. All the inequalities hold under the condition that n is sufficiently large.

2.1. The lower bound.

Let X be a nonnegative integer valued random variable and suppose we want to show $\mathbf{P}(X(n) > k) \to 0$ when $n \to \infty$. By Markov's inequality, i.e. $\mathbf{P}(X(n) > k) \le \mathbf{E}(X(n))/k$, we only need to show $\mathbf{E}(X(n)) \to 0$. For our case, let $X_r^{(2)}$ denote the number of 2-tuple dominating sets of size r, where $r = \lfloor \log_b n - \log_b \ln n + \log_b 2p \rfloor + 1$. It is easy to see that

$$\mathbf{P}\left(i_2(G(n,p)) \le r\right) \le \mathbf{P}\left(X_r^{(2)} \ge 1\right).$$

So by Markov's inequality, we only need to show that $\mathbf{E}(X_r^{(2)}) \to 0$.

To simplify notation, let $q = 1-p$. Let $S_1, S_2, \ldots, S_{\binom{n}{r}}$ be all the subsets of vertices with size r. Define A_k to be the event that S_k is a 2-tuple dominating independent set, and I_k to be the corresponding indicator random variable. Clearly,

$$X_r^{(2)} = \sum_{k=1}^{\binom{n}{r}} I_k.$$

Then it is easy to see that

$$\mathbf{E}\left(X_r^{(2)}\right) = \binom{n}{r} q^{\binom{r}{2}} \left(1 - q^r - rpq^{r-1}\right)^{n-r},$$

where $\left(1-q^r-rpq^{r-1}\right)^{n-r}$ is the probability that every vertex outside of S_i is connected to at least two vertices of S_i and $q^{\binom{r}{2}}$ is the probability that S_i is an independent set. By the inequality $1-x\le e^{-x}$ for any real number x, we have

$$\begin{aligned}
&\mathbf{E}\left(X_r^{(2)}\right)=\binom{n}{r}q^{\binom{r}{2}}\left(1-q^r-rpq^{r-1}\right)^{n-r}\\
&\le\left(\frac{en}{r}\right)^r q^{\binom{r}{2}}\exp\left\{-(n-r)(q^r+rpq^{r-1})\right\}\\
&=\exp\left\{r\ln n+r-r\ln r+\frac{r(r-1)}{2}\ln q-(n-r)q^r-(n-r)rpq^{r-1}\right\}.
\end{aligned}$$

Rewrite $r=\log_b n-\log_b\ln n+\log_b 2p+1-\epsilon$, where

$$\epsilon:=\log_b n-\log_b\ln n+\log_b 2p-\lfloor\log_b n-\log_b\ln n+\log_b 2p\rfloor, \tag{2.1}$$

which is in $[0,1)$. Then

$$\begin{aligned}
q^r&=\frac{q^{1-\epsilon}\ln n}{2np};\\
nrpq^{r-1}&=\frac{1}{2q^\epsilon}\left(\log_b n-\log_b\ln n+\log_b 2p+1-\epsilon\right)\ln n;\\
\frac{r^2}{2}\ln q&=-\frac{(\log_b n)\ln n}{2}-\frac{(\log_b\ln n)\ln\ln n}{2}\\
&\quad+\ln n\cdot\log_b\ln n-(\log_b 2p+1-\epsilon+o(1))\ln n.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\mathbf{E}\left(X_r^{(2)}\right)\\
&\le\exp\left\{r\ln n+r-r\ln r-\frac{r(r-1)}{2}\ln q-(n-r)q^r-(n-r)rpq^{r-1}\right\}\\
&\le\exp\{(\log_b n-\log_b\ln n+\log_b 2p+1-\epsilon)\ln n\\
&\qquad-(1-o(1))\log_b n\cdot\ln\log_b n-\frac{(\log_b n)\ln n}{2}-\frac{(\log_b\ln n)\ln\ln n}{2}\\
&\qquad+\ln n\cdot\log_b\ln n-q^{-\epsilon}\left(\log_b n-\log_b\ln n\right)\ln n/2\\
&\qquad-\left(\log_b 2p+1-\epsilon+o(1)\right)\ln n\}\\
&=\exp\left\{-\left(\frac{1}{2q^\epsilon}-\frac{1}{2}\right)\ln n\cdot\log_b n-\left(1-q^{1-\epsilon}+o(1)\right)\ln n\cdot\ln\log_b n\right\}\\
&\to 0.
\end{aligned}$$

By Markov's inequality,

$$\mathbf{P}\left(X_r^{(2)}\ge 1\right)\le\mathbf{E}\left(X_r^{(2)}\right)\to 0.$$

Therefore,

$$\begin{aligned}\mathbf{P}\{i_2\left(G\left(n,p\right)\right) &\leq \lfloor \log_b n - \log_b \ln n + \log_b 2p \rfloor + 1\} \\ &\leq \mathbf{P}\left(X_r^{(2)} \geq 1\right) \leq \mathbf{E}\left(X_r^{(2)}\right) \to 0,\end{aligned}$$

which implies that a.a.s.,

$$i_2\left(G\left(n,p\right)\right) \geq \lfloor \log_b n - \log_b \ln n + \log_b 2p \rfloor + 2.$$

□

So far, we have obtained the lower bound. In the next subsection we will prove that a.a.s. its upper bound is $\lfloor \log_b n - \log_b \ln n + \log_b 2p \rfloor + 3$.

2.2. **The upper bound.** Let $X(n)$ be a nonnegative integer valued random variable and suppose we want to deduce that $X(n) > 0$ asymptotically almost surely. By Chebyshev's inequality,

$$\mathbf{P}(X(n) = 0) \leq \mathbf{P}\left[|X(n) - \mathbf{E}(X(n))| \geq \mathbf{E}(X(n)\right] \leq \frac{\mathbf{Var}(X(n))}{\mathbf{E}^2(X(n))},$$

we only need to prove that $\mathbf{E}\left(X(n)\right) \to \infty$ and $\mathbf{Var}\left(X(n)\right) = o\left(\mathbf{E}^2(X(n))\right)$. In our case, recall that $X_r^{(2)}$ denotes the number of 2-tuple dominating sets of size r, where

$$r = \lfloor \log_b n - \log_b \ln n + \log_b 2p \rfloor + 3$$

and note that

$$\mathbf{P}\left\{i_2\left(G\left(n,p\right)\right) > \lfloor \log_b n - \log_b \ln n + \log_b 2p \rfloor + 3\right\} \leq \mathbf{P}\left(X_r^{(2)} = 0\right).$$

To show

$$\mathbf{P}\left\{i_2\left(G\left(n,p\right)\right) > \lfloor \log_b n - \log_b \ln n + \log_b 2p \rfloor + 3\right\} \to 0$$

as $n \to \infty$, it suffices to prove $\mathbf{P}(X_r^{(2)} = 0) \to 0$. By Chebyshev's inequality, that is to check

$$\mathbf{E}\left(X_r^{(2)}\right) \to \infty \quad \text{and} \quad \mathbf{Var}\left(X_r^{(2)}\right) = o\left(\mathbf{E}^2\left(X_r^{(2)}\right)\right).$$

Rewrite $r = \log_b n - \log_b \ln n + \log_b 2p + 3 - \epsilon$, where ϵ is defined in (2.1). Then

$$\begin{aligned} q^r &= \frac{\ln n}{n}\frac{q^{3-\epsilon}}{2p}; \\ nrpq^{r-1} &= (1+o(1))\frac{q^{2-\epsilon}}{2}\ln n \cdot \log_b n; \\ \frac{r^2}{2}\ln q &= -\frac{1+o(1)}{2}\ln n \cdot \log_b n; \\ r\ln n &= (1+o(1))\ln n \cdot \log_b n. \end{aligned}$$

Note $1-x \geq e^{-\frac{x}{1-x}}$ for $x \in (0,1)$, and $r! = (1+o(1))\sqrt{2\pi r}\left(\frac{r}{e}\right)^r$. So we obtain

$$\begin{aligned}
\mathbf{E}\left(X_r^{(2)}\right) &= \binom{n}{r} q^{\binom{r}{2}} \left(1-q^r - rpq^{r-1}\right)^{n-r} \\
&\geq \binom{n}{r} \exp\left\{-\frac{nrpq^{r-1}}{1-rpq^{r-1}} + \binom{r}{2}\ln q\right\} \\
&\geq (1+o(1))\frac{n^r}{r!}\exp\left\{-\frac{nrpq^{r-1}}{1-rpq^{r-1}} + \binom{r}{2}\ln q\right\} \\
&\geq (1+o(1))\left(\frac{en}{r}\right)^r (2\pi r)^{-\frac{1}{2}} \exp\left\{-\frac{nrpq^{r-1}}{1-rpq^{r-1}} + \binom{r}{2}\ln q\right\} \\
&\geq (1+o(1))\exp\left\{r\ln n + r + r\ln r - \frac{\lg(2\pi r)}{2}\right. \\
&\qquad \left. + \frac{r(r-1)}{2}\ln q - \frac{nrpq^{r-1}}{1-rpq^{r-1}}\right\} \\
&\geq (1+o(1))\exp\left\{(1+o(1))\ln n\cdot\log_b n - \frac{1+o(1)}{2}\ln n\cdot\log_b n\right. \\
&\qquad \left. -\frac{q^{2-\epsilon}}{2}\ln n\cdot\log_b n\right\} \\
&\geq (1+o(1))\exp\left\{\left(\frac{1}{2} - \frac{q^{2-\epsilon}}{2} + o(1)\right)\ln n\cdot\log_b n\right\} \to \infty.
\end{aligned}$$

For the variance of $X_r^{(2)}$, we have

$$\begin{aligned}
\mathbf{Var}\left(X_r^{(2)}\right) = \mathbf{Var}\left(\sum_{j=1}^{\binom{n}{r}} I_j\right) &= \sum_{j=1}^{\binom{n}{r}} \mathbf{Var}(I_j) + \sum_{i\neq j}\mathbf{Cov}\left(I_i, I_j\right) \\
&= \sum_{j=1}^{\binom{n}{r}} \mathbf{E}(I_j)\left(1-\mathbf{E}(I_j)\right) + 2\sum_{i=1}^{\binom{n}{r}}\sum_{j<i}\left[\mathbf{E}\left(I_iI_j\right) - \mathbf{E}\left(I_i\right)\mathbf{E}\left(I_j\right)\right] \\
&= \binom{n}{r}\sum_{s=0}^{r-1}\binom{r}{s}\binom{n-r}{r-s}\mathbf{E}\left(I_iI_j\right) + \mathbf{E}\left(X_r^{(2)}\right) - \mathbf{E}^2\left(X_r^{(2)}\right). \qquad (2.2)
\end{aligned}$$

Here $s = |S_i \cap S_j|$ and

$$\begin{aligned}
\mathbf{E}\left(I_iI_j\right) &= \mathbf{P}\left\{S_i \text{ and } S_j \text{ are the 2-tuple dominating independent sets}\right\} \\
&\leq \mathbf{P}\left\{\text{Each } v \in \overline{S_i \cup S_j} \text{ has at least two neighbors both in } S_i \text{ and } S_j;\right. \\
&\qquad \left. S_i \text{ and } S_j \text{ are independent sets of size } r\right\}.
\end{aligned}$$

For each $v \in \overline{S_i \cup S_j}$, denote by $B_{ij}(v)$ the event that v has exactly one neighbor both in $S_i \setminus S_j$ and in $S_j \setminus S_i$; by $C_{ij}(v)$ the event that x has at most one neighbor in $S_i \cup S_j$; and by $D_{ij}(v)$ the event that v has at most one neighbor in S_i but at least two neighbors in $S_j \setminus S_i$. Then

$$\begin{aligned}
\mathbf{P}\left(B_{ij}(v)\right) &= (r-s)pq^{r-s-1}(r-s)pq^{r-s-1}q^s = (r-s)^2p^2q^{2r-s-2}, \\
\mathbf{P}\left(C_{ij}(v)\right) &= q^{2r-s} + (2r-s)pq^{2r-s-1} = \left(1+p(2r-s-1)\right)q^{2r-s-1}, \\
\mathbf{P}\left(D_{ij}(v)\right) &= \left\{q^r + rpq^{r-1}\right\} \cdot \left\{1 - q^{r-s} - (r-s)pq^{r-s-1}\right\} \\
&= [1+(r-1)p]q^{r-1} - [1+(r-1)p][1+(r-s-1)p]q^{2r-s-2},
\end{aligned}$$

which means

$$\begin{aligned}
&\mathbf{E}\left(I_iI_j\right) \\
&\le q^{2\binom{r}{2}-\binom{s}{2}} \prod_{v\in\overline{S_i\cup S_j}} \left[1-\mathbf{P}\left(B_{ij}(v)\right) - \mathbf{P}\left(C_{ij}(v)\right) - \mathbf{P}\left(D_{ij}(v)\right) - \mathbf{P}\left(D_{ji}(v)\right)\right] \\
&= q^{2\binom{r}{2}-\binom{s}{2}} \times \Big\{1 - 2(1+(r-1)p)q^{r-1} \\
&\qquad + \left[p^2(r^2-s^2-2r+s+1) + p(2r-s-2)+1\right] q^{2r-s-2}\Big\}^{n-2r+s} \\
&:= m(s).
\end{aligned}$$

In order to get $\mathbf{Var}(X_r^{(2)}) = o\big(\mathbf{E}^2(X_r^{(2)})\big)$, define

$$\Lambda_1 := \binom{n}{r}\sum_{s=1}^{r-1}\binom{r}{s}\binom{n-r}{r-s}m(s),\ \ \Lambda_2 := \binom{n}{r}\binom{r}{0}\binom{n-r}{r}m(0).$$

Then

$$\mathbf{Var}\left(X_r^{(2)}\right) \le \Lambda_1 + \Lambda_2 + \mathbf{E}\left(X_r^{(2)}\right) - \mathbf{E}^2\left(X_r^{(2)}\right).$$

Notice that

$$\begin{aligned}
f(s) &:= \binom{r}{s}\binom{n-r}{r-s} q^{2\binom{r}{2}-\binom{s}{2}} \times \Big\{1-2(1+(r-1)p)q^{r-1} \\
&\qquad + \left[p^2(r^2-s^2-2r+s+1)+p(2r-s-2)+1\right] q^{2r-s-2}\Big\}^{n-2r+s} \\
&\le 2\binom{r}{s}\frac{n^{r-s}}{(r-s)!} q^{2\binom{r}{2}-\binom{s}{2}} \\
&\qquad \times \exp\Big\{nq^{2r-s-2}\left[p^2(r^2-s^2-2r+s+1)+p(2r-s-2)+1\right] \\
&\qquad\qquad -2n(1+(r-1)p)q^{r-1}\Big\}.
\end{aligned}$$

Define

$$\begin{aligned}
g(s) := {}& 2\binom{r}{s}\frac{n^{r-s}}{(r-s)!} q^{2\binom{r}{2}-\binom{s}{2}} \\
&\times \exp\Big\{nq^{2r-s-2}\left[p^2(r^2-s^2-2r+s+1)+p(2r-s-2)+1\right] \\
&\qquad -2n(1+(r-1)p)q^{r-1}\Big\}.
\end{aligned}$$

In the following, we shall prove

$$\sum_{s=1}^{r-1} f(s) \le rg(1).$$

The above inequality holds naturally if we can show that

(i) $s \in [1, \log_b n - (1+\eta(n))\log_b \ln n,]$ $g(s)$ is first decreasing and then increasing, where $\eta(n)$ is a positive function on n which satisfies that $\eta(n) \to 0$ and $\eta(s)\log_b \ln n \to \infty$ as $n \to \infty$;

(ii) $g(1) \ge g(s)$ when $s = \log_b n - (1+\eta(n))\log_b \ln n$;

(iii) $g(1) \ge g(s)$ when $s = \log_b n - \log_b \ln n + c_3$, where c_3 is a constant and $c_3 < \log_b 2p + 3 - \epsilon$.

Proof of (i). In fact,

$$\begin{aligned}
&\frac{g(s+1)}{g(s)} \\
&= \frac{(r-s)^2}{n(s+1)} b^s \exp\left\{np^2q^{2r-s-3}\left[p\left(r^2 - s^2 - 2r + s + 1\right) + 2r - 3s - 2\right]\right\} \\
&\ge 1
\end{aligned}$$

if and only if

$$\begin{aligned}
&s \ln b + np^2q^{2r-s-3}\left[p\left(r^2 - s^2 - 2r + s + 1\right) + 2r - 3s - 2\right] \\
&\ge \ln\left(\frac{n(s+1)}{(r-s)^2}\right). \qquad (2.3)
\end{aligned}$$

Write $\ln\left(n(s+1)/(r-s)^2\right) := (1+\delta(s))\ln n$, where $\delta(s) = \Theta(\ln r/\ln n)$ which tends to 0 as $n \to \infty$. In the following, we will show the monotonicity of $g(s)$ through checking inequality (2.3).

Case 1: $s \le c_1 \log_b n$, where $0 < c_1 < 1$.

Define

$$h(s) = p\left(r^2 - s^2 - 2r + s + 1\right) + 2r - 3s - 2.$$

It is easy to see that $h'(s) = -2ps - (3-p) < 0$, which means $h(s)$ is a deceasing function on s. Therefore, when n is large enough,

$$\begin{aligned}
&s \ln b + np^2q^{2r-s-3}\left[p\left(r^2 - s^2 - 2r + s + 1\right) + 2r - 3s - 2\right] \\
&\le s \ln b + np^2q^{2r-s-3} \cdot \left(pr^2 + (2-2p)r + p - 5\right) \\
&\le c_1 \ln n + np^2 \cdot \frac{\ln^2 n}{n^{2-c_1-o(1)}} \cdot \frac{q^{3-\epsilon}}{4p^2} \cdot \left(pr^2 + (2-2p)r + p - 5\right) \\
&\le c_1 \ln n + \frac{q^{3-\epsilon}}{4}\frac{\ln^2 n}{n^{1-c_1-o(1)}} \cdot 2pc_1^2 \log_b^2 n \\
&= c_1 \ln n + o(\ln n) < (1+\delta(s))\ln n.
\end{aligned}$$

Case 2: $s = \log_b n - c_2 \log_b \ln n + o(\log_b \ln n)$, where c_2 is a constant and $c_2 > 1$.

$$\begin{aligned}
& s\ln b + np^2q^{2r-s-3}\left[p\left(r^2 - s^2 - 2r + s + 1\right) + 2r - 3s - 2\right] \\
&= \ln n - (c_2 + o(1))\ln\ln n \\
&\qquad + \frac{q^{3-2\epsilon}(\ln n)^{c_2-2+o(1)}}{4} \cdot [2p(c_2-1) + o(1)]\log_b \ln n \cdot \log_b n \\
&= \ln n - c_2 \ln\ln n + o(\ln\ln n) \\
&\qquad + \frac{p(c_2-1)q^{3-2\epsilon} + o(1)}{2\ln^2 b} \cdot (\ln n)^{c_2-1+o(1)} \cdot \ln\ln n \\
&\geq \ln n + 2\ln\ln n \geq \ln n + (1+o(1))\ln(\ln n) = \ln\left(\frac{n(s+1)}{(r-s)^2}\right).
\end{aligned}$$

Case 3: $s = \log_b n - \log_b \ln n - \eta(n)\log_b \ln n$, where $\eta(n)$ is a positive function on n which satisfies that $\eta(n) \to 0$ and $\eta(s)\log_b \ln n \to \infty$ as $n \to \infty$.

$$\begin{aligned}
& s\ln b + np^2q^{2r-s-3}\left[p\left(r^2 - s^2 - 2r + s + 1\right) + 2r - 3s - 2\right] \\
&= \ln n - \ln\ln n + c_3 \ln b \\
&\qquad + \frac{q^{3-2\epsilon}(\ln n)^{1-\eta(n)}}{4} \cdot (2p + o(1))\,\eta(n)\log_b \ln n \cdot \log_b n \\
&= \ln n - \ln\ln n + c_3 \ln b \\
&\qquad + \frac{pq^{3-2\epsilon} + o(1)}{2\ln b} \cdot \eta(n)\log_b \ln n \cdot (\ln n)^{2-\eta(n)} \\
&> (\ln n)^{2-\eta(n)} > \left(1 + \Theta\left(\frac{\ln r}{\ln n}\right)\right)\ln n = \ln\left(\frac{n(s+1)}{(r-s)^2}\right).
\end{aligned}$$

By the discussions above, when n is large enough, $g(s)$ is first decreasing and then increasing for $s \in [1, \log_b n - (1+\eta(n))\log_b \ln n]$.

Proof of (ii). When $s = \log_b n - (1+\eta(n))\log_b \ln n$,

$$\begin{aligned}
\frac{g(1)}{g(s)} &= \frac{r\frac{n^{r-1}}{(r-1)!}q^{2\binom{r}{2}}}{\binom{r}{s}\frac{n^{r-s}}{(r-s)!}q^{2\binom{r}{2}-\binom{s}{2}}} \\
&\qquad \times \frac{\exp\left\{nq^{2r-3}\left[p^2(r^2-2r+1) + p(2r-3) + 1\right]\right\}}{\exp\left\{nq^{2r-s-2}\left[p^2(r^2-s^2-2r+s+1) + p(2r-s-2) + 1\right]\right\}} \\
&\geq \frac{n^{s-1}q^{\frac{s^2}{2}}}{r!r^s} \cdot \frac{\exp\left\{n \cdot \frac{q^{3-\epsilon}\ln^2 n}{n^2} \cdot p^2(1+o(1))\log_b^2 n\right\}}{\exp\left\{\frac{q^{4-2\epsilon}}{4p^2} \cdot (2p+o(1))\,\eta(n)\log_b \ln n \cdot \log_b n\right\}} \\
&\geq \frac{(1+o(1))n^{\frac{s}{2}-1}(\ln n)^{\frac{s(1+\eta(n))}{2}}}{2\sqrt{2\pi r}\left(\frac{r}{e}\right)^r \cdot r^s} \cdot \frac{1}{n^{\frac{q^{4-2\epsilon}(1+o(1))}{2p^2\ln b}\eta(n)\log_b \ln n}} > 1.
\end{aligned}$$

Here, the last inequality holds as, noting $s = (1+o(1))\log_b n$, $r = (1+o(1))\log_b n$ and $\eta(n) \to 0$,

$$\begin{aligned}
&\ln\left(\frac{(1+o(1))n^{\frac{s}{2}-1}(\ln n)^{\frac{s(1+\eta(n))}{2}}}{2\sqrt{2\pi r}\left(\frac{r}{e}\right)^r \cdot r^s} \cdot \frac{1}{n^{\frac{q^{4-2\epsilon}(1+o(1))}{2p^2\ln b}\eta(n)\log_b \ln n}}\right) \\
&\geq (1+o(1))\frac{\log_b n}{2}\cdot \ln n + (1+o(1))\frac{\log_b n}{4}\cdot \ln\ln n \\
&\quad - 3((1+o(1)))\log_b n \cdot \ln\log_b n - \log_b \ln n \cdot \ln n \\
&> 0.
\end{aligned}$$

Proof of (iii). When $s = \log_b n - \log_b \ln n + c_3$, where c_3 is a constant and $c_3 \leq \log_b 2p + 3 - \epsilon$, noting that $r! = (1+o(1))\sqrt{2\pi r}\left(\frac{r}{e}\right)^r$ and $\epsilon \in [0,1)$, it is easy to check that

$$q^{2r-s-2} = \frac{q^{6-c_3-2\epsilon}}{4p^2 n}, \qquad q^s = \frac{q^{c_3}\ln n}{n},$$

and

$$p^2(r^2 - s^2 - 2r + s + 1) + p(2r - s - 2) + 1 = \tilde{c}\log_b n,$$

where

$$\tilde{c} := p^2\left(2\log_b 2p + 5 - 2c_3 - 2\epsilon\right) + p + o(1) \geq -p^2 + p + o(1) > 0.$$

So far we have

$$\begin{aligned}
&\frac{g(1)}{g(s)} \\
&= \frac{r\frac{n^{r-1}}{(r-1)!}q^{2\binom{r}{2}}}{\binom{r}{s}\frac{n^{r-s}}{(r-s)!}q^{2\binom{r}{2}-\binom{s}{2}}} \\
&\quad \times \frac{\exp\left\{nq^{2r-3}\left[p^2(r^2-2r+1)+p(2r-3)+1\right]\right\}}{\exp\left\{nq^{2r-s-2}\left[p^2(r^2-s^2-2r+s+1)+p(2r-s-2)+1\right]\right\}} \\
&\geq \frac{n^{s-1}q^{\frac{s^2}{2}}}{r!r^s} \\
&\quad \times \frac{\exp\left\{n\cdot\frac{q^{3-\epsilon}\ln^2 n}{n^2}\cdot p^2(1+o(1))\log_b^2 n\right\}}{\exp\left\{n\cdot\frac{q^{4-c_3-2\epsilon}}{4p^2 n}\cdot\left[p^2\left(2\log_b\frac{2p}{q}+3-2c_3-2\epsilon\right)+p+o(1)\right]\log_b n\right\}} \\
&\geq \frac{(1+o(1))n^{\frac{s}{2}-1}\left(q^{c_3}\ln n\right)^s}{2\sqrt{2\pi r}\left(\frac{r}{e}\right)^r\cdot r^s}\cdot\frac{1}{n^{\frac{\tilde{c}q^{4-c_3-2\epsilon}}{4p^2\ln b}}} > 1.
\end{aligned}$$

Here, we also get that the last inequality holds as, noting $s = (1+o(1))\log_b n$ and $r = (1+o(1))\log_b n$,

$$\begin{aligned}
&\ln\left(\frac{(1+o(1))n^{\frac{s}{2}-1}\left(q^{c_3}\ln n\right)^s}{2\sqrt{2\pi r}\left(\frac{r}{e}\right)^r\cdot r^s}\cdot\frac{1}{n^{\frac{\tilde{c}q^{4-c_3-2\epsilon}}{4p^2\ln b}}}\right)\\
&\geq \frac{(1+o(1))\log_b n}{4}\ln n + (1+o(1))\log_b n\cdot\ln\ln n\\
&\qquad -2(1+o(1))\log_b n\cdot\ln\log_b n - \frac{\tilde{c}q^{4-c_3-2\epsilon}}{4p^2\ln b}\ln n\\
&> 0.
\end{aligned}$$

By (i)–(iii) we can conclude that

$$f(s)\leq g(s)\leq g(1),\ \sum_{s=1}^{r-1}f(s)\leq rg(1).$$

Now we can make estimates for Λ_1 and Λ_2.

$$\begin{aligned}
\frac{\Lambda_1}{\mathbf{E}^2\left(X_r^{(2)}\right)} &= \frac{\binom{n}{r}\sum_{s=1}^{r-1}f(s)}{\mathbf{E}^2\left(X_r^{(2)}\right)} \leq \frac{\binom{n}{r}rg(1)}{\binom{n}{r}^2 q^{2\binom{r}{2}}\left(1-q^r-rpq^{r-1}\right)^{2n-2r}}\\
&\leq \frac{\binom{n}{r}r\frac{2rn^{r-1}q^{2\binom{r}{2}}}{(r-1)!}}{\binom{n}{r}^2 q^{2\binom{r}{2}}\left(1-q^r-rp)q^{r-1}\right)^{2n-2r}}\\
&\quad\times\exp\left\{nq^{2r-3}\left[p^2(r^2-2r+1)+p(2r-3)+1\right]-2n(1+(r-1)p)q^{r-1}\right\}\\
&= \frac{2(1+o(1))r^2n^{r-1}r!}{(r-1)!n^r}\cdot\frac{\exp\left\{-2n(1+(r-1)p)q^{r-1}\right\}}{\left\{1-(1+(r-1)p)q^{r-1}\right\}^{2n-2r}}\\
&\leq \frac{3(\log_b n)^3}{n}\to 0.\\
\frac{\Lambda_2}{\mathbf{E}^2\left(X_r^{(2)}\right)} &= \frac{\binom{n}{r}\binom{n-r}{r}q^{2\binom{r}{2}}}{\binom{n}{r}^2 q^{2\binom{r}{2}}\left(1-q^r-rpq^{r-1}\right)^{2n-2r}}\\
&\quad\times\left\{1-2(1+(r-1)p)q^{r-1}+\left[p^2(r^2-2r+1)+p(2r-2)+1\right]q^{2r-2}\right\}^{n-2r}\\
&= \frac{\binom{n-r}{r}\left(1-(2+o(1))(1+(r-1)p)q^{r-1}\right)^{n-2r}}{\binom{n}{r}\left\{1-(1+(r-1)p)q^{r-1}\right\}^{2n-2r}} = 1+o(1).
\end{aligned}$$

Therefore,

$$\mathbf{Var}\left(X_r^{(2)}\right)\leq\Lambda_1+\Lambda_2-\mathbf{E}^2\left(X_r^{(2)}\right)+\mathbf{E}\left(X_r^{(2)}\right)=o\left(\mathbf{E}^2\left(X_r^{(2)}\right)\right).$$

By Chebyshev's inequality,

$$\begin{aligned}\mathbf{P}\{i_2(G(n,p)) > \lfloor \log_b n - \log_b \ln n + \log_b 2p \rfloor + 3\} \\ \le \mathbf{P}\left(X_r^{(2)} = 0\right) \le \mathbf{P}\left(\left|X_r^{(2)} - \mathbf{E}X_r^{(2)}\right| \ge \mathbf{E}X_r^{(2)}\right) \\ \le \frac{\mathbf{Var}\left(X_r^{(2)}\right)}{\mathbf{E}^2\left(X_r^{(2)}\right)} \to 0.\end{aligned}$$

Thus a.a.s.,

$$i_2(G(n,p)) \le \lfloor \log_b n - \log_b \ln n + \log_b 2p \rfloor + 3.$$

□

3. Conclusions

In this paper, by Markov's inequality and Chebyshev's inequality we showed that 2-tuple dominating independent number of the Erdős–Rényi graph $G(n,p)$ a.a.s. has a two-point concentration when p is a constant.

References

1. N. Alon and J. Spencer, *The Probabilistic Method*, John Wiley Sons, New Jersey, 2008.
2. A. Bonato and C. Wang, *A note on domination parameters in random graphs*, Discuss. Math. Graph Theory **28** (2008), 335–343.
3. L. Clark and D. Johnson, *The independent domination number of a random graph*, Discuss. Math. Graph Theory **31** (2011), 129–142.
4. M. R. Garey and D. S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, Freeman, New York, 1979.
5. W. Goddard and M. A. Henning, *Independent domination in graphs: A survey and recent results*, Discrete Math. **313** (2013), 839–854.
6. Z. L. Nagy, *On the number of k-dominating independent sets*, J. Graph Theory **84** (2017), 566–580.
7. B. Wang and K. Xiang, *On k-tuple domination of random graphs*, Appl. Math. Lett. **22** (2009), 1513–1517.
8. B. Wieland and A. P. Godbole, *On the domination number of a random graph*, Electron. J. Combin. **8** (2001), R37.
9. A. Włoch, *On 2-dominating kernels in graphs*, Australas. J. Combin. **53** (2012), 272–284.

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