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ON THE 2-ADIC BEHAVIOR OF THE NUMBER OF DOMINO TILINGS ON A TORUS

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Abstract. We study the 2-adic behavior of the number of domino tilings of a $2(2n+1)\times 2(2n+1)$ torus. We show that this number is of the form $2^{4n+2}g(n)^2+2^{8n+2}(2n+1)^{4n}h(n)$, where $g(n)$ and $h(n)$ are odd positive integers. Moreover, we prove that $g(n)$ and $h(n)$ are uniformly continuous under the 2-adic metric and invariant under interchanging n and $-1-n$. This paper is an analog of Henry Cohn's results for $2n\times 2n$ squares (Electron. J. Combin. 6 (1999)).

1. Introduction

A *domino* is a 1×2 (or 2×1) rectangle, and a *tiling* of a region by dominos is a way of covering that region with dominos so that there are no gaps or overlaps [2]. In 1961, Kasteleyn [5] found formulas for the numbers of domino tilings of the finite quadratic lattice (with edges or with periodic boundary conditions) by a combinatorial method involving Pfaffians. This kind of combinatorial problems relating to a regular space lattice play a very important role in the theory of various physical phenomena.

In 1999, Cohn [1] studied the 2-adic behavior of the number of domino tilings for the $2n\times 2n$ square planar quadratic lattice based on Kasteleyn's formula. Cohn found that this number is of the form $2^n f(n)^2$, where $f(n)$ is an odd positive integer, and f is uniformly continuous under the 2-adic metric, and its unique extension to a function from $\mathbb{Z}_2$ to $\mathbb{Z}_2$ satisfies the functional equation

$$(1.1)\qquad f(-1-n)=\begin{cases} f(n), & \text{if } n\equiv 0,3 \pmod 4;\\ -f(n), & \text{if } n\equiv 1,2 \pmod 4.\end{cases}$$

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Here and subsequently, we say a function f is *uniformly continuous under the* 2*-adic metric* means that for every k, there exists an l such that if $n \equiv m \pmod{2^l}$, then $f(n) \equiv f(m) \pmod{2^k}$. This definition agrees with the one given in [1].

On account of Kasteleyn's configuration generating function for toroidal quadratic lattice [5, §4, formula (25)], the number of domino tilings of a $k \times l$ toroidal quadratic lattice (with kl even) is equal to

$$\begin{aligned} &\frac{1}{2}\prod_{i=1}^{\frac{1}{2}k}\prod_{j=1}^{l} 2\left(\sin^2\frac{2i\pi}{k}+\sin^2\frac{(2j-1)\pi}{l}\right)^{\frac{1}{2}} \\ &+\frac{1}{2}\prod_{i=1}^{\frac{1}{2}k}\prod_{j=1}^{l} 2\left(\sin^2\frac{(2i-1)\pi}{k}+\sin^2\frac{2j\pi}{l}\right)^{\frac{1}{2}} \\ &+\frac{1}{2}\prod_{i=1}^{\frac{1}{2}k}\prod_{j=1}^{l} 2\left(\sin^2\frac{(2i-1)\pi}{k}+\sin^2\frac{(2j-1)\pi}{l}\right)^{\frac{1}{2}}. \end{aligned} \tag{1.2}$$

The purpose of this paper is to study the 2-adic behavior of the number of domino tilings of a $2(2n+1) \times 2(2n+1)$ torus for $n \geq 0$. We will show that Kasteleyn's formula (1.2) can be written as $2^{4n+2}g(n)^2 + 2^{8n+2}(2n+1)^{4n}h(n)$ in our situation, where $g(n)$ and $h(n)$ are odd positive integers. Moreover, we prove the following main theorems (see Theorems 3.5 and 3.6 in Subsection 3.2 for details):

Theorem 1.1. *The function $g(n)$ is uniformly continuous under the* 2*-adic metric, and its unique extension to a function from $\mathbb{Z}_2$ to $\mathbb{Z}_2$ satisfies the functional equation $g(-1-n) = g(n)$.*

Theorem 1.2. *The function $h(n)$ is uniformly continuous under the* 2*-adic metric, and its unique extension to a function from $\mathbb{Z}_2$ to $\mathbb{Z}_2$ satisfies the functional equation $h(-1-n) = h(n)$.*

Let $\mu_m := e^{2\pi i/m}$ for $m \geq 1$. In our proof, we will view the number of domino tilings as an element of the cyclotomic field $\mathbb{Q}(\mu_{2n+1})$ (note that $\mathbb{Q}(\mu_{4n+2}) = \mathbb{Q}(\mu_{2n+1})$, because $\mu_{4n+2} = -\mu_{2n+1}^{n+1}$). Since $2n+1$ is odd, the extension $\mathbb{Q}_2(\mu_{2n+1})/\mathbb{Q}_2$ is unramified and the rational prime number 2 remains prime in $\mathbb{Q}_2(\mu_{2n+1})$ (see for example [1, 3, 6]). And then we can discuss the 2-adic behaviour of (1.2) for a $2(2n+1) \times 2(2n+1)$ torus.

Moreover, it is natural to study the same problem for the torus with all side lengths equal $4n$. In this case, the corresponding number given by (1.2) belongs to $\mathbb{Q}(\mu_{4n})$. Although the rational prime number 2 is no longer a prime in $\mathbb{Q}(\mu_{4n})$, one may turn to consider the divisibility of (1.2) by some $\mathfrak{p}$, where $\mathfrak{p}|2$ is a prime ideal in the ring of integers of $\mathbb{Q}(\mu_{4n})$. Note that the 2-adic valuation still has a unique extension to $\mathbb{Q}_2(\mu_{4n})$. One may conjecture that some results similar to Theorems 1.1 and 1.2 still hold, we will not consider this situation here but leave it as an open problem.

We remark that the question to study the 2-adic behaviour of the number of domino tilings relating to a regular space lattice is far from being solved. For example, Cohn [1] solved the square case for $k \times l$ planar lattice. But no relevant results have appeared until now when $k \neq l$. Kasteleyn's results [5] told us that to discuss such a question for both planer and toroidal lattices with $k \neq l$, one will involve two distinguish cyclotomic fields, then the number of domino tilings should be viewed as an element of their composition field. By a similar argument as in the previous paragraph, it does make sense to study the 2-adic behaviour for these general situations.

In Section 3, we first give some lemmas that will be used in our proofs. And then we prove Theorems 3.5 and 3.6 in Subsection 3.2. Our method of proof was motivated by Cohn [1] and some results there are directly cited, especially the results on quasi-polynomials.

2. Notes

In what follows, we let $\zeta := e^{2\pi i/(4n+2)}$ and $\xi := e^{2\pi i/(2n+1)}$, where i is the imaginary unit $\sqrt{-1}$. Then we have $\zeta = -\xi^{n+1}$ and the cyclotomic fields $\mathbb{Q}_2(\zeta)=\mathbb{Q}_2(\xi)$. We use $|\cdot|_2$ to denote the unique extension of the 2-adic absolute value to $\mathbb{Q}_2(\xi)$ (see [6, Chapter II] for more details).

We call a polynomial in n and $(-1)^n$ a *quasi-polynomial.* A well-known fact says that every quasi-polynomial over $\mathbb{Q}$ is uniformly continuous under the 2-adic metric (see [1, page 5]). We will apply this fact many times in Subsection 3.2.

3. Main results and proofs

3.1. Some lemmas.

In this subsection, we prove three lemmas which are similar to [1, Lemmas 2 and 3].

Lemma 3.1. *For any* $1 \le i, j \le 2n+1$*, let* $\alpha_{i,j} := \xi^i + \xi^{-i} + \xi^j + \xi^{-j}$ *and* $\beta_{i,j} := \xi^i + \xi^{-i} - \xi^j - \xi^{-j}$*. Then we have*

$$|4+\alpha_{i,j}|_2 = \begin{cases} \frac{1}{8}, & \text{if } i=j=2n+1; \\ \frac{1}{2}, & \text{if } i=j<2n+1; \\ \frac{1}{2}, & \text{if } i+j=2n+1; \\ 1, & \text{otherwise.} \end{cases}$$

and

$$|4-\beta_{i,j}|_2 = \begin{cases} \frac{1}{4}, & \text{if } i=j; \\ \frac{1}{4}, & \text{if } i+j=2n+1; \\ 1, & \text{otherwise.} \end{cases}$$

Proof. (1) The number $4+\alpha_{i,j}$ is an algebraic integer in $\mathbb{Q}_2(\xi)$, so its 2-adic absolute value is at most 1. We first notice that $\alpha_{i,j} = \xi^i + \xi^{-i} + \xi^j + \xi^{-j} = (\xi^i + \xi^j)(\xi^{i+j}+1)\xi^{-i}\xi^{-j}$. In order for $4+\alpha_{i,j}$ to reduce to 0 modulo 2, there must be $\xi^i + \xi^j \equiv 0 \pmod 2$ or $\xi^{i+j} + 1 \equiv 0 \pmod 2$, which implies that $i = j$ or $i + j = 2n + 1$.

On one hand, $i = j$ implies that $4 + \alpha_{i,j} = 4 + 2(\xi^i + \xi^{-i})$. In order to have $|4 + \alpha_{i,j}|_2 < 1/2$, there must be $\xi^i + \xi^{-i} \equiv 0 \pmod 2$, which implies that $i = 2n + 1$, i.e., $|4 + \alpha_{2n+1,2n+1}|_2 = 1/8$. On the other hand, $i + j = 2n + 1$ implies that $4 + \alpha_{i,j} = 4 + (\xi^i + \xi^{-i} + \xi^{2n+1-i} + \xi^{-(2n+1-i)}) = 4 + 2(\xi^i + \xi^{-i})$. In order to have $|4 + \alpha_{i,j}|_2 < 1/2$, there must be $\xi^i + \xi^{-i} \equiv 0 \pmod 2$, which implies that $i = 2n + 1$ as above. But this is impossible since $i + j = 2n + 1$ and $i, j \geq 1$.

(2) The number $4 - \beta_{i,j}$ is also an algebraic integer in $\mathbb{Q}_2(\xi)$, so its 2-adic absolute value is at most 1. We first notice that $\beta_{i,j} = \xi^i + \xi^{-i} - \xi^j - \xi^{-j} = (\xi^i - \xi^j)(\xi^{i+j} - 1)\xi^{-i}\xi^{-j}$. In order for $4 - \beta_{i,j}$ to reduce to 0 modulo 2, there must be $\xi^i - \xi^j \equiv 0 \pmod 2$ or $\xi^{i+j} - 1 \equiv 0 \pmod 2$, which implies that $i = j$ or $i + j = 2n + 1$. In these two cases, we have $\beta_{i,j} = 0$. Hence $|4 + \beta_{i,j}|_2 = (1/2)^2 = 1/4$. □

Lemma 3.2. *In the notation of Lemma 3.1, the number of domino tilings of a* $2(2n + 1) \times 2(2n + 1)$ *torus can be written as* $2^{4n+2} g(n)^2 + 2^{8n+2} h'(n)$, *where*

$$g(n) = \prod_{\substack{1 \leq i < j \leq 2n+1, \\ i+j \neq 2n+1}} (4 + \alpha_{i,j}), \tag{3.1}$$

$$h'(n) = \prod_{\substack{1 \leq i \neq j \leq 2n+1, \\ i+j \neq 2n+1}} (4 - \beta_{i,j}). \tag{3.2}$$

Furthermore, we have $g(n), h'(n) \in \mathbb{Z}$, *and* $g(n) \equiv h'(n) \equiv 1 \pmod 2$.

Remark: Lemma 3.4 will show that $g(n) \equiv 1 \pmod 4$.

Proof. By taking $k = l = 4n + 2$ in Kasteleyn's formula (1.2), we get that the number of domino tilings in our situation is

$$\begin{aligned} &\frac{1}{2} \prod_{i=1}^{2n+1} \prod_{j=1}^{4n+2} 2 \left(\sin^2 \frac{2i\pi}{4n+2} + \sin^2 \frac{(2j-1)\pi}{4n+2} \right)^{\frac{1}{2}} \\ &+ \frac{1}{2} \prod_{i=1}^{2n+1} \prod_{j=1}^{4n+2} 2 \left(\sin^2 \frac{(2i-1)\pi}{4n+2} + \sin^2 \frac{2j\pi}{4n+2} \right)^{\frac{1}{2}} \\ &+ \frac{1}{2} \prod_{i=1}^{2n+1} \prod_{j=1}^{4n+2} 2 \left(\sin^2 \frac{(2i-1)\pi}{4n+2} + \sin^2 \frac{(2j-1)\pi}{4n+2} \right)^{\frac{1}{2}}. \end{aligned} \tag{3.3}$$

We replace $\sin \frac{(2j-1)\pi}{4n+2}$ and $\sin \frac{2j\pi}{4n+2}$ by $-\sin \frac{(2j-4n-3)\pi}{4n+2}$ and $-\sin \frac{(2j-4n-2)\pi}{4n+2}$ respectively whenever $2n + 2 \leq j \leq 4n + 2$ in (3.3). Note that when j runs over $2n + 2, 2n + 3, \ldots, 4n + 2$, the corresponding $2j - 1 - (4n + 2)$ runs over $1, 3, 5, \ldots, 4n + 1$. Do a similar analysis for $2j - (4n + 2)$. Thus formula (3.3) becomes

$$
\begin{aligned}
&\frac{1}{2}\prod_{i=1}^{2n+1}\prod_{j=1}^{2n+1}4\left(\sin^2\frac{2i\pi}{4n+2}+\sin^2\frac{(2j-1)\pi}{4n+2}\right)\\
&+\frac{1}{2}\prod_{i=1}^{2n+1}\prod_{j=1}^{2n+1}4\left(\sin^2\frac{(2i-1)\pi}{4n+2}+\sin^2\frac{2j\pi}{4n+2}\right)\\
&+\frac{1}{2}\prod_{i=1}^{2n+1}\prod_{j=1}^{2n+1}4\left(\sin^2\frac{(2i-1)\pi}{4n+2}+\sin^2\frac{(2j-1)\pi}{4n+2}\right).
\end{aligned}
\tag{3.4}
$$

Note that the first two summands in above summation are equal. According to the Euler formula $e^{i\theta} = \cos\theta + i\sin\theta$, and our notation $\zeta = -\xi^{n+1}$, formula (3.4) can be simplified as

(3.5)

$$
\prod_{i=1}^{2n+1}\prod_{j=1}^{2n+1}\left(4-\beta_{(2n+2)i,(n+1)(2j-1)}\right)+\frac{1}{2}\prod_{i=1}^{2n+1}\prod_{j=1}^{2n+1}\left(4+\alpha_{(n+1)(2i-1),(n+1)(2j-1)}\right),
$$

where $\alpha_{i,j}, \beta_{i,j}$ are the same as Lemma 3.1. By $\xi^{2n+1} = 1$, we have $\xi^{(2n+2)i} = \xi^i$, and $\xi^{(n+1)(2j-1)} = \xi^{-(n+1)+j}$. It is easy to check that when j runs over $1, 2, \ldots, 2n+1$, the corresponding $-(n+1)+j$ runs over $-n, -n+1, \ldots, -1, 0, 1, \ldots, n-1, n$ which constitutes a complete set of residues to the modulus $2n+1$. Thus formula (3.5) is equal to

$$
\prod_{i=1}^{2n+1}\prod_{j=1}^{2n+1}(4-\beta_{i,j})+\frac{1}{2}\prod_{i=1}^{2n+1}\prod_{j=1}^{2n+1}(4+\alpha_{i,j}). \tag{3.6}
$$

Now by application of Lemma 3.1, we move the divisor 2 forward in each summand of (3.6). It follows that

$$
\prod_{i=1}^{2n+1}\prod_{j=1}^{2n+1}(4-\beta_{i,j}) = 2^{8n+2}\prod_{\substack{1\le i\ne j\le 2n+1,\\ i+j\ne 2n+1}}(4-\beta_{i,j}),
$$

and

$$
\prod_{i=1}^{2n+1}\prod_{j=1}^{2n+1}(4+\alpha_{i,j}) \tag{3.7}
$$

$$
= 2^{4n+3}\prod_{\substack{1\le i\ne j\le 2n+1,\\ i+j\ne 2n+1}}(4+\alpha_{i,j}) = 2^{4n+3}\prod_{\substack{1\le i< j\le 2n+1,\\ i+j\ne 2n+1}}(4+\alpha_{i,j})^2, \tag{3.8}
$$

where we used the fact $\alpha_{i,j} = \alpha_{j,i}$ for the second equal sign. Hence formula (3.6) becomes

$$
2^{4n+2}g(n)^2 + 2^{8n+2}h'(n),
$$

where $g(n) \equiv h'(n) \equiv 1 \pmod 2$.

Moreover, note that both $g(n)$ and $h'(n)$ are algebraic integers in $\mathbb{Q}(\xi)$, and they are invariant under the action of every automorphism of the Galois extension $\mathbb{Q}(\xi)/\mathbb{Q}$. Thus we actually have $g(n), h'(n) \in \mathbb{Z}$. □

To determining the 2-adic behavior of g and h', we start by examining them modulo 4. In that case, we have formulas

$$g(n) \equiv \prod_{\substack{1\le i<j\le 2n+1,\\ i+j\neq 2n+1}} \alpha_{i,j} \pmod 4, \tag{3.9}$$

$$h'(n) \equiv \prod_{\substack{1\le i\neq j\le 2n+1,\\ i+j\neq 2n+1}} -\beta_{i,j} = \prod_{\substack{1\le i\neq j\le 2n+1,\\ i+j\neq 2n+1}} \beta_{i,j} \pmod 4, \tag{3.10}$$

where there exist $4n^2$ product factors $-\beta_{i,j}$ in (3.10). The following lemma tells us that the right products appearing in above two equations can actually be evaluated explicitly.

Lemma 3.4. *Following the notation as before, we have*

$$\prod_{\substack{1\le i<j\le 2n+1,\\ i+j\neq 2n+1}} \alpha_{i,j} = 1, \text{ and} \tag{3.11}$$

$$\prod_{\substack{1\le i\neq j\le 2n+1,\\ i+j\neq 2n+1}} \beta_{i,j} = \prod_{\substack{1\le i<j\le 2n+1,\\ i+j\neq 2n+1}} \beta_{i,j}^2 = (2n+1)^{4n}. \tag{3.12}$$

Proof. To prove (3.11), we first see that

$$\begin{aligned}
\prod_{\substack{1\le i<j\le 2n+1,\\ i+j\neq 2n+1}} \alpha_{i,j} &= \prod_{i=1}^{2n} \prod_{\substack{j=i+1,\\ j\neq 2n+1-i}}^{2n+1} \alpha_{i,j} = \prod_{i=1}^{2n} \alpha_{i,2n+1} \cdot \prod_{i=1}^{2n} \prod_{\substack{j=i+1,\\ j\neq 2n+1-i}}^{2n} \alpha_{i,j} \\
&= \prod_{i=1}^{2n} \alpha_{i,2n+1} \cdot \prod_{i=1}^{n} \prod_{\substack{j=i+1,\\ j\neq 2n+1-i}}^{n} \alpha_{i,j} \cdot \prod_{i=1}^{n} \prod_{\substack{j=n+1,\\ j\neq 2n+1-i}}^{2n} \alpha_{i,j} \cdot \prod_{i=n+1}^{n-1} \prod_{\substack{j=i+1,\\ j\neq 2n+1-i}}^{2n} \alpha_{i,j}.
\end{aligned}$$

Here and subsequenty, we use the symbol $\cdot$ and write $\prod * \cdot \prod *'$ to mean that $(\prod *)\cdot(\prod *')$. Note that if $1 \le i, j \le n$ then $i + j < 2n + 1$, and if $i \ge n+1$ then $i + j \ge 2n + 3$ since $j > i$. For the factors when $i \ge n + 1$ (resp. $j \ge n+1$), we replace ξ^i by $\xi^{-(2n+1-i)}$ (resp. replace ξ^j by $\xi^{-(2n+1-j)}$). Then the above equation becomes

$$\begin{aligned}
\prod_{\substack{1\le i<j\le 2n+1,\\ i+j\neq 2n+1}} \alpha_{i,j} &= \prod_{i=1}^{n} \alpha_{i,2n+1}^2 \cdot \prod_{i=1}^{n} \prod_{j=i+1}^{n} \alpha_{i,j} \cdot \prod_{i=1}^{n} \prod_{\substack{j=1,\\ j\neq i}}^{n} \alpha_{i,j} \cdot \prod_{i=1}^{n} \prod_{j=1}^{i-1} \alpha_{i,j} \\
&= \prod_{i=1}^{n} \alpha_{i,2n+1}^2 \cdot \prod_{i=1}^{n} \prod_{\substack{j=1,\\ j\neq i}}^{n} \alpha_{i,j}^2,
\end{aligned}$$

where we have combined the factors $\prod_{i=1}^{n}\prod_{j=i+1}^{n}\alpha_{i,j}$ and $\prod_{i=1}^{n}\prod_{j=1}^{i-1}\alpha_{i,j}$ for the last equal sign. According to [1, page 2, formula (2)], because $\alpha_{i,2n+1} = 2+\xi^i+\xi^{-i} = (1+\xi^i)(1+\xi^{-i}) = (1+\xi^i)(1+\xi^{2n+1-i})$, we see that

$$\prod_{i=1}^{n}\alpha_{i,2n+1} = \prod_{i=1}^{n}(2+\xi^i+\xi^{-i}) = \prod_{i=1}^{2n}(1+\xi^i) = 1, \tag{3.13}$$

where for the third equal sign in (3.13) we have substituted $z=-1$ into the equality $z^{2n+1}-1 = \prod_{i=0}^{2n}(z-\xi^i)$. Now it follows that

$$\prod_{\substack{1\le i<j\le 2n+1,\\ i+j\neq 2n+1}}\alpha_{i,j} = \prod_{i=1}^{n}\prod_{\substack{j=1,\\ j\neq i}}^{n}\alpha_{i,j}^2 = \frac{\prod_{i=1}^{n}\prod_{j=1}^{n}\alpha_{i,j}^2}{\prod_{i=1}^{n}\alpha_{i,i}^2}. \tag{3.14}$$

According to [1, page 5] and since $\alpha_{i,i} = 2(\xi^i+\xi^{-i})$, we see that

$$\prod_{i=1}^{n}\alpha_{i,i} = 2^n\prod_{i=1}^{n}(\xi^i+\xi^{-i}) = 2^n(-1)^{\lfloor\frac{n+1}{2}\rfloor}, \tag{3.15}$$

and

$$\prod_{i=1}^{n}\prod_{j=1}^{n}\alpha_{i,j} = 2^n(-1)^{\lfloor\frac{n+1}{2}\rfloor}. \tag{3.16}$$

Therefore equation (3.11) holds by substituting formulas (3.15) and (3.16) into formula (3.14).

To prove (3.12), note that $\beta_{i,j} = -\beta_{j,i}$, and we see that

$$\prod_{\substack{1\le i\neq j\le 2n+1,\\ i+j\neq 2n+1}}\beta_{i,j} = (-1)^{2n^2}\prod_{\substack{1\le i<j\le 2n+1,\\ i+j\neq 2n+1}}\beta_{i,j}^2 = \prod_{\substack{1\le i<j\le 2n+1,\\ i+j\neq 2n+1}}\beta_{i,j}^2, \tag{3.17}$$

where the first equal sign holds by the same reasons as in (3.10). Similarly to the proof of (3.11), we can write

$$\prod_{\substack{1\le i<j\le 2n+1,\\ i+j\neq 2n+1}}\beta_{i,j} = \prod_{i=1}^{2n}\beta_{i,2n+1}\cdot\prod_{i=1}^{n}\prod_{j=i+1}^{n}\beta_{i,j}\cdot\prod_{i=1}^{n}\prod_{\substack{j=n+1,\\ j\neq 2n+1-i}}^{2n}\beta_{i,j}\cdot\prod_{i=n+1}^{2n}\prod_{j=i+1}^{2n}\beta_{i,j}.$$

Note that when i (resp. j) runs over $n+1, n+2, \ldots, 2n$, the corresponding $2n+1-i$ (resp. $2n+1-j$) runs over $n, n-1, \ldots, 2, 1$. And $\xi^{-(2n+1-l)} = \xi^l$ for $l = i, j$ implies that $\beta_{i,j} = \beta_{2n+1-i,j} = \beta_{i,2n+1-j} = \beta_{2n+1-i,2n+1-j}$. Thus we get that

$$\prod_{\substack{1\le i<j\le 2n+1,\\ i+j\neq 2n+1}}\beta_{i,j} = \prod_{i=1}^{n}\beta_{i,2n+1}^2\cdot\prod_{i=1}^{n}\prod_{\substack{j=1,\\ j\neq i}}^{n}\beta_{i,j}^2. \tag{3.18}$$

From $\beta_{i,2n+1} = \xi^i + \xi^{-i} - 2 = -(\xi^i - 1)(\xi^{-i} - 1) = -(\xi^i - 1)(\xi^{2n+1-i} - 1)$, it follows that

$$\prod_{i=1}^{n} \beta_{i,2n+1} = (-1)^n \prod_{i=1}^{n}(\xi^i - 1) \cdot \prod_{i=1}^{n}(\xi^{2n+1-i} - 1) = (-1)^n \prod_{i=1}^{2n}(\xi^i - 1).$$

Since $z^{2n+1} - 1 = \prod_{i=0}^{2n}(z - \xi^i) = (z - 1)\prod_{i=1}^{2n}(z - \xi^i)$, $\prod_{i=1}^{2n}(z - \xi^i) = z^{2n} + z^{2n-1} + \cdots + z + 1$. By substituting $z = 1$ into this formula, we see that

$$\prod_{i=1}^{2n}(\xi^i - 1) = 2n + 1, \tag{3.19}$$

and then

$$\prod_{i=1}^{n} \beta_{i,2n+1} = (-1)^n(2n + 1). \tag{3.20}$$

Now combining formulas (3.20) and (3.18), we have

$$\prod_{\substack{1\le i<j\le 2n+1,\\ i+j\neq 2n+1}} \beta_{i,j} = (2n + 1)^2 \prod_{i=1}^{n} \prod_{\substack{j=1,\\ j\neq i}}^{n} \beta_{i,j}^2. \tag{3.21}$$

Note that $\beta_{i,j} = \xi^i + \xi^{-i} - \xi^j - \xi^{-j} = (\xi^{i-j} - 1)(\xi^{i+j} - 1)\xi^{-i}$ and $\beta_{j,i} = -\beta_{i,j}$ in above (3.21), which indicates that

$$\begin{aligned}
\prod_{i=1}^{n} \prod_{\substack{j=1,\\ j\neq i}}^{n} \beta_{i,j} &= \prod_{1\le i<j\le n} \beta_{i,j} \cdot \prod_{1\le j<i\le n} \beta_{i,j} = (-1)^{\frac{n(n-1)}{2}} \prod_{1\le i<j\le n} \beta_{i,j}^2 \\
&= \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} ((\xi^{i-j} - 1)(\xi^{i+j} - 1)\xi^{-i})^2 \\
&= \xi^* \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} ((\xi^{2n+1+(i-j)} - 1)(\xi^{i+j} - 1))^2,
\end{aligned} \tag{3.22}$$

where we write ξ^* to indicate an unspecified power of ξ. Because the product in question is real and the only real power of ξ is $\xi^{2n+1} = 1$, we will in several cases see that ξ^* equals 1 without having to count the ξ's. Moreover, when the first index i in (3.22) is fixed, the second index j runs over $i + 1, i + 2, \ldots, n$. This yields the exponent $i + j$ on the right hand side of (3.22) runs over $2i + 1, 2i + 2, \ldots, n + i$; and the exponent $2n + 1 + (i - j)$ runs over $2n, 2n - 1, \ldots, n + i + 1$. So we get that

$$\prod_{i=1}^{n} \prod_{\substack{j=1,\\ j\neq i}}^{n} \beta_{i,j} = \xi^* \prod_{i=1}^{n-1} \prod_{s=2i+1}^{2n} (\xi^s - 1)^2. \tag{3.23}$$

Now we divide the remaining proof into two cases according to the parity of n as below.
Case (i): $n \equiv 1 \pmod 2$. We divide the set of indices i into $\{1, 2, \ldots, \frac{n-1}{2}\}$ and $\{\frac{n+1}{2}, \frac{n+3}{2}, \ldots, n-1\}$. Replacing $\xi^s - 1$ by $-\xi^s(\xi^{2n+1-s} - 1)$ on the right hand side of (3.23) for $\frac{n+1}{2} \leq i \leq n-1$, it follows that

$$\begin{aligned}
\prod_{i=1}^{n-1} \prod_{s=2i+1}^{2n} (\xi^s - 1) &= \prod_{i=1}^{\frac{n-1}{2}} \prod_{s=2i+1}^{2n} (\xi^s - 1) \cdot \prod_{i=\frac{n+1}{2}}^{n-1} \prod_{s=2i+1}^{2n} (-\xi^s(\xi^{2n+1-s} - 1)) \\
&= (-1)^* \xi^* \prod_{i=1}^{\frac{n-1}{2}} \prod_{s=2i+1}^{2n} (\xi^s - 1) \cdot \prod_{i=\frac{n+1}{2}}^{n-1} \prod_{s'=1}^{2n-2i} (\xi^{s'} - 1) \\
&= (-1)^* \xi^* \prod_{i=1}^{\frac{n-1}{2}} \prod_{s=2i+1}^{2n} (\xi^s - 1) \cdot \prod_{i'=1}^{\frac{n-1}{2}} \prod_{s'=1}^{2i'} (\xi^{s'} - 1) \\
&= (-1)^* \xi^* \prod_{i=1}^{\frac{n-1}{2}} \prod_{j=1}^{2n} (\xi^j - 1) \\
&\overset{(3.19)}{=\!=} (-1)^* \xi^* (2n+1)^{\frac{n-1}{2}},
\end{aligned}$$

where $s' = 2n+1-s, i' = n-i$, and $(-1)^*$ denotes an unspecified power of -1. Substituting above result into formula (3.23), we get that

$$\prod_{i=1}^{n} \prod_{\substack{j=1, \\ j \neq i}}^{n} \beta_{i,j} = (-1)^* \xi^* (2n+1)^{n-1}. \tag{3.24}$$

Furthermore, by combining formulas (3.24), (3.21), and (3.17), we get that

$$\prod_{\substack{1 \leq i \neq j \leq 2n+1, \\ i+j \neq 2n+1}} \beta_{i,j} = \xi^* (2n+1)^{4n}. \tag{3.25}$$

Note that the left hand side of formula (3.25) is invariant under the action of any automorphisms belong to the Galois group $\mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q})$. Thus $\prod_{\substack{1 \leq i \neq j \leq 2n+1, \\ i+j \neq 2n+1}} \beta_{i,j}$ is actually a rational integer, i.e., it belongs to $\mathbb{Z}$. On the other hand, since $2n+1$ is odd, there is no power of ξ belong to $\mathbb{Q}$ except $\xi^{2n+1} = 1$. This implies that the ξ^* in (3.25) is equal to 1. Hence we prove formula (3.12) in this case.
Case (ii): $n \equiv 0 \pmod 2$. We divide the set of indices i into $\{1, 2, \ldots, \frac{n}{2} - 1\}$, $\{\frac{n}{2}\}$ and $\{\frac{n}{2} + 1, \frac{n}{2} + 2, \ldots, n-1\}$. Similarly to the proof of Case (i), by replacing $\xi^s - 1$ with $-\xi^s(\xi^{2n+1-s} - 1)$ on the right hand side of (3.23) for

$\frac{n}{2}+1 \leq i \leq n-1$, we can prove that

$$\prod_{i=1}^{n-1} \prod_{s=2i+1}^{2n} (\xi^s - 1) = (-1)^* \xi^* (2n+1)^{\frac{n}{2}-1} \prod_{s=n+1}^{2n} (\xi^s - 1). \tag{3.26}$$

To determine the factor $\prod_{n+1 \leq s \leq 2n} (\xi^s - 1)$ on the right hand side of (3.26), note that

$$\begin{aligned}
\prod_{s=n+1}^{2n} (\xi^s - 1)^2 &= \prod_{s=n+1}^{2n} -\xi^s (\xi^{2n+1-s} - 1) \cdot \prod_{s=n+1}^{2n} (\xi^s - 1) \\
&= (-1)^n \xi^* \prod_{s=1}^{n} (\xi^s - 1) \cdot \prod_{s=n+1}^{2n} (\xi^s - 1) \\
&= \xi^* \prod_{s=1}^{2n} (\xi^s - 1) \\
&\overset{(3.19)}{=\!=} \xi^* (2n+1). \qquad (3.27)
\end{aligned}$$

Now by combining formulas (3.27), (3.26), (3.23), (3.21), and (3.17), we also get a formula which can be written as

$$\prod_{\substack{1 \leq i \neq j \leq 2n+1, \\ i+j \neq 2n+1}} \beta_{i,j} = \xi^* (2n+1)^{4n}. \tag{3.28}$$

For the same reason as the proof of Case (i), the ξ^* in (3.28) is equal to 1. Hence we also prove the formula (3.12). This completes the proof of our lemma. ☐

3.2. **Main results for g and h'.** Now we turn to prove our main results for g and h', which immediately give Theorems 1.1 and 1.2 stated in the introduction section.

Theorem 3.5. *Let $g(n)$ be the function given by* (3.1)*. Then $g(n)$ is uniformly continuous under the 2-adic metric, and its unique extension to a function from $\mathbb{Z}_2$ to $\mathbb{Z}_2$ satisfies the functional equation $g(-1-n) = g(n)$.*

Proof. Note that we have dealt with the behavior of g modulo 4 in Lemma 3.4. Since $+1$ and -1 are unequal in the set of residues modulo 4, we can simplify the proof considerably by working with g^2 rather than g. Therefore, if we can show that g^2 is uniformly continuous 2-adically and satisfies $g(-1-n)^2 = g(n)^2$, then we will prove Theorem 3.5.

From formula (3.7), we see that

$$g(n)^2 = \prod_{\substack{1 \leq i \neq j \leq 2n+1, \\ i+j \neq 2n+1}} (4 + \alpha_{i,j}). \tag{3.29}$$

Obviously, the right hand side of (3.29) can be rewritten as

$$\frac{\prod_{i=1}^{2n+1}\prod_{j=1}^{2n+1}(4+\alpha_{i,j})}{\prod_{i=1}^{2n+1}(4+\alpha_{i,i})\cdot\prod_{\substack{1\le i,j\le 2n+1,\\ i+j=2n+1}}(4+\alpha_{i,j})}.$$

According to formula (3.13),we have

$$\prod_{i=1}^{2n+1}(4+\alpha_{i,i}) = 2^{2n+3}\prod_{i=1}^{2n}(2+\xi^i+\xi^{-i}) = 2^{2n+3}\prod_{i=1}^{n}(2+\xi^i+\xi^{-i})^2 = 2^{2n+3},$$

and

$$\prod_{\substack{1\le i,j\le 2n+1,\\ i+j=2n+1}}(4+\alpha_{i,j}) = 2^{2n}\prod_{i=1}^{2n}(2+\xi^i+\xi^{-i}) = 2^{2n}.$$

Hence (3.29) becomes

$$g(n)^2 = 2^{-(4n+3)}\prod_{i=1}^{2n+1}\prod_{j=1}^{2n+1}(4+\alpha_{i,j}). \tag{3.30}$$

We compute the product on the right hand side of (3.30) as follows. By combining all terms when $i = 2n+1$, we have

$$\begin{aligned}\prod_{i=1}^{2n+1}\prod_{j=1}^{2n+1}(4+\alpha_{i,j}) = &\prod_{i=1}^{n}\prod_{j=1}^{n}(4+\alpha_{i,j})\cdot\prod_{i=1}^{n}\prod_{j=n+1}^{2n}(4+\alpha_{i,j})\cdot\prod_{i=1}^{n}(4+\alpha_{i,2n+1})\\ &\cdot\prod_{i=n+1}^{2n}\prod_{j=1}^{n}(4+\alpha_{i,j})\cdot\prod_{i=n+1}^{2n}\prod_{j=n+1}^{2n}(4+\alpha_{i,j})\cdot\prod_{i=n+1}^{2n}(4+\alpha_{i,2n+1})\\ &\cdot\prod_{j=1}^{2n}(4+\alpha_{2n+1,j})\cdot(4+\alpha_{2n+1,2n+1}),\end{aligned}$$

where $4+\alpha_{2n+1,2n+1} = 2^3$, $4+\alpha_{2n+1,j} = 6+\xi^j+\xi^{-j}$ and $4+\alpha_{i,2n+1} = 6+\xi^i+\xi^{-i}$. From now on, we denote $\xi^j+\xi^{-j}$ by γ_j for simplicity. Remember that $\alpha_{i,j} = \alpha_{2n+1-i,j} = \alpha_{i,2n+1-j} = \alpha_{2n+1-i,2n+1-j}$. Then it follows that

$$\prod_{i=1}^{2n+1}\prod_{j=1}^{2n+1}(4+\alpha_{i,j}) = 2^3\prod_{j=1}^{n}(6+\gamma_j)^4\cdot\prod_{i=1}^{n}\prod_{j=1}^{n}(4+\alpha_{i,j})^4.$$

And thus

$$g(n)^2 = 2^{-4n}\prod_{j=1}^{n}(6+\gamma_j)^4\cdot\prod_{i=1}^{n}\prod_{j=1}^{n}(4+\alpha_{i,j})^4 = f(n)^8\prod_{j=1}^{n}(6+\gamma_j)^4, \tag{3.31}$$

where

$$f(n) := \prod_{1\le i<j\le n}(4+\alpha_{i,j})$$

was studied by Cohn in [1]. Cohn proved that f is uniformly continuous under the 2-adic metric, and its unique extension to a function from $\mathbb{Z}_2$ to $\mathbb{Z}_2$ satisfies the functional equation (1.1). This implies that

$$f(-1-n)^2 = f(n)^2. \tag{3.32}$$

In what follows, we write

$$g'(n) := \prod_{i=1}^{n}(6+\gamma_i) \text{ for } \gamma_i = \xi^i + \xi^{-i}. \tag{3.33}$$

So formula (3.31) can be witten as

$$g(n)^2 = f(n)^8 g'(n)^4. \tag{3.34}$$

Thus in order to prove Theorem 3.5, it suffices to prove that g' is uniformly continuous 2-adically and satisfies a functional equation $g'(-1-n) = g'(n)$.

We first note that the function g' satisfies

$$\prod_{i=1}^{n}(6+\gamma_i) = \prod_{i=1}^{n}\gamma_i\left(1+\frac{6}{\gamma_i}\right) = \prod_{i=1}^{n}\gamma_i \cdot \prod_{i=1}^{n}\left(1+\frac{6}{\gamma_i}\right), \tag{3.35}$$

and according to formula (3.15), where

$$\prod_{i=1}^{n}\gamma_i = (-1)^{\lfloor\frac{n+1}{2}\rfloor}. \tag{3.36}$$

Secondly, it is easy to check that

$$\prod_{i=1}^{n}\left(1+\frac{6}{\gamma_i}\right) = \sum_{r=0}^{n} 6^r P_r\left(1/\gamma_1, 1/\gamma_2, \cdots, 1/\gamma_n\right), \tag{3.37}$$

where $P_0 = 1$ and P_r for $1 \le r \le n$ are the rth elementary symmetric polynomials in $1/\gamma_1, 1/\gamma_2, \ldots, 1/\gamma_n$ (see for example [4, Section 2.13]). Combining formulas (3.33)–(3.37), we see that

$$g'(n) = (-1)^{\lfloor\frac{n+1}{2}\rfloor}\sum_{r=0}^{n} 6^r P_r. \tag{3.38}$$

Note that the function $n \mapsto (-1)^{\lfloor\frac{n+1}{2}\rfloor}$ is uniformly continuous 2-adically and invariant under interchanging n with $-1-n$ (see [1, page 5]). So to prove these properties for g' we only need to prove them for $\sum_{r=0}^{n} 6^r P_r$ in (3.38).

Since $\gamma_i = \xi^i + \xi^{-i} = \xi^{-i}(1+\xi^{2i})$ for $1 \le i \le n$, γ_i can not be congruent to 0 modulo 2. It follows that $|\gamma_i|_2 = 1$, and thus $|\frac{1}{\gamma_i}|_2 = 1$ and $|\frac{1}{\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_r}}|_2 = 1$. Hence $|P_r|_2 \le 1$ by the ultrametric absolute inequality (see for example [6, Chapter II]). So P_r for $1 \le r \le n$ are actually 2-adic integers in $\mathbb{Q}_2(\xi)$.

We remark that for each 6 on the right hand side of (3.38) contributes at least one prime factor 2, thus for any positive integer k, to determine

$\sum_{r=0}^{n} 6^r P_r$ modulo 2^k we need only look at the first k terms, i.e., the terms when $r = 0, 1, 2, \ldots, k-1$. Let

$$S_r(n) := \sum_{i=1}^{n} \frac{1}{\gamma_i^r}.$$

Newton's identities (see [1, page 6] or [4, page 140, exercises 3]) say that the elementary symmetric polynomials P_r can be expressed by these symmetric polynomials S_r. Thus to study P_r, it is sufficient to study these S_r. Now we define

$$U_r(n) := \sum_{i=0}^{2n} \frac{1}{\gamma_i^r}.$$

Note that $\gamma_i = \xi^i + \xi^{-i} = \xi^{-(2n+1-i)} + \xi^{2n+1-i}$, and when i runs over $1, 2, \ldots, n$, the correspongding $2n+1-i$ runs over $n+1, n+2, \ldots, 2n$. This implies that $S_r(n) = \sum_{i=1}^{n} \gamma_i^{-r} = \sum_{i=n+1}^{2n} \gamma_i^{-r}$. Therefore, we get that

$$U_r(n) = \gamma_0^{-r} + 2S_r(n) = 2^{-r} + 2S_r(n).$$

According to [1, Proposition 5], the function $U_r(n)$ is a quasi-polynomial over $\mathbb{Q}$ and satisfies the functional equation $U_r(-1-n) = U_r(n)$. It follows that S_r is uniformly continuous 2-adically and satisfies a similar functional equation. By combining this with formulas (3.32) and (3.34), we see that $g(n)$ is uniformly continuous under the 2-adic metric and satisfies the functional equation $g(-1-n) = g(n)$, which completes the proof. □

Theorem 3.6. *Let $h'(n)$ be the function given by* (3.2)*. Then $h'(n)$ is of the form $(2n+1)^{4n} h(n)$, where $h(n)$ is uniformly continuous under the 2-adic metric, and its unique extension to a function from $\mathbb{Z}_2$ to $\mathbb{Z}_2$ satisfies the functional equation $h(-1-n) = h(n)$.*

Proof. First of all, note that $4 - \beta_{i,j} = -\beta_{i,j}(1 - 4/\beta_{i,j})$ in (3.2), and it is easy to check that the cardinal number $\#\{(i,j) : 1 \le i \ne j \le 2n+1, i+j \ne 2n+1\}$ equals $4n^2$, which is even. Thus we have

$$h'(n) = \prod_{\substack{1 \le i \ne j \le 2n+1, \\ i+j \ne 2n+1}} \beta_{i,j} \cdot \prod_{\substack{1 \le i \ne j \le 2n+1, \\ i+j \ne 2n+1}} \left(1 - \frac{4}{\beta_{i,j}}\right).$$

Now Lemma 3.4 says that the first factor $\prod_{\substack{1 \le i \ne j \le 2n+1, \\ i+j \ne 2n+1}} \beta_{i,j} = (2n+1)^{4n}$. For the second factor, we write

$$h(n) := \prod_{\substack{1 \le i \ne j \le 2n+1, \\ i+j \ne 2n+1}} \left(1 - \frac{4}{\beta_{i,j}}\right).$$

Similar to formula (3.37), we see that

$$\prod_{\substack{1 \le i \ne j \le 2n+1, \\ i+j \ne 2n+1}} \left(1 - \frac{4}{\beta_{i,j}}\right) = \sum_{0 \le r \le 4n^2} (-4)^r Q_r,$$

where $Q_0 = 1$ and Q_r are the rth elementary symmetric polynomials in $1/\beta_{i,j}$'s with subscripts (i,j) run over $\{(i,j) : 1 \le i \ne j \le 2n+1, i+j \ne 2n+1\}$. By using Newton's identities again, it suffices to study

$$T_r(n) := \sum_{\substack{1\le i\ne j\le 2n+1,\\ i+j\ne 2n+1}} \frac{1}{\beta_{i,j}^r}.$$

Define

$$V_r(n) := \sum_{1\le i\le 2n} \frac{1}{(\xi^i - \xi^{-i})^r}.$$

Note that when i runs over $1, 2, \ldots, 2n$, the corresponding ξ^i runs over all $(2n+1)$st roots of unity except $\xi^{2n+1} = 1$. So we can write above $V_r(n)$ as the following form

$$\sum_{\phi\ne 1} \frac{1}{(\phi - \phi^{-1})^r},$$

where ϕ runs over all $(2n+1)$st roots of unity except $\phi = 1$. For this $V_r(n)$, we actually have

(3.39)
$$V_r^2(n) = \sum_{\phi\ne 1} \frac{1}{(\phi-\phi^{-1})^r} \cdot \sum_{\varphi\ne 1} \frac{1}{(\varphi-\varphi^{-1})^r} = \sum_{\substack{\phi\ne 1\\ \varphi\ne 1}} \frac{1}{(\phi\varphi + (\phi\varphi)^{-1} - \frac{\phi}{\varphi} - \frac{\varphi}{\phi})^r}.$$

Note that $\phi\varphi = \phi/\varphi$ if and only if $\varphi = 1$ since φ can not equal -1, and $\phi\varphi = \varphi/\phi$ if and only if $\phi = 1$ for the same reason. It follows that $\phi\varphi \ne (\phi/\varphi)^{\pm 1}$ on the left hand side of (3.39). We now write $\phi\varphi = \xi^i$ and $\phi/\varphi = \xi^j$, and then we have $i \ne j$ and $i + j \ne 2n+1$. Therefore, we get that

$$V_r^2(n) = T_r(n).$$

Hence it remains to show that $V_r(n)$ is a quasi-polynomial and satisfies the functional equation $V_r(n) = V_r(-1-n)$.

We apply the following significant identity [1, page 7]

$$\frac{d}{dx}\log\prod_{k=1}^{m}(x - x_k) = -\sum_{k=1}^{m}\left(\frac{1}{x_k} + \frac{1}{x_k^2}x + \frac{1}{x_k^3}x^2 + \cdots\right) \tag{3.40}$$

by taking $x_k = \xi^k - \xi^{-k}$ for $k = 1, 2, \ldots, 2n$. Then the coefficient of x^{r-1} on the right hand side of (3.40) is exactly $-V_r(n)$ with $r = 1, 2, \ldots$. For the left side, we have

$$\prod_{k=1}^{2n}\left(x - 2i\sin\frac{2k\pi}{2n+1}\right) = \frac{8\cos((2n+1)\arcsin(\frac{x}{2i}) - 1)}{(2n+1)x^2}, \tag{3.41}$$

where i is the imaginary unit. We obtain equality (3.41) by comparing roots of both sides and their limits when x tends to 0, where we also use the identity

$$\prod_{1\le i\le 2n}(\xi^i-\xi^{-i})=\prod_{1\le i\le 2n}\xi^{-i}\cdot\prod_{1\le i\le 2n}(\xi^{2i}-1)\xlongequal{(3.19)}2n+1.$$

It is easy to compute that

$$\frac{d}{dx}\log\prod_{k=1}^{2n}\left(x-2i\sin\frac{2k\pi}{2n+1}\right) \tag{3.42}$$

$$=\frac{(2n+1)i}{2(1+\frac{x^2}{4})^{1/2}}\frac{\sin((2n+1)\arcsin(\frac{x}{2i}))}{\cos((2n+1)\arcsin(\frac{x}{2i}))-1}-\frac{2}{x}, \tag{3.43}$$

Denote the right hand side of formula (3.43) by $D_n(x)$. On one hand, it is immediate that $D_n(x)=D_{-1-n}(x)$. This implies that $V_r(n)=V_r(-1-n)$ for $r=1,2,\dots$. On the other hand, by computing the Taylor expansion of $D_n(x)$ directly, we see that every coefficient $V_r(n)$ in (3.40) is a polynomial with respect to n over $\mathbb{Q}$. Hence the functions $V_r(n)$ are quasi-polynomials and thus uniformly continuous under the 2-adic metric. Here we remark that the imaginary unit on the right hand side of (3.43) is eliminated by the Taylor expansion of $\sin((2n+1)\arcsin(x/2i))$. This finishes the proof. □

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References

1. H. Cohn, *2-adic behavior of numbers of domino tilings*, Electron. J. Combin. **6** (1999), Research Paper 14, 7.
2. H. Cohn, R. Kenyon, and J. Propp, *A variational principle for domino tilings*, J. Amer. Math. Soc. **14** (2001), no. 2, 297–346.
3. F. Q. Gouvêa, *p-adic numbers: An introduction*, second ed., Universitext, Springer-Verlag, Berlin, 1997.
4. N. Jacobson, *Basic algebra. I*, second ed., W. H. Freeman and Company, New York, 1985.
5. P. W. Kasteleyn, *The statistics of dimers on a lattice : I. the number of dimer arrangements on a quadratic lattice*, Physica **27** (1961), no. 12, 1209–1225.
6. J. Neukirch, *Algebraic number theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 322, Springer-Verlag, Berlin, 1999, Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder.

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