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DECOMPOSITION OF COMPLETE TRIPARTITE GRAPHS INTO CYCLES AND PATHS OF LENGTH THREE

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Abstract. Let C_k and P_k denote a cycle and a path on k vertices, respectively. In this paper, we obtain necessary and sufficient conditions for the decomposition of $K_{r,s,t}$ into p copies of C_3 and q copies of P_4 for all possible values of p, $q \geq 0$.

1. Introduction

We consider only finite undirected simple graphs. Let $K_{n_1,n_2,\ldots,n_r}$ denote a complete r-partite graph with part sizes $n_1, n_2, \ldots, n_r$, where each $n_i > 0$ is an integer. A partition of a graph G into edge disjoint subgraphs $G_1, G_2, G_3, \ldots, G_n$ such that their union gives G is called a *decomposition* of G. Let C_k and P_k respectively denote a cycle and a path on k vertices. They are also called a k-cycle and k-path, respectively. The problem of finding necessary and sufficient conditions to decompose complete n-partite graphs into k-cycles has been considered for many values of n and k. The case $n = 2$ was completely solved by Sotteau [13]. Smith [12] proved that the necessary conditions for the decomposition of complete equipartite graphs into cycles of length $2p$ (where $p \geq 3$ is a prime) are also sufficient. In the case of complete tripartite graphs, Cavenagh [5] has shown that $K_{m,m,m}$ can be decomposed into k-cycles if and only if $k \leq 3m$ and k divides $3m^2$. Billington [2] gave necessary and sufficient conditions for the existence of a decomposition of any complete tripartite graph into specified number of 3-cycles and 4-cycles. Mahmoodian and Mirzakhani [10] proved the existence of a C_5-decomposition of $K_{r,s,t}$ whenever the necessary conditions are satisfied and two of the partite sets have equal size, except when $r = s = 0 \pmod 5$ and $t \neq 0 \pmod 5$. The authors of [1, 3, 6, 7] also studied this problem. Billington et al. [4] gave necessary and sufficient conditions for the path and cycle decomposition of complete equipartite graphs with 3 and 5 parts. Priyadharsini and Muthusamy [11] gave necessary and sufficient conditions for the existence of (G_n, H_n)-decomposition of λK_n and $\lambda K_{n,n}$, where G_n,

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$H_n \in \{C_n, P_n, S_{n-1}\}$. Jeevadoss and Muthusamy [8] gave necessary and sufficient conditions for the existence of $\{P_{k+1}, C_k\}_{p,q}$-decomposition of $K_{m,n}$ and K_n, when $m \geq k/2$, $n \geq \lceil (k+1)/2 \rceil$ for $k \equiv 0 \pmod 4$ and when m, $n \geq 2k$ for $k \equiv 2 \pmod 4$.

In this paper we give necessary and sufficient conditions for decomposing $K_{r,s,t}$ with $r \leq s \leq t$ into p copies of C_3 and q copies of P_4 for all possible values of p, $q \geq 0$. Definitions and notation not defined here can be referred to in [9].

Lemma 1.1 ([7]). *Let r, s, and t be integers such that $r \leq s \leq t$. A Latin rectangle of order $r \times s$ based on t elements is equivalent to the existence of rs edge-disjoint triangles sitting inside the complete tripartite graph $K_{r,s,t}$.*

The triangle (i, j, k) in the 3-partite graph $K_{r,s,t}$ is the subgraph of $K_{r,s,t}$ induced by the ith vertex of part 1, jth vertex of part 2, and kth vertex of part 3.

Definition 1.2 ([7]). *Consider a rectangular array of order $r \times s$ with entries from the set $T = \{1, 2, \ldots, t\}$. If each element of T appears at most once in each row and at most once in each column, we call such an array a Latin rectangle of order $r \times s$ on t elements.*

Definition 1.3 ([7]). *Let r, s, and t be integers such that $r \leq s \leq t$. A Latin representation of the complete tripartite graph $K_{r,s,t}$ is a Latin rectangle of order $r \times s$ on t elements, together with a set of $t - s$ elements at the end of each row and a set of $t - r$ elements at the bottom of every column so that each element from the set $T = \{1, 2, 3, \ldots, t\}$ occurs once in each of the r rows and once in each of the s columns.*

Remark: To construct a Latin representation of the complete tripartite graph $K_{r,s,t}$ we first take a Latin rectangle of order $r \times s$ on t elements. We then adjoin to the end of each row a set of remaining elements from the set $\{1, 2, 3, \ldots, t\}$ not already used in that row and to the bottom of each column we adjoin a set of remaining elements from the set $\{1, 2, 3, \ldots, t\}$ not already used in that column as in Figure 1.

Each entry k of the set appended at the end of the ith row represents an edge from the ith element of the partite set of size r to the element k of the partite set of size t. Similarly, each entry k of the set appended at the bottom of the jth column represents an edge from the jth element of the partite set of size s to the element k of the partite set of size t. So a Latin representation of $K_{r,s,t}$ is in fact equivalent to a decomposition of $K_{r,s,t}$ into rs triangles and $rK_{1,t-s} + sK_{1,t-r}$.

Here we define *trade* to be a set of elements in the Latin representation, corresponding to a set of triangles and edges in $K_{r,s,t}$ which are P_4-decomposable. We define *relabelling* of the elements of a trade to be a bijection ϕ from the set of elements of $T = \{1, 2, \ldots, t\}$ onto itself. Thus every occurrence of $i \in T$ in the trade is replaced by $\phi(i)$. The relabelling of

1	2	$\cdots$	s	$s+1$	$\cdots$	t
2	3	$\cdots$	$s+1$	$s+2$	$\cdots$	1
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
r	$r+1$	$\cdots$	$r+s-1$	$r+s$	$\cdots$	$r-1$

$r+1$	$r+2$	$\cdots$	$r+s$
$\vdots$	$\vdots$		$\vdots$
t	1	$\cdots$	$s-1$

FIGURE 1.

the elements in a trade does not change the structure of the corresponding set of edges in $K_{r,s,t}$.

Construction 1.5. *Two copies of C_3, with a common vertex, is equivalent to two copies of P_4. Let (a_1, b_1, c_1), (a_1, b_2, c_2) be two copies of C_3 with a common vertex a_1; then it can be written as two copies of P_4, $P(c_1, a_1, b_2, c_2)$, $P(c_2, a_1, b_1, c_1)$. In general, n copies of C_3 with a common vertex is equivalent to n copies of P_4. Let $(a_1, b_1, c_1), (a_1, b_2, c_2), \ldots, (a_1, b_{(n-1)}, c_{(n-1)})$, (a_1, b_n, c_n) be n copies of C_3 with a common vertex a_1; then it can be written n copies of P_4 as $P(c_1, a_1, b_2, c_2), P(c_2, a_1, b_3, c_3), \ldots, P(c_{(n-1)}, a_1, b_n, c_n)$, $P(c_n, a_1, b_1, c_1)$.*

Construction 1.6. *Here we define two types of trades, in the first type we use elements from outside the Latin rectangle which are P_4-decomposable. The trades of first type are T_1, T_2, T_3, T_4, as shown in Figure 2 from the elements outside the Latin rectangle in which each copy of trades in $K_{r,s,t}$ are all edge-disjoint and P_4-decomposable.*

The trade T_1 can be obtained from the newly adjoined elements on the right side of the Latin rectangle which can be decomposed into three copies of P_4 as follows: $P(c_{(s+1)}, a_i, c_{(s+2)}, a_j)$, $P(a_i, c_{(s+3)}, a_k, c_{(s+5)})$, $P(c_{(s+3)}, a_j, c_{(s+4)}, a_k)$. Similarly, by relabelling we can obtain the trade T_1 from the newly adjoined elements on the bottom of the Latin rectangle.

The trade T_2 can be obtained from the newly adjoined elements on the right side of the Latin rectangle which can be decomposed into two copies of P_4 as $P(c_{(s+1)}, a_i, c_{(s+2)}, a_j)$, $P(a_j, c_{(s+3)}, a_k, c_{(s+4)})$. Similarly, by relabelling we can obtain the trade T_2 from the newly adjoined elements on the bottom of the Latin rectangle.

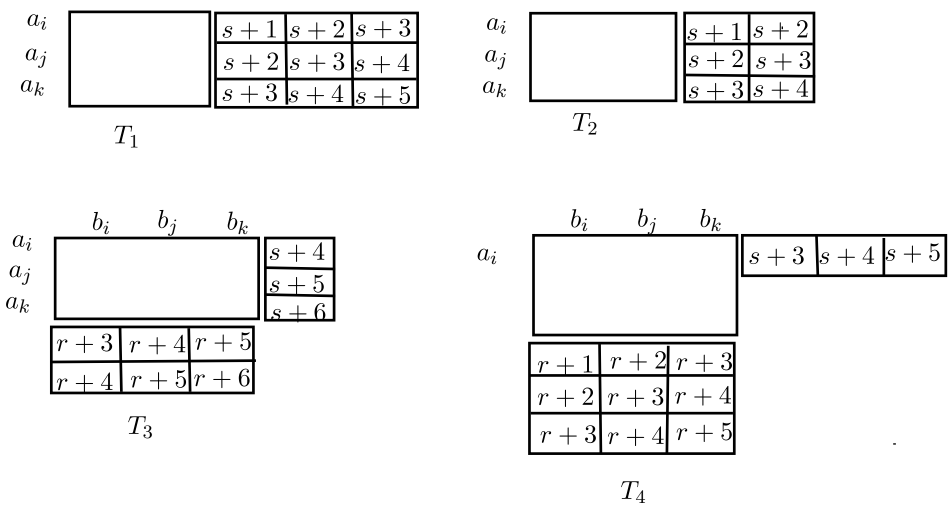

FIGURE 2.

The trade T_3 can be obtained from the newly adjoined elements on the right side and the bottom of the Latin rectangle which can be decomposed into three copies of P_4 as $P(c_{(r+3)}, b_i, c_{(r+4)}, a_i)$, $P(c_{(r+4)}, b_j, c_{(s+5)}, a_j)$, $P(c_{(r+5)}, b_k, c_{(r+6)}, a_k)$, where $s+4$, $s+5$, $s+6$ in the right side of the Latin rectangle are equivalent to $r+4$, $r+5$, $r+6$ respectively.

The trade T_4 can be obtained from the newly adjoined elements on the right side and the bottom of the Latin rectangle which can be decomposed into four copies of P_4 as $P(c_{(r+1)}, b_i, c_{(r+2)}, b_j)$, $P(b_i, c_{(s+3)}, b_j, c_{(s+4)})$, $P(c_{(s+3)}, b_k, c_{(s+4)}, a_i)$, $P(b_k, c_{(s+5)}, a_i, c_{(s+3)})$ where $s+3$, $s+4$, $s+5$ in the right side of the Latin rectangle are equivalent to $r+3$, $r+4$, $r+5$ respectively.

In the second type, the elements from both inside and outside of the Latin rectangle are used. We use these two types of suitable trades untill all the edges in $K_{r,s,t}$ are used.

2. NECESSARY CONDITIONS

Theorem 2.1. *If the complete tripartite graph $K_{r,s,t}$, where $r \leq s \leq t$, has a decomposition into p copies of C_3 and q copies of P_4, then the following holds:*

(i) $3|(rs+st+tr)$,
(ii) $q \neq 1$.

Proof. By a counting argument, we get the required condition (i). We prove (ii) by a contradiction. Suppose that $q = 1$. Then the end vertices of the only path P_4 have odd degree in $(K_{r,s,t} - E(P_4))$. Therefore the resulting graph $(K_{r,s,t} - E(P_4))$ cannot be decomposed into C_3, a contradiction. Hence $q \neq 1$. □

1	2	3
2	3	1
3	1	2

FIGURE 3.

Corollary 2.2. *If the complete tripartite graph $K_{r,s,t}$ can be decomposed into pC_3 and qP_4, where $r \leq s \leq t$, then r, s, and t must satisfy one of the following:*

(a) *any two of r, s, t are congruent to $0 \pmod 3$,*
(b) *all of r, s, t are congruent to $1 \pmod 3$,*
(c) *all of r, s, t are congruent to $2 \pmod 3$.*

Proof. The proof follows from the fact that the number of edges of $K_{r,s,t}$ is divisible by 3. □

3. SUFFICIENT CONDITIONS

Lemma 3.1. *The graph $K_{3,3,3}$ can be decomposed into p copies of C_3 and q copies of P_4, where $0 \leq p \leq 9$ and $0 \leq q \leq 9$, $q \neq 1$.*

Proof. Form a Latin square of order 3×3 on 3 elements as shown in Figure 3. By Lemma 1.1, we have nine edge-disjoint 3-cycles as follows:

$$(a_1, b_1, c_1), (a_1, b_2, c_2), (a_1, b_3, c_3), (a_2, b_1, c_2), (a_2, b_2, c_3),$$
$$(a_2, b_3, c_1), (a_3, b_1, c_3), (a_3, b_2, c_1), (a_3, b_3, c_2).$$

In fact, this gives the required decomposition when $p = 9$, $q = 0$. The required decomposition for the other choices of p and q can be obtained by using Construction 1.5. □

Lemma 3.2. *The graph $K_{3,3,4}$ can be decomposed into pC_3 and qP_4, where $0 \leq p \leq 7$ and $4 \leq q \leq 11$.*

Proof. We form a Latin rectangle of order 3×3 on 4 elements. By Lemma 1.1, we have nine copies of C_3 as follows:

$$(a_1, b_1, c_1), (a_1, b_2, c_2), (a_1, b_3, c_3), (a_2, b_1, c_2), (a_2, b_2, c_3),$$
$$(a_2, b_3, c_4), (a_3, b_1, c_3), (a_3, b_2, c_4), (a_3, b_3, c_1).$$

The newly added element to the right side of each row in the Latin rectangle represents a single edge which cannot be decomposed into P_4. Similarly the newly added element to the bottom of each column of the Latin rectangle represents a single edge which cannot be decomposed into P_4. Here we use trades of the second type to get required number of copies of P_4. The single

1	2	3	4
2	3	4	1
3	**4**	**1**	2
4	1	2	

FIGURE 4.

edges outside the Latin rectangle along with the two copies of C_3 indicated by bold letters in Figure 4 give four copies of P_4:

$$P(a_1, c_4, b_2, a_3), P(b_1, c_4, a_3, b_3), P(a_2, c_1, a_3, c_2), P(b_2, c_1, b_3, c_2).$$

Also, when $p = 6$, $q = 5$, we have six copies of C_3:

$$(a_1, b_1, c_1), (a_1, b_2, c_2), (a_1, b_3, c_3), (a_2, b_1, c_2), (a_2, b_2, c_3), (a_2, b_3, c_1)$$

and five copies of P_4:

$$P(c_3, b_3, c_1, a_2), P(c_1, a_3, b_1, c_3), P(c_2, a_3, b_2, c_4),$$
$$P(a_1, c_4, b_2, c_1), P(b_1, c_4, a_3, b_3).$$

The other choices of p and q can be obtained by using Construction 1.5. Hence the graph $K_{3,3,4}$ has the desired decomposition. □

Theorem 3.3. *If $r \equiv 0 \pmod 3$, $s \equiv 0 \pmod 3$, and for any t, then the complete tripartite graph $K_{r,s,t}$, $r \leq s \leq t$, can be decomposed into p copies of C_3 and q copies of P_4, where $q \neq 1$.*

Proof. The proof is separated into three cases.

CASE 1: $r \equiv 0 \pmod 3, s \equiv 0 \pmod 3, t \equiv 0 \pmod 3$.

The Latin rectangle of order $r \times s$ on t elements give rs triangles. The other choices of p and q can be obtained by Construction 1.5. Now the newly added elements to the right side of the Latin rectangle form $(r/3)[(t-s)/3]$ copies of 3×3 arrays each representing the trade T_1. Similarly the newly added elements to the bottom of the Latin rectangle form $(s/3)[(t-r)/3]$ copies of 3×3 arrays each representing the trade T_1. By Construction 1.6, the copies of trade T_1 are all edge-disjoint and P_4-decomposable. Hence we have the required decomposition, where $0 \leq p \leq rs$,

$$r\left(\frac{t-s}{3}\right) + s\left(\frac{t-r}{3}\right) \leq q \leq rs + r\left(\frac{t-s}{3}\right) + s\left(\frac{t-r}{3}\right).$$

CASE 2: $r \equiv 0 \pmod 3, s \equiv 0 \pmod 3, t \equiv 1 \pmod 3$.

For the graph $K_{r,r,r+1}$, the newly added elements to the right side of Latin rectangle form an $r \times 1$ array which cannot be decomposed into P_4. Similarly the newly added elements to the bottom of Latin rectangle form a $1 \times s$ array which cannot be decomposed into P_4. Therefore we use trades of the second type to obtain the required number of copies of P_4. The single edges on the both side of the Latin rectangle along with $2r/3$ copies of C_3 give $4r/3$ copies of P_4. These $4r/3$ copies of P_4 and the remaining $(r^2 - (2r/3))$ copies of C_3 give the maximum number of copies of P_4 by Construction 1.5. Hence the graph has the required decomposition, where $0 \leq p \leq (r^2 - \frac{2r}{3})$,

$$\frac{4r}{3} \leq q \leq r^2 + \frac{2r}{3}.$$

For the graph $K_{r,s,s+1}$, the newly added elements to the right side of the Latin rectangle is an $r \times 1$ array. The newly added elements to the bottom of the Latin rectangle form $(s/3)[((t-r)-4)/3]$ copies of 3×3 arrays which represents the trade T_1 and the remaining elements form $2s/3$ copies of 2×3 arrays which represents the trade T_2. The elements of $r/3$ copies of 3×1 array in the right side of the Latin rectangle along with the elements of $r/3$ copies of 2×3 array at the bottom of the Latin rectangle which contain the same elements of a 3×1 array form the trade T_3. By Construction 1.6, the edge-disjoint copies of T_1, T_2, T_3 are P_4-decomposable. The remaining possible choices of p and q can be obtained by using Construction 1.5. Hence we have the required decomposition, where $0 \leq p \leq rs$,

$$s\left[\frac{(t-r)-4}{3}\right] + \frac{4s}{3} - \frac{2r}{3} + r \leq q$$
$$\leq s\left[\frac{(t-r)-4}{3}\right] + \frac{4s}{3} - \frac{2r}{3} + r + rs.$$

For $t > s+1$, the newly added elements to the right side of the Latin rectangle form $(r/3)[((t-s)-4)/3]$ copies of 3×3 arrays which represents the trade T_1 and the remaining elements form $2r/3$ copies of 3×2 arrays which represents the trade T_2. The newly added elements to the bottom of the Latin rectangle form $(s/3)[((t-r)-4)/3]$ copies of 3×3 arrays which represents the trade T_1 and $2s/3$ copies of 2×3 arrays which represents the trade T_2. By Construction 1.6, the edge-disjoint copies of T_1, T_2 are P_4-decomposable. The remaining possible choices of p and q can be obtained by using Construction 1.5. Hence we have the required decomposition, where $0 \leq p \leq rs$,

$$s\left[\frac{(t-r)-4}{3}\right] + r\left[\frac{(t-s)-4}{3}\right] + 4\left(\frac{r}{3} + \frac{s}{3}\right) \leq q$$
$$\leq s\left[\frac{(t-r)-4}{3}\right] + r\left[\frac{(t-s)-4}{3}\right] + 4\left(\frac{r}{3} + \frac{s}{3}\right) + rs.$$

Case 3: $r \equiv 0 \pmod 3, s \equiv 0 \pmod 3, t \equiv 2 \pmod 3$.

In this graph, the newly added elements to the right side of Latin rectangle form $(r/3)[((t-s)-2)/3]$ copies of 3×3 arrays which represents the trade T_1 and the remaining elements form $r/3$ copies of 3×2 arrays which represents the trade T_2. Similarly the newly added elements to the bottom of the Latin rectangle form $(s/3)[((t-r)-2)/3]$ copies of 3×3 arrays which represents the trade T_1 and the remaining elements form $s/3$ copies of 2×3 arrays which represents the trade T_2. Hence we have the required decomposition, where $0 \leq p \leq rs$,

$$s\left[\frac{(t-r)-2}{3}\right]+r\left[\frac{(t-s)-2}{3}\right]+\frac{2(r+s)}{3} \leq q \leq s\left[\frac{(t-r)-2}{3}\right]+r\left[\frac{(t-s)-2}{3}\right]+\frac{2(r+s)}{3}+rs.$$

□

Theorem 3.4. *If $r \equiv 0 \pmod 3$, $s \equiv 1 \pmod 3$, and $t \equiv 0 \pmod 3$, then the complete tripartite graph $K_{r,s,t}$, $r < s < t$, can be decomposed into pC_3 and qP_4, where $q \neq 1$.*

Proof. The Latin rectangle of order $r \times s$ on t elements form rs triangles. The other choices of p and q can be obtained by Construction 1.5. The newly added elements to the right side of the Latin rectangle form $(r/3)[((t-s)-2)/3]$ copies of 3×3 arrays each representing the trade T_1 and the remaining elements form $r/3$ copies of 3×2 arrays which represents the trade T_2. Similarly the newly added elements to the bottom of the Latin rectangle form $[(t-r)/3][(s-4)/3]$ copies of 3×3 arrays each representing the trade T_1 and the remaining elements form $[2(t-r)/3]$ copies of 3×2 arrays which represents the trade T_2. By Construction 1.6, all the trades are edge-disjoint and P_4-decomposable. Hence we have the required decomposition, where $0 \leq p \leq rs$,

$$r\left[\frac{(t-s)-2}{3}\right]+\frac{4(t-r)}{3}+\frac{2r}{3}+\left[\frac{(t-r)(s-4)}{3}\right] \leq q \leq rs+r\left[\frac{(t-s)-2}{3}\right]+\frac{4(t-r)}{3}+\frac{2r}{3}+\frac{(t-r)(s-4)}{3}.$$

□

Theorem 3.5. *If $r \equiv 0 \pmod 3$, $s \equiv 2 \pmod 3$, and $t \equiv 0 \pmod 3$, then the complete tripartite graph $K_{r,s,t}$, $r < s < t$, can be decomposed into pC_3 and qP_4, where $q \neq 1$.*

Proof. We consider $s = r + 2$, then the newly added elements to the right side of the Latin rectangle is an $r \times 1$ array which cannot be decomposed into P_4. Therefore, we use trades of the second type. These r single edges along with $2r/3$ triangles give $5r/3$ copies of P_4. Now the newly added elements to the bottom of the Latin rectangle have $(s-2)/3$ copies of 3×3 arrays and one copy of a 3×2 array. Hence we have the required decomposition, where $0 \leq p \leq r(s-(2/3))$,

$$\frac{5r}{3}+s \leq q \leq \frac{5r}{3}+s+r\left(s-\frac{2}{3}\right).$$

For $t = s + 1$, the newly added elements to the right side of the Latin rectangle form $r/3$ copies of 3×1 arrays which cannot be decomposed into copies of P_4. Now the elements of $r/3$ copies of 3×1 arrays in the right side of the Latin rectangle along with the elements of $r/3$ copies of 2×3 arrays in the bottom of the Latin rectangle which contain the same elements of a 3×1 array form the trade T_3. The newly added elements to the bottom of the Latin rectangle form $([((t-r)-6)/3][(s-2)/3]$ copies of 3×3 arrays, $(t-r)/3$ copies of 3×2 arrays and $3(s-2)/3$ copies of 2×3 arrays. Therefore we get $[((t-r)-6)/3][(s-2)/3]$ copies of the trade T_1, $(t-r)/3$, $3(s-2)/3$, and $(3s-6-r)/3$ copies of T_2 in which all are edge-disjoint. Hence we have the required decomposition, where $0 \le p \le rs$ and $q \neq 1$,

$$\begin{aligned}(s-2)\left[\frac{(t-r)-6}{3}\right] + r + \frac{2(t-r)}{3} + 2(s-2) + \frac{2(3s-6-r)}{3} \le q \\ \le (s-2)\left[\frac{(t-r)-6}{3}\right] + r + \frac{2(t-r)}{3} + 2(s-2) + \frac{2(3s-6-r)}{3} + rs.\end{aligned}$$

Now for $t > s + 1$, the newly added elements to the right side of the Latin rectangle form $\frac{r}{3}[\frac{(t-s)-4}{3}]$ copies of 3×3 arrays and $2r/3$ copies of 3×2 arrays which represents the trade T_1 and T_2 respectively. The newly added elements to the bottom of the Latin rectangle form $[(t-r)/3][(s-2)/3]$ copies of 3×3 arrays and $[(t-r)/3]$ copies of 3×2 arrays which represents the trade T_1 and T_2 respectively. Hence we have the required decomposition, where $0 \le p \le rs$ and $q \neq 1$,

$$\begin{aligned}\frac{4r}{3} + \frac{2(t-r)}{3} + \frac{(t-r)(s-2)}{3} + r\left[\frac{(t-s)-4}{3}\right] \le q \\ \le \frac{4r}{3} + \frac{2(t-r)}{3} + \frac{(t-r)(s-2)}{3} + r\left[\frac{(t-s)-4}{3}\right] + rs.\end{aligned}$$

□

Theorem 3.6. *If $r \equiv 1 \pmod 3$, $s \equiv 1 \pmod 3$, and $t \equiv 1 \pmod 3$, then the complete tripartite graph $K_{r,s,t}$, $r \le s \le t$, can be decomposed into pC_3 and qP_4, where $q \neq 1$.*

Proof. We consider three cases:
CASE 1: $r > 1$.

In this case the Latin rectangle of order $r \times s$ on t elements give rs triangles. The newly added elements to the right side of the Latin rectangle form $[(t-s)/3][(r-4)/3]$ copies of 3×3 arrays which represents the trade T_1 and the remaining elements form $2(t-s)/3$ copies of 2×3 arrays which represents the trade T_2. The newly added elements to the bottom of the Latin rectangle form $[(t-r)/3][(s-4)/3]$ copies of 3×3 arrays each representing the trade T_1 and the remaining elements form $2(t-r)/3$ copies of 3×2 arrays which represents the trade T_2. The other choices of p and q can be obtained by Construction 1.5. Hence we have the required decomposition, where $0 \leq p \leq rs$,

$$\frac{(t-s)(r-4)}{3} + \frac{(t-r)(s-4)}{3} + \frac{4(t-s)}{3} + \frac{4(t-r)}{3} \leq q$$
$$\leq rs + \frac{(t-s)(r-4)}{3} + \frac{(t-r)(s-4)}{3} + \frac{4(t-s)}{3} + \frac{4(t-r)}{3}.$$

Case 2: $r = 1$, $s = 1$.

We have one copy of C_3 and $(t-1)/3$ copies of $K_{2,3}$ which can be decomposed into two copies of P_4. Therefore in this case we get $p = 1$ and $q = 2(t-1)/3$.

Case 3: $r = 1$, $s > 1$.

In this case we have $p = s$. The newly added elements to the bottom of the Latin rectangle form $[(t-1)/3][(s-4)/3]$ copies of 3×3 arrays which represents the trade T_1 and the remaining elements form $2(t-1)/3$ copies of 3×2 arrays which represents the trade T_2. The newly added elements to the right side of the Latin rectangle form $(t-s)/3$ copies of 1×3 arrays which cannot be decomposed into copies of P_4. The elements of $(t-s)/3$ copies of 1×3 arrays in the right side of the Latin rectangle along with the elements of $(t-s)/3$ copies of 3×3 arrays in the bottom of the Latin rectangle which contain the same elements of 1×3 arrays form the trade T_4. Therefore we get $([(t-1)/3][(s-4)/3] - [(t-s)/3])$ copies of 3×3 arrays which represents the trade T_1. Hence we have the required decomposition, where $0 \leq p \leq rs$,

$$\frac{4(t-s)}{3} + \left[\frac{(t-1)(s-4) - 3(t-s)}{3}\right] + \frac{4(t-1)}{3} \leq q$$
$$\leq s + \frac{4(t-s)}{3} + \left[\frac{(t-1)(s-4) - 3(t-s)}{3}\right] + \frac{4(t-1)}{3}.$$

□

Theorem 3.7. *If* $r \equiv 1 \pmod 3$, $s \equiv 0 \pmod 3$, *and* $t \equiv 0 \pmod 3$, *then the complete tripartite graph* $K_{r,s,t}$, $r < s \leq t$, *can be decomposed into* pC_3 *and*qP_4, *where* $q \neq 1$.

Proof. We consider two cases:

Case 1: $r = 1$.

The Latin rectangle of order $1 \times s$ on t elements give s triangles. The newly added elements to the bottom of the Latin rectangle have $(s/3)[(t-3)/3]$ copies of 3×3 arrays and $s/3$ copies of 2×3 arrays. The newly added elements to the right side of the Latin rectangle have $(t-s)/3$ copies of 1×3 arrays. The elements of $(t-s)/3$ copies of 1×3 arrays on the right side of the Latin rectangle along with the elements of $(t-s)/3$ copies of 3×3 array in the bottom of the Latin rectangle which contain the same elements of a 1×3 array form the trade T_4. Therefore we have $((s/3)[(t-3)/3] - [(t-s)/3])$ copies of T_1, $(t-s)/3$ copies of T_4

and $s/3$ copies of T_2. Hence we have the required decomposition, where $0 \leq p \leq s$,

$$\left[\frac{s(t-3)-3(t-s)}{3}\right] + \frac{2s}{3} + \frac{4(t-s)}{3} \leq q \leq s + \left[\frac{s(t-3)-3(t-s)}{3}\right] + \frac{2s}{3} + \frac{4(t-s)}{3}.$$

CASE 2: $r > 1$.

The Latin rectangle of order $r \times s$ on t elements give rs triangles. The newly added elements to the right side of the Latin rectangle form $(t-s)(r-4)/9$ copies of 3×3 arrays and $2(t-s)/3$copies of 2×3 arrays. The newly added elements to the bottom of the Latin rectangle form $[s(t-r-2)/9]$ copies of 3×3 arrays and $s/3$ copies of 2×3 arrays. Therefore each copy of 3×3 arrays and 2×3 arrays representing the trades T_1 and T_2 respectively. Hence we have the required decomposition, where $0 \leq p \leq rs$,

$$\frac{s(t-r-2)}{3} + \frac{(t-s)(r-4)}{3} + \frac{4(t-s)}{3} + \frac{2s}{3} \leq q \leq rs + \frac{s(t-r-2)}{3} + \frac{(t-s)(r-4)}{3} + \frac{4(t-s)}{3} + \frac{2s}{3}.$$

□

Theorem 3.8. *If* $r \equiv 2 \pmod 3$, $s \equiv 2 \pmod 3$, *and* $t \equiv 2 \pmod 3$, *then the complete tripartite graph* $K_{r,s,t}$, $r \leq s \leq t$, *can be decomposed into* pC_3 *and* qP_4, $q \neq 1$.

Proof. The Latin rectangle of order $r \times s$ on t elements give rs triangles. The other choices of p and q can be obtained by using Construction 1.5. The newly added elements to the right side of the Latin rectangle form $[(t-s)/3][(r-2)/3]$ copies of 3×3 arrays and $(t-s)/3$ copies of 2×3 arrays each representing the trades T_1 and T_2 respectively. Similarly the newly added elements to the bottom of the Latin rectangle form $[(t-r)/3][(s-2)/3]$ copies of 3×3 arrays and $(t-r)/3$ copies of 3×2 arrays each representing the trades T_1 and T_2 respectively. Hence we have the required decomposition, where $0 \leq p \leq rs$,

$$\frac{(t-s)(r-2)}{3} + \frac{2(t-s)}{3} + \frac{(t-r)(s-2)}{3} + \frac{2(t-r)}{3} \leq q \leq \frac{(t-s)(r-2)}{3} + \frac{2(t-s)}{3} + \frac{(t-r)(s-2)}{3} + \frac{2(t-r)}{3} + rs.$$

□

Theorem 3.9. *If* $r \equiv 2 \pmod 3$, $s \equiv 0 \pmod 3$, *and* $t \equiv 0 \pmod 3$, *then the complete tripartite graph* $K_{r,s,t}$, $r < s \leq t$, *can be decomposed into* pC_3 *and* qP_4, $q \neq 1$.

Proof. We consider two cases:
CASE 1: $s = t$.

The Latin rectangle of order $r \times s$ on s elements give rs triangles. The newly added elements to the bottom of the Latin rectangle is a $1 \times (s/3)$ array which cannot be decomposed into P_4. Here we use trades of the second type to decompose P_4. The edges in the bottom of the Latin rectangle along with $2s/3$ triangles give s copies of P_4. Therefore we get $0 \leq p \leq (rs - (2s/3))$ and $s \leq q \leq (rs + s/3)$.

CASE 2: $s < t$.

The newly added elements to the right side of the Latin rectangle form $[(t-s)/3][(r-2)/3]$ copies of 3×3 arrays and the remaining elements form $[(t-s)/3]$ copies of 2×3 arrays. Similarly the newly added elements to the bottom of the Latin rectangle form $[(t-r-4)/3][s/3]$ copies of 3×3 arrays and the remaining elements form $2s/3$ copies of 2×3 arrays. Therefore each copy of 3×3 arrays and 2×3 arrays represent the trades T_1 and T_2, respectively. The other choices of p and q can be obtained by using Construction 1.5. Hence we get the required decomposition, where $0 \leq p \leq rs$,

$$\frac{(t-s)(r-2)}{3} + \frac{2(t-s)}{3} + \frac{s(t-r-4)}{3} + \frac{4s}{3} \leq q$$
$$\leq \frac{(t-s)(r-2)}{3} + \frac{2(t-s)}{3} + \frac{s(t-r-4)}{3} + \frac{4s}{3} + rs.$$

□

4. CONCLUSION

Main Theorem. *Let p and q be nonnegative integers and let r, s, t be positive integers. There exists a decomposition of $K_{r,s,t}$, $r \leq s \leq t$, into pC_3 and qP_4 if and only if $3(p+q) = rs + st + tr$, $q \neq 1$, where r, s, t satisfy the following conditions:*

(a) *any two of r, s, t are congruent to* $0 \pmod 3$,
(b) *all of r, s, t are congruent to* $1 \pmod 3$,
(c) *all of r, s, t are congruent to* $2 \pmod 3$.

Proof. This follows from the Theorems 2.1, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, and 3.9. □

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