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ON THE ILLUMINATION OF A CLASS OF CONVEX BODIES

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Abstract. We study Boltyanski's illumination problem (or Hadwiger's covering problem) for the class of convex bodies in $\mathbb{R}^n$ consisting of convex hulls of a pair of compact convex sets contained in two parallel hyperplanes of $\mathbb{R}^n$. This special case of the problem is completely solved when $n = 3$.

1. Introduction

We denote by $\mathcal{K}^n$ the family of nonempty compact convex subsets of $\mathbb{R}^n$, and by $\mathcal{K}^n_m$ ($m \leq n$) the set of compact convex subsets of $\mathbb{R}^n$ whose affine dimension is m. Each member of $\mathcal{K}^n_n$ is called a *convex body*, i.e., a convex body in $\mathbb{R}^n$ is a compact convex set having interior points.

For each $K \in \mathcal{K}^n$, we denote by $\operatorname{int} K$, $\operatorname{relint} K$, $\operatorname{bd} K$, and $\operatorname{relbd} K$ the *interior*, *relative interior*, *boundary*, and *relative boundary* of K, respectively. A unit vector in $\mathbb{R}^n$ is called a *direction*. Suppose that $K \in \mathcal{K}^n$, $x \in \operatorname{relbd} K$, and u is a direction. If there exists a positive number λ such that $x + \lambda u \in \operatorname{relint} K$, then we say that *$x$ is illuminated by u*. Let A be a subset of $\operatorname{relbd} K$ and D be a set of directions. If each point in A is illuminated by a direction in D, then we say that *A is illuminated by D*. When $\operatorname{relbd} K \neq \emptyset$, the *illumination number* $c(K)$ of K is defined by

$$c(K) = \min \{\operatorname{card} D : \ D \text{ is a set of directions illuminating } \operatorname{relbd} K\},$$

where $\operatorname{card} D$ is the *cardinality* of D. It is not difficult to see that $c(K)$ is affinely invariant. Concerning the least upper bound of $c(K)$ for all $K \in \mathcal{K}^n_n$, there is a long-standing conjecture (see the monographs [11], [12], and [4], and the surveys [9], [14], [3], [5] for the history, known results, and relevant references of this conjecture):

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Conjecture 1.1 (Boltyanski's illumination conjecture)**.** *For each $K \in \mathcal{K}_n^n$, we have*

$$c(K) \leq 2^n;$$

equality holds if and only if $K \in \mathcal{K}_n^n$ is a parallelotope.

Since, when $K \in \mathcal{K}_n^n$, $c(K)$ equals the least number of translates of $\operatorname{int} K$ needed to cover K, Conjecture 1.1 is also called *Hadwiger's covering conjecture*. A "dual" version of Conjecture 1.1 is called *Bezdek's separation conjecture*, cf. [4, p. 24].

Conjecture 1.2 (Bezdek's separation conjecture)**.** *Let $K \in \mathcal{K}_n^n$ ($n \geq 3$) and p be an arbitrary interior point of K. Then there exists a collection $\mathcal{H}$ of 2^n hyperplanes such that each exposed face of K and p can be strictly separated by at least one hyperplane in $\mathcal{H}$. Furthermore, 2^n hyperplanes are necessary only if K is the convex hull of n line segments having linearly independent directions which intersect at the common relative interior point p.*

See, e.g., [2], [6], [8], and [7] for progress toward the solution of Conjecture 1.2.

Conjecture 1.1 is completely solved only when $n = 2$. More precisely, it is known that $c(K) = 4$ when $K \in \mathcal{K}_2^2$ is a parallelogram and $c(K) = 3$ holds for the rest. Even when $n = 3$, Conjecture 1.1 is widely open. Thus it is natural to consider how to make efficient use of the knowledge of $c(K)$ when $K \in \mathcal{K}^{n-1}$ to solve Conjecture 1.1 in $\mathbb{R}^n$. It is more natural to study the value of $c(K)$ when K is constructed from lower dimensional convex bodies. For example, it is shown in [10] that

$$c(M_1 \oplus M_2 \oplus \cdots \oplus M_k) \leq \prod_{i=1}^{k} c(M_i),$$

where M_i, $i = 1, \ldots, k$, are compact convex sets and $M_1 \oplus M_2 \oplus \cdots \oplus M_k$ is their direct sum. Moreover, when $M_1, M_2 \in \mathcal{K}_2^2$, the set of all possible values of $c(M_1 \oplus M_2)$ is $\{7, 8, 9, 12, 16\}$, cf. [1]. Also, solving Conjecture 1.1 for centrally symmetric convex bodies in $\mathbb{R}^{n+1}$ (the case when $n = 2$ is solved by Lassak in [13]) would lead to the solution of Conjecture 1.1 for all convex bodies in $\mathbb{R}^n$ if the answer to the following problem (possibly due to P. Soltan) is positive:

Problem 1.3. *Suppose that $T = -B \in \mathcal{K}_n^n$ and*

$$K = \operatorname{conv}\left((T \times \{1\}) \cup (B \times \{0\})\right).$$

Is it true that $c(K) = c(T) + c(B) = 2c(T)$?

In this paper we estimate the illumination number for the class $\mathcal{A}$ of convex bodies in $\mathcal{K}_n^n$ constructed from lower dimensional convex bodies, where each member of $\mathcal{A}$ is the convex hull of two compact convex sets contained in two parallel hyperplanes of $\mathbb{R}^n$. In Section 2 we introduce several fundamental and useful lemmas. The illumination numbers of convex bodies in $\mathcal{A}$ will be studied in Section 3.

2. Several lemmas

For each positive integer m, we put

$$[m] = \left\{ k \in \mathbb{Z}^+ :\ 1 \le k \le m \right\}.$$

Let a and b be two distinct points in $\mathbb{R}^n$. We denote by $[a,b]$ the *segment connecting a and b*, and by $\langle a,b\rangle$ the *line passing through a and b*. Suppose that $K \in \mathcal{K}_n^n$ and $[a,b] \subseteq K$. If there exists a pair of parallel supporting hyperplanes H_1 and H_2 of K such that $a \in H_1$ and $b \in H_2$, then we say that $[a,b]$ is an *affine diameter* of K. In this situation, each segment contained in K and parallel to $\langle a,b\rangle$ is not longer than $[a,b]$. See the excellent survey [15] for more information about affine diameters of convex bodies.

Lemma 2.1. *Let $K \in \mathcal{K}_n^n$ and $[a,b]$ be an affine diameter of K. If $u \in \mathbb{R}^n$ is a direction illuminating a, then u does not illuminate b.*

Proof. Suppose the contrary that u also illuminates b. Then there exists a positive number λ such that

$$a + \lambda u,\ b + \lambda u \in \operatorname{int} K.$$

It follows that there exists a segment contained in K parallel to $\langle a,b\rangle$ that is strictly longer than $[a,b]$, a contradiction to the fact that $[a,b]$ is an affine diameter of K. □

Lemma 2.2. *Suppose that $K \in \mathcal{K}^n$, $c \in \operatorname{relint} K$, and $0 < \lambda_1 \le \lambda_2 < 1$. Then*

$$(1-\lambda_1)c + \lambda_1 K \subseteq (1-\lambda_2)c + \lambda_2 K.$$

Proof. Let z be an arbitrary point in $(1-\lambda_1)c + \lambda_1 K$. There exists a point $x \in K$ such that $z = (1-\lambda_1)c + \lambda_1 x$. We have

$$\begin{aligned} z - (1-\lambda_2)c &= z - (1-\lambda_1)c + (\lambda_2 - \lambda_1)c \\ &= \lambda_1 x + (\lambda_2 - \lambda_1)c \\ &= \lambda_2\left(\frac{\lambda_1}{\lambda_2}x + \frac{\lambda_2 - \lambda_1}{\lambda_2}c\right) \in \lambda_2 K, \end{aligned}$$

from which it follows that $z \in (1-\lambda_2)c + \lambda_2 K$. □

Lemma 2.3. *Suppose that $K \in \mathcal{K}^n$, $p \in \operatorname{relint} K$, C is a nonempty compact subset of $\operatorname{relbd} K$, and $D = \{u_i :\ i \in [m]\}$ is a set of directions illuminating C. Then there exist two numbers $\lambda > 0$ and $\gamma \in (0,1)$ such that, for each point $x \in C$, there exists $i \in [m]$ satisfying*

$$x + \lambda u_i \in \gamma(\operatorname{relint} K) + (1-\gamma)p.$$

Proof. For each pair of $k, j \in \mathbb{Z}^+$, and each $i \in [m]$, put

$$C_{i,j,k} = \left\{ x \in C :\ x + \frac{1}{j}u_i \in \left(1 - \frac{1}{k}\right)K + \frac{1}{k}p \subset \operatorname{relint} K \right\}.$$

It suffices to show that there exists $j_0 \in \mathbb{Z}^+$ satisfying

$$C \subseteq \bigcup_{i \in [m]} C_{i,j_0,j_0^2}.$$

Otherwise, for each $j \in \mathbb{Z}^+$, there exists a point

$$x_j \in C \setminus \left(\bigcup_{i \in [m]} C_{i,j,j^2} \right). \tag{2.1}$$

By choosing a subsequence if necessary, we may assume that $\{x_j\}_{j \in \mathbb{Z}^+}$ converges to a point $x_0 \in C$. Then there exist an $i_0 \in [m]$ and a number $\lambda > 0$ such that

$$x_0 + \lambda u_{i_0} \in \operatorname{relint} K.$$

Hence, there exists a positive number δ such that (cf. Theorem 2.23 on p. 85 in [16] for the last equality)

$$B(x_0 + \lambda u_{i_0}, \delta) \cap \operatorname{aff} K \subset \operatorname{relint} K = \bigcup_{k \in \mathbb{Z}^+} \left(\left(1 - \frac{1}{k}\right) K + \frac{1}{k} p \right),$$

where $B(x_0 + \lambda u_{i_0}, \delta)$ is *the closed ball centered at* $x_0 + \lambda u_{i_0}$ *having radius* δ. Then there exists a number $\eta \in (0, 1)$ such that

$$B(x_0 + \lambda u_{i_0}, \delta) \cap \operatorname{aff} K \subset (1 - \eta) K + \eta p.$$

Let j_0 be an integer in $\mathbb{Z}^+$ such that

$$\|x_{j_0} - x_0\| < \delta \text{ and that } j_0 > \max \left\{ \frac{\lambda}{\eta}, \frac{1}{\lambda} \right\}.$$

Then

$$\begin{aligned}
x_{j_0} + \frac{1}{j_0} u_{i_0} &= \left(1 - \frac{1}{\lambda j_0}\right) x_{j_0} + \frac{1}{\lambda j_0} (x_{j_0} + \lambda u_{i_0}) \\
&\in \left(1 - \frac{1}{\lambda j_0}\right) K + \frac{1}{\lambda j_0} (B(x_0 + \lambda u_{i_0}, \delta) \cap \operatorname{aff} K) \\
&\subseteq \left(1 - \frac{1}{\lambda j_0}\right) K + \frac{1}{\lambda j_0} ((1 - \eta) K + \eta p) \\
&= \left(1 - \frac{\eta}{\lambda j_0}\right) K + \frac{\eta}{\lambda j_0} p \\
&\subseteq \left(1 - \frac{1}{j_0^2}\right) K + \frac{1}{j_0^2} p,
\end{aligned}$$

where the last step follows from Lemma 2.2, a contradiction to (2.1). □

Corollary 2.4. *Let* $K \in \mathcal{K}_n^n$ *and* $m = c(K)$. *Then there exist a set* $D = \{u_i : i \in [m]\}$ *of* m *directions in* $\mathbb{R}^n$ *and a number* $\lambda > 0$ *such that, for each point* $x \in \operatorname{bd} K$, *there exists* $i \in [m]$ *satisfying* $x + \lambda u_i \in \operatorname{int} K$.

We will use several fundamental results related to relative interior of convex sets. First we have:

Lemma 2.5 (cf. Theorem 2.34 on p. 92 in [16]). *If $T:\ \mathbb{R}^n \mapsto \mathbb{R}^m$ is an affine transformation and $K \in \mathcal{K}^n$, then*

$$\operatorname{relint}(T(K)) = T(\operatorname{relint} K). \tag{2.2}$$

From Lemma 2.5 it follows that

$$\operatorname{relint}(K+x) = \operatorname{relint} K + x,\ \forall K \in \mathcal{K}^n,\ x \in \mathbb{R}^n. \tag{2.3}$$

By Theorem 2.63 in [16, p. 109], we have

$$\operatorname{relbd}(K+x) = \operatorname{relbd} K + x,\ \forall K \in \mathcal{K}^n,\ x \in \mathbb{R}^n.$$

Finally, the following lemma will also be used in the next section.

Lemma 2.6 (cf. Theorem 2.29 on p. 89 in [16]). *If $K_1, \ldots, K_m$ are convex sets in $\mathbb{R}^n$ and $\mu_1, \ldots, \mu_m$ are in $\mathbb{R}$, then*

$$\operatorname{relint}\left(\sum_{i=1}^{m} \mu_i K_i\right) = \sum_{i=1}^{m} \mu_i \operatorname{relint} K_i.$$

3. Illuminations numbers of convex bodies in $\mathcal{A}$

Throughout this section n is an integer not less than 3, T and B are two sets in $\mathcal{K}^{n-1}$ such that

$$K = \operatorname{conv}((T \times \{1\}) \cup (B \times \{0\})) \in \mathcal{K}^n_n. \tag{3.1}$$

For each $\eta \in [0,1]$, put $H_\eta = \mathbb{R}^{n-1} \times \{\eta\}$ and $K_\eta = K \cap H_\eta$. It is clear that

- H_1 and H_0 are the two supporting hyperplanes of K parallel to $\mathbb{R}^{n-1} \times \{0\}$,
- $T \times \{1\}$, $B \times \{0\} \subset \operatorname{bd} K$,
- for each $\eta \in (0,1)$, H_η intersects $\operatorname{int} K$.

Lemma 3.1. *For each $\eta \in (0,1)$, we have*

(1) $K_\eta = (\eta T) \times \{\eta\} + ((1-\eta)B) \times \{0\}$,
(2) $\operatorname{relint} K_\eta = \operatorname{int} K \cap H_\eta$.

Proof. (1). First suppose that $x \in K \cap H_\eta$. Then there exist two positive integers m_1 and m_2, two sets of positive numbers $\{\alpha_i :\ i \in [m_1]\}$ and $\{\beta_j :\ j \in [m_2]\}$, and two sets of points

$$\{p_i :\ i \in [m_1]\} \subset T \text{ and } \{q_j :\ j \in [m_2]\} \subset B$$

such that

$$\sum_{i\in[m_1]} \alpha_i + \sum_{j\in[m_2]} \beta_j = 1 \text{ and } x = \sum_{i\in[m_1]} \alpha_i (p_i, 1) + \sum_{j\in[m_2]} \beta_j (q_j, 0).$$

It follows that

$$\sum_{i\in[m_1]} \alpha_i = \eta \text{ and } \sum_{j\in[m_2]} \beta_j = (1-\eta).$$

Therefore,

$$\begin{aligned} x &= \eta \sum_{i\in[m_1]} \frac{\alpha_i}{\eta}(p_i,1) + (1-\eta)\sum_{j\in[m_2]} \frac{\beta_j}{1-\eta}(q_j,0) \\ &\in \eta(T\times\{1\}) + (1-\eta)(B\times\{0\}) \\ &= (\eta T)\times\{\eta\} + ((1-\eta)B)\times\{0\}. \end{aligned}$$

Conversely, suppose that

$$x \in (\eta T)\times\{\eta\} + ((1-\eta)B)\times\{0\}.$$

Then there exist two points $p\in T$ and $q\in B$ such that

$$x = (\eta p,\eta) + ((1-\eta)q,0) = \eta(p,1) + (1-\eta)(q,0) \in K\cap H_\eta.$$

(2). Since $\eta\in(0,1)$, $\operatorname{int} K\cap H_\eta \neq \emptyset$. For each point $x\in \operatorname{int} K\cap H_\eta$, there exists a number $\delta>0$ such that $B(x,\delta)\subset K$. It follows that

$$B(x,\delta)\cap \operatorname{aff} K_\eta \subset B(x,\delta)\cap H_\eta \subset K\cap H_\eta = K_\eta.$$

Thus $x\in \operatorname{relint} K_\eta$.

Fix a point $x_0 \in \operatorname{int} K\cap K_\eta$. Then $x_0\in \operatorname{relint} K_\eta$. For each point $y\in \operatorname{relint} K_\eta$, there exists a number $\gamma>0$ such that

$$z = y + \gamma(y-x_0)\in K_\eta.$$

It follows that

$$y = \frac{1}{1+\gamma}z + \frac{\gamma}{1+\gamma}x_0 \in \operatorname{int} K\cap H_\eta.$$

This completes the proof. □

Lemma 3.2. *The boundary* $\operatorname{bd} K$ *of* K *can be illuminated by a set* D *of directions if and only if* D *illuminates both* $T\times\{1\}$ *and* $B\times\{0\}$.

Proof. Clearly, if D illuminates $\operatorname{bd} K$ then D has to illuminate both $T\times\{1\}$ and $B\times\{0\}$. Conversely, suppose that D illuminates $T\times\{1\}$ as well as $B\times\{0\}$. Denote by D_T and D_B the subsets of D that can illuminate $T\times\{1\}$ and $B\times\{0\}$, respectively. Let (z,η) be an arbitrary point in $\operatorname{bd} K\setminus(T\times\{1\}\cup B\times\{0\})$. Then $\eta\in(0,1)$ and, by Lemma 2.6,

$$(z,\eta)\in K_\eta\setminus(\eta \operatorname{relint} T\times\{1\} + (1-\eta)\operatorname{relint} B\times\{0\}).$$

We distinguish two cases.

Case 1: There exist $x\in \operatorname{relbd} T$ and $y\in B$ such that $z=\eta x+(1-\eta)y$.

There exist a direction $(u,\gamma)\in D_T$ and $\lambda>0$ such that $(x,1)+\lambda(u,\gamma)\in \operatorname{int} K$. Thus

$$\begin{aligned} (z,\eta)+\eta\lambda(u,\gamma) &= (\eta x+(1-\eta)y,\eta)+\eta\lambda(u,\gamma) \\ &= \eta((x,1)+\lambda(u,\gamma)) + (1-\eta)(y,0)\in \operatorname{int} K. \end{aligned}$$

Case 2: There exist $x\in T$ and $y\in \operatorname{relbd} B$ such that $z=\eta x+(1-\eta)y$.

As in Case 1, there exists a direction in D_B illuminating (z,η).

Thus $\operatorname{bd} K$ is illuminated by D. □

Lemma 3.3. *Suppose that T is not a singleton. Let $m = c(T)$, $\{v_i : i \in [m]\} \subset \mathbb{R}^{n-1}$ be a set of directions illuminating T, p be a point in* relint T, *and q be a point in* relint B. *Then there exist a positive number λ and a number $\gamma \in (0,1)$ such that $T \times \{1\}$ can be illuminated by*

$$D = \left\{ \frac{(\lambda v_i, 0) + u_0}{\|(\lambda v_i, 0) + u_0\|} : \ i \in [m] \right\},$$

where $u_0 = (1-\gamma)(-p+q,-1)$.

Proof. By Lemma 2.3, there exist a positive number λ and a number $\gamma \in (0,1)$ such that for each $x \in \operatorname{relbd} T$, there exists $i \in [m]$ satisfying

$$x + \lambda v_i \in \gamma(\operatorname{relint} T) + (1-\gamma)p.$$

We claim that λ and γ have the desired property. Let $(x,1)$ be an arbitrary point in $T \times \{1\}$. We distinguish two cases.

Case 1: $x \in \operatorname{relbd} T$.

Without loss of generality we may assume that

$$x + \lambda v_1 \in \gamma(\operatorname{relint} T) + (1-\gamma)p.$$

Then

$$\begin{aligned}
&(x,1) + \|(\lambda v_1, 0) + u_0\| \frac{(\lambda v_1, 0) + u_0}{\|(\lambda v_1, 0) + u_0\|} \\
&= (x,1) + (\lambda v_1, 0) + u_0 \\
&= (x + \lambda v_1, 1) + (1-\gamma)(-p+q,-1) \\
&= (x + \lambda v_1 + (1-\gamma)(-p+q), \gamma) \\
&\in (\gamma(\operatorname{relint} T) + (1-\gamma)(\operatorname{relint} B)) \times \{\gamma\} \\
&\subset \ \operatorname{int} K.
\end{aligned}$$

Case 2: $x \in \operatorname{relint} T$.

There exist two points $y, z \in \operatorname{relbd} T$ and a number $\eta \in (0,1)$ such that $x = \eta y + (1-\eta)z$. Without loss of generality we may assume that

$$y + \lambda v_1 \in \gamma(\operatorname{relint} T) + (1-\gamma)p.$$

As in Case 1, we have

$$(y,1) + \|(\lambda v_1, 0) + u_0\| \frac{(\lambda v_1, 0) + u_0}{\|(\lambda v_1, 0) + u_0\|} \in \operatorname{int} K.$$

Thus

$$\begin{aligned}
&(x,1) + \eta \|(\lambda v_1, 0) + u_0\| \frac{(\lambda v_1, 0) + u_0}{\|(\lambda v_1, 0) + u_0\|} \\
=&\eta\left((y,1) + \|(\lambda v_1, 0) + u_0\| \frac{(\lambda v_1, 0) + u_0}{\|(\lambda v_1, 0) + u_0\|} \right) + (1-\eta)(z,1) \\
\in& \operatorname{int} K.
\end{aligned}$$

It follows that $T \times \{1\}$ can be illuminated by D. $\square$

In a similar way we can prove the following lemma.

Lemma 3.4. *Suppose that B is not a singleton. Let $m = c(B)$, $\{v_i : i \in [m]\} \subset \mathbb{R}^{n-1}$ be a set of directions illuminating B, p be a point in* $\operatorname{relint} B$*, and q be a point in* $\operatorname{relint} T$*. Then there exist a positive number λ and a number $\gamma \in (0,1)$ such that $B \times \{0\}$ can be illuminated by*

$$D = \left\{ \frac{(\lambda v_i, 0) + u_0}{\|(\lambda v_i, 0) + u_0\|} : \ i \in [m] \right\},$$

where $u_0 = (1-\gamma)(-p+q, 1)$.

Theorem 3.5. *$c(K) \le c(T) + c(B)$.*

Proof. Suppose that one of T and B, say T, is a singleton. Then K is a convex cone having B as base. By Theorem 4 in [17] or by Theorem 3.11 below, we have

$$c(K) \le 1 + c(B) = c(T) + c(B).$$

If neither T nor B is a singleton, then the desired inequality follows from Lemma 3.2, Lemma 3.3, and Lemma 3.4. □

In general, the estimation given in Theorem 3.5 is best possible. See the following examples.

Example 3.6. *Let*

$$T = [(-1,0),(1,0)] \subset \mathbb{R}^2 \text{ and } B = [(0,-1),(0,1)] \subset \mathbb{R}^2.$$

Then $K \in \mathcal{K}_3^3$ is a tetrahedron and $c(K) = c(T) + c(B)$.

Example 3.7. *Let $T = B = [0,1] \times [0,1] \subset \mathbb{R}^2$. Then $K \in \mathcal{K}_3^3$ is a cube and $c(K) = c(T) + c(B)$.*

Lemma 3.8. *If B is contained in a translate of T, then the minimal cardinality m of a set of directions in $\mathbb{R}^n$ that can illuminate* $\operatorname{relbd} T \times \{1\}$ *equals $c(T)$.*

Proof. By Lemma 3.3, we only need to prove that $m \ge c(T)$.

Let $t \in \mathbb{R}^{n-1}$ be a point such that $B \subset T + t$, p be an arbitrary point in $\operatorname{int} K$, and $D = \{(u_i, \gamma_i) : \ i \in [m]\}$ be a set of directions in $\mathbb{R}^n$ illuminating $\operatorname{relbd} T \times \{1\}$. Then it is clear that each direction in D illuminates at least one point in $\operatorname{relbd} T \times \{1\}$. It is not difficult to see that $\gamma_i < 0, \ \forall i \in [m]$. By Lemma 2.3, there exist two positive numbers λ and $\gamma \in (0,1)$ such that, for each point $(x,1) \in \operatorname{relbd} T \times \{1\}$, there exists $i \in [m]$ satisfying

$$(x,1) + \lambda(u_i, \gamma_i) \in \gamma \operatorname{int} K + (1-\gamma)p.$$

Let

$$\eta = \max\{(y|e_n) : \ y \in \gamma K + (1-\gamma)p\},$$

where $(y|e_n)$ is the inner product of y and the nth member of the standard basis of $\mathbb{R}^n$. Then $\eta \in (0,1)$ and $1 + \lambda\gamma_i \le \eta$ holds for each $i \in [m]$.

For each point $x \in \operatorname{relbd} T$, there exists $i \in [m]$ such that

$$(x, 1) + \lambda(u_i, \gamma_i) \in \gamma \operatorname{int} K + (1 - \gamma)p.$$

Let λ_i be the number such that $1 + \lambda_i\lambda\gamma_i = \eta$. Clearly, $\lambda_i \in (0, 1]$.

$$\begin{aligned}
(x, 1) + \lambda_i\lambda(u_i, \gamma_i) &= (x + \lambda_i\lambda u_i, 1 + \lambda_i\lambda\gamma_i) \\
&= (x + \lambda_i\lambda u_i, \eta) \\
&\in [(x, 1), (x, 1) + \lambda(u_i, \gamma_i)] \setminus \{(x, 1)\} \cap H_\eta \\
&\subset \operatorname{int} K \cap H_\eta \\
&= (\eta \operatorname{relint} T + (1 - \eta) \operatorname{relint} B) \times \{\eta\} \\
&\subset (\eta \operatorname{relint} T + (1 - \eta)B) \times \{\eta\} \\
&\subset (\eta \operatorname{relint} T + (1 - \eta)(T + t)) \times \{\eta\} \\
&\subset (\operatorname{relint} T + (1 - \eta)t) \times \{\eta\}.
\end{aligned}$$

It follows that

$$x + \lambda_i\lambda\left(u_i - \frac{1 - \eta}{\lambda_i\lambda}t\right) \in \operatorname{relint} T.$$

Since $x \in \operatorname{relbd} T$,

$$v_i := u_i - \frac{1 - \eta}{\lambda_i\lambda}t \neq o.$$

Hence

$$\left\{\frac{v_i}{\|v_i\|} : \ i \in [m]\right\}$$

is a set of directions in $\mathbb{R}^{n-1}$ illuminating $\operatorname{relbd} T$. Therefore $c(T) \leq m$. $\square$

In a similar way, we can prove the following lemma.

Lemma 3.9. *If T is contained in a translation of B, then the minimal cardinality m of a set of directions in $\mathbb{R}^n$ illuminating* $\operatorname{relbd} B \times \{0\}$ *equals $c(B)$.*

Corollary 3.10. *If T is a translate of B, then $c(K) = c(T) + c(B) = 2c(T)$.*

For some special cases, the estimation of $c(K)$ can be improved. In the proof of the next theorem, we shall use the following result (see (6) of Theorem 2.15 in [16, p. 79]): if A and B are subsets of $\mathbb{R}^n$ such that a translate of B lies in $\operatorname{aff} A$, then $\operatorname{relint} A + B \subset \operatorname{relint}(A + B)$.

Theorem 3.11. *Suppose that $T + c \subseteq$* $\operatorname{relint} B$ *holds for some point $c \in \mathbb{R}^{n-1}$. Then $c(K) = 1 + c(B)$.*

Proof. By Lemma 3.9, the minimal cardinality m of a set of directions in $\mathbb{R}^n$ illuminating $\operatorname{relbd} B \times \{0\}$ equals $c(B)$. By Lemma 2.1, $c(K) \geq c(B) + 1$.

Put $u = (c, -1)/2$. Next we show that each point in $K_1 = T \times \{1\}$ can be illuminated by $u/\|u\|$:

$$\begin{aligned}
K_1 + u &= T \times \{1\} + \frac{1}{2}(c, -1) \\
&= \left(\frac{1}{2}T\right) \times \{1\} + \left(\frac{1}{2}T\right) \times \{0\} + \frac{1}{2}(c, 0) + \frac{1}{2}(0, -1) \\
&= \frac{1}{2}(T \times \{1\}) + \left(\frac{1}{2}(T + c)\right) \times \{0\} \\
&\subseteq \frac{1}{2}(T \times \{1\}) + \frac{1}{2}(\operatorname{relint} B \times \{0\}) \\
&= \left(\frac{1}{2}T + \frac{1}{2}\operatorname{relint} B\right) \times \left\{\frac{1}{2}\right\} \\
&\subseteq \left(\operatorname{relint}\left(\frac{1}{2}T + \frac{1}{2}B\right)\right) \times \left\{\frac{1}{2}\right\} \\
&\subseteq \operatorname{relint} K_{\frac{1}{2}} \subseteq \operatorname{int} K.
\end{aligned}$$

Hence each point in K_1 can be illuminated by $u/\|u\|$. By Lemma 3.2, $c(K) \leq 1 + c(B)$. Therefore $c(K) = c(B) + 1$. $\square$

Remark. Both Example 3.7 and Corollary 3.10 show that the condition $T + x \subseteq \operatorname{relint} B$ in Theorem 3.11 cannot be replaced by $T + x \subseteq B$.

It is easy to verify the following:

Lemma 3.12. *A set D of directions in $\mathbb{R}^n$ illuminates $T \times \{1\}$ (resp. $B \times \{0\}$) if and only if D illuminates the set of extreme points of $T \times \{1\}$ (resp. of $B \times \{0\}$).*

Theorem 3.13. *If $n = 3$ then $c(K) \leq 8$; equality holds if and only if K is a parallelepiped.*

Proof. Since $n = 3$, T and B are in $\mathcal{K}^2$. Thus $c(K) \leq c(T) + c(B) \leq 4 + 4 = 8$.

Suppose that $c(K) = 8$ holds. It follows that $c(T) = c(B) = 4$. Therefore, T and B are two parallelograms, and each one of them has 4 extreme points. By a suitable translation if necessary, we may assume that B is centered at the origin o of $\mathbb{R}^{n-1}$. Let t be the center of T. It is clear that $(t/2, 1/2) \in \operatorname{int} K$. To show that K is a parallelepiped it sufficies to show that $T - t = B$.

Otherwise $T - t \not\subseteq B$ or $B \not\subseteq T - t$. We may assume, without loss of generality, that one vertex v of $T - t$ is exterior to B. We distinguish two cases.

Case 1: The segment $[o, v]$ contains a vertex w of B.

There exists a number $\lambda \in (0,1)$ such that $w = \lambda v$. In this case we have

$$\begin{aligned}(w,0)+\frac{1}{2}(t,1) &= \lambda(v,0)+\frac{1}{2}(t,1)\\ &= \frac{2\lambda}{\lambda+1}\cdot\frac{1}{2}((w,0)+(v+t,1))+\frac{1-\lambda}{\lambda+1}\cdot\frac{1}{2}(t,1)\\ &\in \operatorname{int} K.\end{aligned}$$

In a similar way, we can show that $(-w,0)+(t/2,1/2)\in \operatorname{int} K$. By Lemma 3.2 and Lemma 3.12, $\operatorname{bd} K$ can be illuminated by at most 7 directions, a contradiction.

Case 2: The segment $[o, v]$ contains a point p which is a relative interior point of an edge $[w, w']$ of B.

There exist two numbers $\lambda, \eta \in (0,1)$ such that $p = \eta v = \lambda w + (1-\lambda)w'$. By interchanging w and w' if necessary, we may assume that $\lambda \le 1-\lambda$. In this case we have

$$\begin{aligned}\frac{1}{2}(t,1)+(p,0) &= \frac{1}{2}(t,1)+\eta(v,0)\\ &= \frac{1-\eta}{1+\eta}\cdot\frac{1}{2}(t,1)+\frac{2\eta}{1+\eta}\cdot\frac{1}{2}((1+\eta)(v,0)+(t,1))\\ &= \frac{1-\eta}{1+\eta}\cdot\frac{1}{2}(t,1)+\frac{2\eta}{1+\eta}\cdot\frac{1}{2}((p,0)+(v,0)+(t,1))\\ &= \frac{1-\eta}{1+\eta}\cdot\frac{1}{2}(t,1)+\frac{2\eta}{1+\eta}\cdot\frac{1}{2}((p,0)+(v+t,1))\\ &\in \frac{1-\eta}{1+\eta}\operatorname{int} K+\frac{2\eta}{1+\eta}K\\ &\subseteq \operatorname{int} K.\end{aligned}$$

In a similar way we can show that

$$\frac{1}{2}(t,1)-(p,0)\in \operatorname{int} K.$$

Put

$$u=\frac{\frac{1}{2}(t,1)+(p-w,0)}{\left\|\frac{1}{2}(t,1)+(p-w,0)\right\|}.$$

Thus

$$(w,0)+\left\|\frac{1}{2}(t,1)+(p-w,0)\right\|u=\frac{1}{2}(t,1)+(p,0)\in \operatorname{int} K,$$

and

$$\begin{aligned}
&-(w',0)+\frac{\lambda\left\|\frac{1}{2}(t,1)+(p-w,0)\right\|}{1-\lambda}u\\
&=-(w',0)+\frac{\lambda}{1-\lambda}\left(\frac{1}{2}(t,1)+(1-\lambda)(w'-w,0)\right)\\
&=-(w',0)+\frac{\lambda}{1-\lambda}\cdot\frac{1}{2}(t,1)+\lambda(w'-w,0)\\
&=-(p,0)+\frac{\lambda}{1-\lambda}\cdot\frac{1}{2}(t,1)\\
&\in \operatorname{int} K;
\end{aligned}$$

i.e., $(w,0)$ and $-(w',0)$ can be illuminated by the direction u. Thus K can be illuminated by at most 7 directions, again a contradiction.

□

Remark. Let $\mathcal{A}^+ \subset \mathcal{K}_n^n$ be the family of convex bodies having a summand in $\mathcal{A}$. Then, by Theorem 34.8 on p. 266 in [11], for each $K' \in \mathcal{A}^+$, we have

$$c(K') \leq \max\{c(K):\ K \in \mathcal{A}\}.$$

Thus our main results also yield good estimations of $c(K)$ for $\mathcal{A}^+$.

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