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SUN TOUGHNESS AND $P_{\geq 3}$-FACTORS IN GRAPHS

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ABSTRACT. A $P_{\geq n}$-factor means a path factor with each component having at least n vertices, where $n \geq 2$ is an integer. A graph G is called a $P_{\geq n}$-factor deleted graph if $G - e$ admits a $P_{\geq n}$-factor for any $e \in E(G)$. A graph G is called a $P_{\geq n}$-factor covered graph if G admits a $P_{\geq n}$-factor containing e for each $e \in E(G)$. In this paper, we first introduce a new parameter $s(G)$, called sun toughness, which is defined as follows:

$$s(G) = \min \left\{ \frac{|X|}{\mathrm{sun}(G-X)} : X \subseteq V(G),\ \mathrm{sun}(G-X) \geq 2 \right\},$$

if G is not a complete graph, and $s(G) = +\infty$, if G is a complete graph, where $\mathrm{sun}(G - X)$ denotes the number of sun components of $G - X$. Then we obtain two sun toughness conditions for a graph to be a $P_{\geq n}$-factor deleted graph or a $P_{\geq n}$-factor covered graph. Furthermore, it is shown that our results are sharp.

1. INTRODUCTION

Many real-world networks can conveniently be modelled by graphs or networks. Examples include a railroad network with nodes representing railroad stations, and links corresponding to railways between two stations, or the World Wide Web with nodes representing web pages, and links corresponding to hyperlinks between web pages, or a communication network with nodes and links modelling cities and communication channels, respectively. In our daily life, many problems on network design and optimization, e.g., building blocks, coding design, scheduling problems, file transfer problems on computer networks, are related to the factors and factorizations of graphs [1]. For example, the file transfer problem in computer networks can be converted into factorizations of graphs, and the problem of telephone network design can be converted into P_2-factors of graphs. Many other applications in this field can be found in a current survey [1]. It is well-known

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that a graph can represent a network. Vertices and edges of a graph models nodes and links between the nodes in the network. Henceforth, we use the term *graph* instead of *network*.

We consider finite undirected graphs without loops or multiple edges. Readers are referred to [4] for undefined terminology and notation in this paper. Let G be a graph, and $V(G)$ and $E(G)$ denote its vertex set and edge set, respectively. For a vertex x in a graph G, $d_G(x)$ denotes the degree of x in G. For any subset S of $V(G)$, we denote by $G-S$, the resulting graph after deleting the vertices of S from G. For any $E' \subseteq E(G)$, we use $G-E'$ to denote the subgraph obtained from G by deleting E'. We write $G-x=G-\{x\}$ for $S=\{x\}$ and $G-e=G-\{e\}$ for $E'=\{e\}$. The number of connected components of a graph G is denoted by $\omega(G)$. Let $n \geq 2$ be an integer. The path with n vertices and $n-1$ edges is denoted by P_n. A path factor of a graph G is defined as a spanning subgraph whose components are paths. A $P_{\geq n}$-factor means a path factor with each component having at least n vertices. A graph G is called a $P_{\geq n}$-factor deleted graph if $G-e$ admits a $P_{\geq n}$-factor for any $e \in E(G)$. A graph G is called a $P_{\geq n}$-factor covered graph if G admits a $P_{\geq n}$-factor containing e for each $e \in E(G)$.

A 1-factor of a graph G is defined as a spanning subgraph F of G with $d_F(x)=1$ for any $x \in V(G)$. A graph R is called a factor-critical graph if $R-x$ admits a 1-factor for each $x \in V(G)$. Let R be a factor-critical graph with $V(R)=\{x_1,x_2,\cdots,x_p\}$. If vertices $y_1,y_2,\cdots,y_p$ together with new edges $x_iy_i(1 \leq i \leq p)$ are added to R, then a new graph H is obtained. The resulting graph H is said to be a sun. Note that K_1 and K_2 are also called suns. It is easy to see that $|V(H)| \geq 6$, and we call H a big sun. A component of G is called a sun component of G if it is isomorphic to a sun. We use $\mathrm{sun}(G)$ to denote the number of sun components of G. Kaneko [9] showed a criterion for a graph having a $P_{\geq 3}$-factor. Kano, Katona, and Király [10] presented a simpler proof.

Theorem 1 (Kaneko [9])**.** *A graph G admits a $P_{\geq 3}$-factor if and only if $\mathrm{sun}(G-S) \leq 2|S|$ for every $S \subseteq V(G)$.*

Zhang and Zhou [16] obtained a necessary and sufficient condition for a graph to be a $P_{\geq 3}$-factor covered graph, which is an extension of Theorem 1.

Theorem 2 (Zhang and Zhou [16])**.** *A connected graph G is a $P_{\geq 3}$-factor covered graph if and only if for every $S \subseteq V(G)$,*

$$\mathrm{sun}(G-S) \leq 2|S| - \varepsilon(S),$$

where $\varepsilon(S)$ is defined as follows:

$$\varepsilon(S)=\begin{cases} 2, & \textit{if } S \textit{ is not an independent set,} \\ 1, & \textit{if } S \neq \emptyset \textit{ and there exists a non-sun component of } G-S, \\ 0, & \textit{otherwise.} \end{cases}$$

Chvátal [5] defined the toughness $t(G)$ of a graph G as

$$t(G) = \min\left\{\frac{|X|}{\omega(G-X)} : X \subseteq V(G),\ \omega(G-X) \geq 2\right\},$$

if G is not a complete graph, and $t(G) = +\infty$, if G is a complete graph. Many authors [6, 7, 12, 14] studied the existence of graph factors depending on toughness. Bazgan, Benhamdine, Li, and Wozniak [3] used toughness to obtain a sufficient condition for a graph to have a $P_{\geq 3}$-factor.

Theorem 3 (Bazgan, Benhamdine, Li, and Wozniak [3])**.** *Let G be a graph with at least three vertices. Then G admits a $P_{\geq 3}$-factor if $t(G) \geq 1$.*

Some other results on the existence of path factors in graphs can be found in [2, 8, 11, 13, 15]. In this paper, we first introduce a new parameter $s(G)$, called sun toughness, which is defined as follows:

$$s(G) = \min\left\{\frac{|X|}{\text{sun}(G-X)} : X \subseteq V(G),\ \text{sun}(G-X) \geq 2\right\},$$

if G is not a complete graph, and $s(G) = +\infty$, if G is a complete graph. We investigate the relationship between sun toughness and the existence of $P_{\geq 3}$-factors in graphs, and obtain some new results on $P_{\geq 3}$-factors in graphs which are shown in Sections 2 and 3.

2. $P_{\geq 3}$-FACTOR DELETED GRAPHS

Theorem 4. *A 2-edge-connected graph G is a $P_{\geq 3}$-factor deleted graph if $s(G) \geq 1$.*

Proof. If G is a complete graph, then it is easy to see that G is a $P_{\geq 3}$-factor deleted graph. Hence, we assume that G is not a complete graph.

Let $H = G - e$ for every $e \in E(G)$. To verify Theorem 4, we only need to testify that H contains a $P_{\geq 3}$-factor. On the contrary, we assume that H has no $P_{\geq 3}$-factor. Then by Theorem 1, there exists some subset $S \subseteq V(H) = V(G)$ such that

$$\text{sun}(H-S) > 2|S|. \tag{1}$$

If $S = \emptyset$, then from (1) we obtain

$$\text{sun}(H) > 0. \tag{2}$$

Note that $H = G - e$ and G is a 2-edge-connected graph. Hence, we have

$$\text{sun}(H) \leq \omega(H) = 1.$$

Combining this with (2) and the integrity of sun(H), we obtain

$$\text{sun}(H) = \omega(H) = 1.$$

Since G is a 2-edge-connected graph, we have $|V(G)| = |V(H)| \geq 3$. And so, $H = G - e$ is a big sun by the definitions of a sun and a big sun. Thus, there exists a factor-critical graph R in $H = G - e$ with $d_H(x) = 1$ for each $x \in V(H) \setminus V(R)$ and $|V(R)| = |V(H)|/2 \geq 3$. Obviously, $G = H + e$

has at least one vertex with degree 1, which contradicts that G is a 2-edge-connected graph.

If $|S| \geq 2$, then by (1) we have

$$\text{sun}(H - S) \geq 2|S| + 1 \geq 5. \tag{3}$$

Note that $\text{sun}(H - S) = \text{sun}(G - e - S) \leq \text{sun}(G - S) + 2$, that is,

$$\text{sun}(G - S) \geq \text{sun}(H - S) - 2 \geq 3. \tag{4}$$

It follows from (1), (3), (4), $s(G) \geq 1$ and the definition of $s(G)$ that

$$1 \leq s(G) \leq \frac{|S|}{\text{sun}(G - S)} \leq \frac{|S|}{\text{sun}(H - S) - 2} \leq \frac{|S|}{2|S| + 1 - 2} = \frac{|S|}{2|S| - 1},$$

which implies

$$|S| \leq 1,$$

and contradicts $|S| \geq 2$.

In the following, we always assume that $|S| = 1$. In terms of (1), we obtain

$$\text{sun}(H - S) \geq 2|S| + 1 = 3. \tag{5}$$

Note that $\text{sun}(H - S) = \text{sun}(G - e - S) \leq \text{sun}(G - S) + 2$. We shall consider two cases.

Case 1: $\text{sun}(G - S) \geq \text{sun}(H - S) - 1$.

According to (5), $\text{sun}(G - S) \geq 2$. In view of (5), $s(G) \geq 1$ and the definition of $s(G)$, we have

$$1 \leq s(G) \leq \frac{|S|}{\text{sun}(G - S)} \leq \frac{|S|}{\text{sun}(H - S) - 1} \leq \frac{|S|}{2|S| + 1 - 1} = \frac{1}{2},$$

which is a contradiction.

Case 2: $\text{sun}(G - S) = \text{sun}(H - S) - 2$.

If $\text{sun}(G - S) \geq 2$, then by $|S| = 1$ and the definition of $s(G)$, we obtain

$$s(G) \leq \frac{|S|}{\text{sun}(G - S)} \leq \frac{1}{2},$$

which contradicts that $s(G) \geq 1$. Hence, we assume that $\text{sun}(G - S) = 1$, and so, $\text{sun}(H - S) = 3$. We use C_1 to denote the sun component of $G - S$. According to $H = G - e$ and $\text{sun}(H - S) = 3$, C_1 is also a sun component of $H - S$, and we use C_2, C_3 to denote the other two sun components of $H - S$.

Subcase 1: $C_2 = K_1$ and $C_3 = K_1$.

In this case, we have $C_2 \cup C_3 + e = K_2$, which is a sun component of $G - S$. Hence, $\text{sun}(G - S) = 2$, which contradicts $\text{sun}(G - S) = 1$.

Subcase 2: $C_2 \neq K_1$ or $C_3 \neq K_1$.

Without loss of generality, suppose that $C_3 \neq K_1$. Clearly, $C_3 = K_2$ or C_3 is a big sun. We write $e = uv$ and $u \in V(C_3)$.

If $C_3 = K_2$, then we have $\text{sun}(G-S\cup\{u\}) = \text{sun}(H-S\cup\{u\}) = 3$. Combining this with $|S| = 1$ and the definition of $s(G)$, we obtain

$$s(G) \leq \frac{|S \cup \{u\}|}{\text{sun}(G - S \cup \{u\})} = \frac{2}{3},$$

which contradicts that $s(G) \geq 1$. In the following, we consider that C_3 is a big sun. Thus, there exists a factor-critical graph R' in C_3 with $d_{C_3}(x) = 1$ for each $x \in V(C_3) \setminus V(R')$ and $|V(R')| = |V(C_3)|/2 \geq 3$. If $u \in V(R')$, then we obtain

$$\text{sun}(G - S \cup V(R')) = \text{sun}(H - S \cup V(R')) = |V(R')| + 2.$$

Combining this with $|S| = 1$ and the definition of $s(G)$,

$$s(G) \leq \frac{|S \cup V(R')|}{\text{sun}(G - S \cup V(R'))} = \frac{|V(R')| + 1}{|V(R')| + 2} < 1,$$

which contradicts $s(G) \geq 1$. If $u \in V(C_3) \setminus V(R')$, then there exists $u' \in V(R')$ with $uu' \in E(C_3)$. Thus, we have

$$\begin{aligned}\text{sun}(G - S \cup (V(R') \setminus \{u'\}) \cup \{u\})& \\ &= \text{sun}(H - S \cup (V(R') \setminus \{u'\}) \cup \{u\}) \\ &= |V(R')| + 2.\end{aligned}$$

Combining this with $|S| = 1$ and the definition of $s(G)$,

$$s(G) \leq \frac{|S \cup (V(R') \setminus \{u'\}) \cup \{u\}|}{\text{sun}(G - S \cup (V(R') \setminus \{u'\}) \cup \{u\})} = \frac{|V(R')| + 1}{|V(R')| + 2} < 1,$$

which contradicts $s(G) \geq 1$. This completes the proof of Theorem 4. □

Remark 1. The condition $s(G) \geq 1$ in Theorem 4 is sharp, which is shown in the following.

Let H_1 be a big sun and R be the factor-critical graph of H_1 with $d_{H_1}(x) = 1$ for any $x \in V(H_1) \setminus V(R)$ and $|V(R)| = |V(H_1)|/2 \geq 3$. Let $v \in V(H_1) \setminus V(R)$ and $u \notin V(H_1)$. We construct a graph H' with $V(H') = V(H_1) \cup \{u\}$ and $E(H') = E(H_1) \cup \{e\}$, where $e = uv$. Furthermore, we construct a graph $G = K_1 \vee (H' \cup K_2)$. We choose $X = V(K_1) \cup (V(R) \setminus \{w\}) \cup \{v\}$, where $w \in V(R)$ and $vw \in E(R)$. Then we have $\text{sun}(G - X) = \text{sun}(G - e - X) = |V(R)| + 2$, $|X| = |V(R)| + 1$, and it is easy to see that $s(G) = |X|/\text{sun}(G - X) = (|V(R)| + 1)/(|V(R)| + 2) < 1$. But $s(G) \to 1$ when $|V(R)| \to +\infty$.

In this graph G, we choose $S = V(K_1)$. Then we have

$$\text{sun}(G - e - S) = 3 > 2 = 2|S|.$$

In terms of Theorem 1, $G-e$ has no $P_{\geq 3}$-factor, that is, G is not a $P_{\geq 3}$-factor deleted graph.

3. $P_{\geq 3}$-factor covered graphs

Theorem 5. *A connected graph G with at least three vertices is a $P_{\geq 3}$-factor covered graph if $s(G) \geq 1$.*

Proof. Note that $|V(G)| \geq 3$. Hence, Theorem 5 holds for a complete graph. In what follows, we always assume that G is not a complete graph.

Assume that G satisfies the hypothesis of Theorem 5, but it is not a $P_{\geq 3}$-factor covered graph. Then from Theorem 2, there exists a vertex subset S of G satisfying

$$\text{sun}(G-S) > 2|S| - \varepsilon(S). \tag{6}$$

Claim 1. $|S| \geq 2$.

Proof. If $S = \emptyset$, then it follows from (6) and the definition of $\varepsilon(S)$ that $\text{sun}(G) > 0$. Combining this with G being a connected graph, $\text{sun}(G) \leq \omega(G)$ and the integrity of $\text{sun}(G)$, we obtain

$$1 \leq \text{sun}(G) \leq \omega(G) = 1,$$

that is,

$$\text{sun}(G) = \omega(G) = 1.$$

Note that $|V(G)| \geq 3$. Hence, G is a big sun. Then there exists a factor-critical graph R in G with $d_G(x) = 1$ for each $x \in V(G) \setminus V(R)$ and $|V(R)| = |V(G)|/2 \geq 3$. We write $X = V(R) \setminus \{u\}$ for any $u \in V(R)$. Obviously, $\text{sun}(G-X) = |V(R)| \geq 3$. In terms of the definition of $s(G)$, we obtain

$$s(G) \leq \frac{|X|}{\text{sun}(G-X)} = \frac{|V(R)|-1}{|V(R)|} < 1,$$

which contradicts $s(G) \geq 1$.

If $|S| = 1$, then $\varepsilon(S) \leq 1$. It follows from (6) that

$$\text{sun}(G-S) \geq 2|S| - \varepsilon(S) + 1 \geq 2|S| \geq 2. \tag{7}$$

According to (7) and the definition of $s(G)$, we obtain

$$s(G) \leq \frac{|S|}{\text{sun}(G-S)} \leq \frac{1}{2},$$

which contradicts $s(G) \geq 1$. Therefore, we have $|S| \geq 2$. Claim 1 is proved. □

It follows from the definition of $\varepsilon(S)$ that $\varepsilon(S) \leq 2$. In terms of (6) and Claim 1, we have

$$\text{sun}(G-S) \geq 2|S| - \varepsilon(S) + 1 \geq 2|S| - 1 \geq |S| + 1 \geq 3. \tag{8}$$

On the other hand, in view of $s(G) \geq 1$ and the definition of $s(G)$, we obtain

$$\frac{|S|}{\text{sun}(G-S)} \geq s(G) \geq 1,$$

which implies

$$|S| \geq \text{sun}(G - S),$$

which contradicts (8). Theorem 5 is proved. □

Remark 2. We now show the condition $s(G) \geq 1$ in Theorem 5 is the best possible. Let G be a big sun. Then there exists a factor-critical graph R in G with $d_G(x) = 1$ for each $x \in V(G)\setminus V(R)$ and $|V(R)| = |V(G)|/2 \geq 3$. We write $X = V(R)\setminus\{u\}$ for any $u \in V(R)$. Obviously, $\text{sun}(G-X) = |V(R)| \geq 3$. It is easy to see that $s(G) = |X|/\text{sun}(G-X) = (|V(R)|-1)/|V(R)| < 1$. But $s(G) \to 1$ when $|V(R)| \to +\infty$. Let $S = \emptyset$, and so $\varepsilon(S) = 0$. Then we obtain

$$\text{sun}(G - S) = 1 > 0 = 2|S| - \varepsilon(S).$$

By Theorem 2, G is not a $P_{\geq 3}$-factor covered graph.

Finally we pose the following problem.

Problem 1. *Find the relationship between sun toughness and other types of graph factors.*

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References

1. B. Alspach, K. Heinrich, and G. Liu, *Contemporary design theory: A collection of surveys*, ch. Orthogonal factorizations of graphs, pp. 13–37, Wiley, New York, 1992.
2. K. Ando, Y. Egawa, A. Kaneko, K. Kawarabayashi, and H. Matsuda, *Path factors in claw-free graphs*, Discrete Math. **243** (2002), 195–200.
3. C. Bazgan, A. H. Benhamdine, H. Li, and M. Wozniak, *Partitioning vertices of 1-tough graph into paths*, Theoret. Comput. Sci. **263** (2001), 255–261.
4. J. A. Bondy and U. S. R. Murty, *Graph theory*, Springer, Berlin, 2008.
5. V. Chvátal, *Tough graphs and Hamiltonian circuits*, Discrete Math. **5** (1973), no. 3, 215–228.
6. W. Gao, L. Liang, T. Xu, and J. Zhou, *Tight toughness condition for fractional (g, f, n)-critical graphs*, J. Korean Math. Soc. **51** (2014), 55–65.
7. W. Gao and W. Wang, *Toughness and fractional critical deleted graph*, Util. Math. **98** (2015), 295–310.
8. M. Johnson, D. Paulusma, and C. Wood, *Path factors and parallel knock-out schemes of almost claw-free graphs*, Discrete Math. **310** (2010), 1413–1423.
9. A. Kaneko, *A necessary and sufficient condition for the existence of a path factor every component of which is a path of length at least two*, J. Combin. Theory Ser. B **88** (2003), no. 2, 195–218.
10. M. Kano, G. Y. Katona, and Z. Király, *Packing paths of length at least two*, Discrete Math. **283** (2004), no. 1–3, 129–135.
11. K. Kawarabayashi, H. Matsuda, Y. Oda, and K. Ota, *Path factors in cubic graphs*, J. Graph Theory **39** (2002), no. 3, 188–193.
12. G. Liu and L. Zhang, *Toughness and the existence of fractional k-factors of graphs*, Discrete Math. **308** (2008), no. 9, 1741–1748.
13. M. D. Plummer, *Graph factors and factorization: 1985-2003: A survey*, Discrete Math. **307** (2007), no. 7–8, 791–821.

14. H. Ye S. Zhou, Z. Sun, *A toughness condition for fractional* (k, m)*-deleted graphs*, Inform. Process. Lett. **113** (2013), no. 8, 255–259.
15. H. Wang, *Path factors of bipartite graphs*, J. Graph Theory **18** (1994), no. 2, 161–167.
16. H. Zhang and S. Zhou, *Characterizations for* $p_{\geq 2}$*-factor and* $p_{\geq 3}$*-factor covered graphs*, Discrete Math. **309** (2009), 2067–2076.

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