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THE COMPLEXITY OF POWER GRAPHS ASSOCIATED WITH FINITE GROUPS

S. KIRKLAND, A. R. MOGHADDAMFAR, S. NAVID SALEHY, S. NIMA SALEHY, AND M. ZOHOURATTAR

ABSTRACT. The power graph $\mathcal{P}(G)$ of a finite group G is the graph whose vertex set is G, and two elements in G are adjacent if one of them is a power of the other. The purpose of this paper is twofold. First, we find the complexity of a clique-replaced graph and study some applications. Second, we derive some explicit formulas concerning the complexity $\kappa(\mathcal{P}(G))$ for various groups G such as the cyclic group of order n, the simple groups $L_2(q)$, the extra-special p-groups of order p^3, the Frobenius groups, etc.

1. Introduction

All graphs considered here are finite simple connected graphs. A *spanning tree* of a connected graph is a subgraph that contains all the vertices and is a tree. Counting the number of spanning trees in a connected graph is a problem of long-standing interest in various fields of science. For a graph Γ, the number of spanning trees of Γ, denoted by $\kappa(\Gamma)$, is known as the *complexity* of Γ.

In this paper, we consider a well-known graph arising from a finite group. This graph, which is called the power graph, is defined as follows.

Definition 1.1. Let G be a finite group and X a nonempty subset of G. The *power graph* $\mathcal{P}(G, X)$, has X as its vertex set and two vertices x and y in X are joined by an edge if $\langle x\rangle \subseteq \langle y\rangle$ or $\langle y\rangle \subseteq \langle x\rangle$.

The term *power graph* was introduced in [11], and after that power graphs have been investigated by many authors, see for instance [1, 4, 16]. The investigation of power graphs associated with algebraic structures is important, because these graphs have valuable applications (see the survey article [12]) and are related to automata theory (see the book [10]).

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In the case $X = G$, we will simply write $\mathcal{P}(G)$ instead of $\mathcal{P}(G, G)$. Clearly, when $1 \in X$, the power graph is connected and we can discuss the complexity of this graph. For convenience, we put $\kappa_G(X) = \kappa(\mathcal{P}(G, X))$ and $\kappa(G) = \kappa(\mathcal{P}(G))$. A well-known result due to Cayley [5] says that the complexity of the complete graph on n vertices is n^{n-2}. In [7] it was shown that a finite group has a complete power graph if and only if it is a cyclic p-group, where p is a prime number. Thus, as an immediate consequence of Cayley's result, we derive $\kappa(\mathbb{Z}_{p^m}) = p^{m(p^m-2)}$. Recently, the authors of [15] obtained a formula to compute the complexity $\kappa(\mathbb{Z}_n)$ for any n (see Corollary 4.3 below). To obtain Corollary 4.3, we will define a class of graphs more general than the power graphs of cyclic groups. Specifically, we start with a graph Γ on vertices $v_1, v_2, \ldots, v_n$. To construct a new graph, we replace each v_i by a complete graph K_{x_i} on x_i vertices and if there is an edge between v_i and v_j in Γ, then we connect each vertex of K_{x_i} with each vertex of K_{x_j}. The new graph will be denoted by $\Gamma_{[x_1,\ldots,x_n]}$. We will derive explicit formulas for the complexity $\kappa(\Gamma_{[x_1,\ldots,x_n]})$ (see Theorem 4.1 and Remark 4.2). Then we will obtain a formula for the complexity $\kappa(\mathbb{Z}_n)$ by choosing a certain graph Γ on k vertices and positive integers $x_1, x_2, \ldots, x_k$ (Corollary 4.3). Finally, the complexities $\kappa(G)$ for certain groups G are presented.

The outline of the paper is as follows. In the next section, we recall some basic definitions and notation and give several auxiliary results to be used later. The main result of Section 4 is Theorem 4.1 and we include some of its applications. In Section 5, we compute $\kappa(G)$ for certain groups G.

2. Terminology and Previous Results

We first establish some notation which will be used repeatedly in the sequel. Given a graph Γ, we denote by $\mathbf{A}_\Gamma$ and $\mathbf{D}_\Gamma$ the adjacency matrix and the diagonal matrix of vertex degrees of Γ, respectively. The Laplacian matrix of Γ is defined as $\mathbf{L}_\Gamma = \mathbf{D}_\Gamma - \mathbf{A}_\Gamma$. Clearly, $\mathbf{L}_\Gamma$ is a real symmetric matrix and its eigenvalues are nonnegative real numbers. The Laplacian spectrum of Γ is

$$\mathrm{Spec}(\mathbf{L}_\Gamma) = (\mu_1(\Gamma), \mu_2(\Gamma), \ldots, \mu_n(\Gamma)),$$

where $\mu_1(\Gamma) \geqslant \mu_2(\Gamma) \geqslant \cdots \geqslant \mu_n(\Gamma)$, are the eigenvalues of L_Γ arranged in weakly decreasing order, and $n = |V(\Gamma)|$. Note that $\mu_n(\Gamma) = 0$, because each row sum of $\mathbf{L}_\Gamma$ is 0. Instead of $\mathbf{A}_\Gamma$, $\mathbf{L}_\Gamma$, and $\mu_i(\Gamma)$ we simply write $\mathbf{A}$, $\mathbf{L}$, and μ_i if it does not lead to confusion. Given a subset Λ of the vertex set of a graph, we let $\mathbf{A}(\Lambda)$ denote the principal submatrix of $\mathbf{A}$ corresponding to the vertices in Λ.

For a graph with n vertices and Laplacian spectrum $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_n = 0$ it has been proved [2, Corollary 6.5] that

$$\kappa(\Gamma) = \frac{\mu_1 \mu_2 \cdots \mu_{n-1}}{n}. \tag{2.1}$$

The vertex-disjoint union of the graphs Γ_1 and Γ_2 is denoted by $\Gamma_1 \oplus \Gamma_2$. Define the *join* of Γ_1 and Γ_2 to be $\Gamma_1 \vee \Gamma_2 = (\Gamma_1^c \oplus \Gamma_2^c)^c$. Evidently this is the graph formed from the vertex-disjoint union of the two graphs Γ_1, Γ_2, by adding edges joining every vertex of Γ_1 to every vertex of Γ_2. Now, one may easily prove the following (see also [14]).

Lemma 2.1. *Let Γ_1 and Γ_2 be two graphs on disjoint sets with m and n vertices, respectively. If*

$$\mathrm{Spec}(\mathbf{L}_{\Gamma_1}) = (\mu_1(\Gamma_1), \mu_2(\Gamma_1), \ldots, \mu_m(\Gamma_1)),$$

and

$$\mathrm{Spec}(\mathbf{L}_{\Gamma_2}) = (\mu_1(\Gamma_2), \mu_2(\Gamma_2), \ldots, \mu_n(\Gamma_2)),$$

then, the following hold.

(1) *The eigenvalues of Laplacian matrix $\mathbf{L}_{\Gamma_1 \oplus \Gamma_2}$ are:*

$$\mu_1(\Gamma_1), \ldots, \mu_m(\Gamma_1), \mu_1(\Gamma_2), \ldots, \mu_n(\Gamma_2).$$

(2) *The eigenvalues of Laplacian matrix $\mathbf{L}_{\Gamma_1 \vee \Gamma_2}$ are:*

$$m+n, \mu_1(\Gamma_1)+n, \ldots, \mu_{m-1}(\Gamma_1)+n, \mu_1(\Gamma_2)+m, \ldots, \mu_{n-1}(\Gamma_2)+m, 0.$$

A *universal* vertex is a vertex of a graph that is adjacent to all other vertices of the graph. Now, we restrict our attention to information about the set of universal vertices of the power graph of a group G. As already mentioned, the identity element of G is a universal vertex in $\mathcal{P}(G)$, and also $\mathcal{P}(G)$ is complete if and only if G is cyclic of prime power order, and in this case G is the set of all universal vertices. However, the following lemma determines the set of universal vertices of the power graph of G, in the general case.

Lemma 2.2 ([4, Proposition 4])**.** *Let G be a finite group and S the set of universal vertices of the power graph $\mathcal{P}(G)$. Suppose that $|S| > 1$. Then one of the following occurs:*

(a) *G is cyclic of prime power order, and $S = G$;*
(b) *G is cyclic of nonprime power order n, and S consists of the identity and the generators of G, so that $|S| = 1 + \phi(n)$, where ϕ is Euler's ϕ-function;*
(c) *G is generalized quaternion, and S contains the identity and the unique involution in G, so that $|S| = 2$.*

We conclude this section with notation and definitions to be used in the paper. All the groups considered here are finite. We denote by $[G, G]$ the commutator subgroup, for any group G. We refer to any element in G of order 2 as an *involution.* An *elementary abelian p-group* of order p^n, denoted by $\mathbb{E}_{p^n}$, is isomorphic to a direct product of n copies of the cyclic group $\mathbb{Z}_p$. The complement of a graph Γ is denoted by Γ^c. The neighborhood of a vertex v in the graph Γ is denoted by $N_\Gamma(v)$. Let K_n denote the *complete graph* (*clique*) with n vertices. Throughout we use the standard notation and terminology introduced in [2, 9] for graph theory and group theory.

3. Auxiliary Results

Lemma 3.1. *Let Γ be any graph on n vertices with Laplacian spectrum*

$$\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_n.$$

If m is an integer, then the product $(\mu_1+m)(\mu_2+m)\cdots(\mu_{n-1}+m)$ is also an integer.

Proof. Consider the characteristic polynomial of the Laplacian matrix $\mathbf{L}$:

$$\sigma(\Gamma;\mu) = \det(\mu\mathbf{I}-\mathbf{L}) = \mu^n + c_1\mu^{n-1} + \cdots + c_{n-1}\mu + c_n.$$

First, we observe that the coefficients c_i are integers [2, Theorem 7.5], and in particular, $c_n = 0$. This forces $\sigma(\Gamma;-m)$ to be an integer, which is divisible by m. Moreover, we have

$$\sigma(\Gamma;\mu) = (\mu-\mu_1)(\mu-\mu_2)\cdots(\mu-\mu_n),$$

and since $\mu_n = 0$, we obtain

$$\sigma(\Gamma;-m) = (-1)^n m(\mu_1+m)(\mu_2+m)\cdots(\mu_{n-1}+m).$$

The result now follows. □

Lemma 3.2. *Let a graph Γ with n vertices contain $m < n$ universal vertices. Then $k(\Gamma)$ is divisible by n^{m-1}.*

Proof. Let W be the set of universal vertices, $\Gamma_0 = \Gamma - W$ and $t = n-m$. Clearly, we have $\Gamma = K_m \vee \Gamma_0$. Let $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_t = 0$, be the eigenvalues of $\mathbf{L}_{\Gamma_0}$. Since the Laplacian matrix for the complete graph K_m has eigenvalue 0 with multiplicity 1 and eigenvalue m with multiplicity $m-1$, it follows by Lemma 2.1 that the eigenvalues of the Laplacian matrix $\mathbf{L}_\Gamma$ are:

$$n,\ \underbrace{n, n, \ldots, n}_{m-1},\ \underbrace{\mu_1+m, \mu_2+m,\ \ldots, \mu_{t-1}+m}_{t-1}, 0.$$

We find immediately using (2.1) that

$$\kappa(\Gamma) = n^{m-1}(\mu_1+m)(\mu_2+m)\cdots(\mu_{t-1}+m).$$

Finally, since $(\mu_1+m)(\mu_2+m)\cdots(\mu_{t-1}+m)$ is an integer by Lemma 3.1, we obtain the result. □

Let Q_{2^n} for $n \geqslant 3$ denote the generalized quaternion group of order 2^n, which can be presented by

$$Q_{2^n} = \langle x, y \mid x^{2^{n-1}} = 1, y^2 = x^{2^{n-2}}, x^y = x^{-1}\rangle.$$

Moreover, the power graph $\mathcal{P}(Q_{2^n})$ has the following form:

$$\mathcal{P}(Q_{2^n}) = K_2 \vee \Big(K_{2^{n-1}-2} \oplus \underbrace{K_2 \oplus K_2 \oplus \cdots \oplus K_2}_{2^{n-2}-\text{times}}\Big).$$

Using Lemma 2.1 and (2.1), we have the following corollary:

Corollary 3.3 ([15, Theorem 5.2])**.** *Let $n \geqslant 3$ be an integer. Then, $\kappa(Q_{2^n}) = 2^{(2^{n-2}-1)(2n+1)+4}$.*

A finite group G is called an *element prime order* group (EPO-group) if every nonidentity element of G has prime order. We can consider the power graph of an EPO-group G as follows:

$$\mathcal{P}(G) = K_1 \vee \left(\bigoplus_{p \in \pi(G)} c_p K_{p-1} \right),$$

where c_p signifies the number of cyclic subgroups of order p in G. Again, using Lemma 2.1 and (2.1), we have the following corollary:

Corollary 3.4 ([15, Corollary 3.4]). *Let G be an EPO-group. Then we have:*

$$\kappa(G) = \prod_{p \in \pi(G)} p^{(p-2)c_p}.$$

In particular, we have

$$\kappa\left(\mathbb{E}_{p^n}\right) = p^{(p-2)(p^n-1)/(p-1)}.$$

4. Clique-Replaced Graphs

Let Γ be a connected graph with vertices $v_1, \ldots, v_k$. Given positive integers $x_1, \ldots, x_k$, we construct the new graph $\Gamma_{[x_1,\ldots,x_k]}$ as follows: Replace vertex v_i in Γ by the complete graph (clique) K_{x_i}, $i = 1, \ldots, k$, and label the vertex set of K_{x_i} for each i as: $u_{i_1}, u_{i_2}, \ldots, u_{i_{x_i}}$. Now, if v_i is adjacent to v_j in Γ, then connect all vertices $u_{i_1}, u_{i_2}, \ldots, u_{i_{x_i}}$ with all vertices $u_{j_1}, u_{j_2}, \ldots, u_{j_{x_j}}$. We call the resulting graph $\Gamma_{[x_1,\ldots,x_k]}$ the *clique-replaced graph*. It is clear that for a fixed i, all vertices $u_{i_1}, u_{i_2}, \ldots, u_{i_{x_i}}$ have the same degree which is equal to

$$n_i = x_i - 1 + \sum_{v_j \in N_\Gamma(v_i)} x_j.$$

Put $m_i = n_i + 1 = x_i + \sum_{v_j \in N_\Gamma(v_i)} x_j$, $\lambda_i = m_i/x_i$, $i = 1, \ldots, k$, and $\Psi = \prod_{i=1}^k \lambda_i$. Suppose that $n = x_1 + \cdots + x_k$.

Theorem 4.1. *With the notation as explained above, we have*

$$\kappa\left(\Gamma_{[x_1,\ldots,x_k]}\right) = \prod_{i=1}^k m_i^{x_i} \Big(\Psi + \sum_{\Lambda} \det \mathbf{A}_{\Gamma^c}(\Lambda) \lambda_1^{t_1} \lambda_2^{t_2} \cdots \lambda_k^{t_k} \Big) / (\Psi n^2), \tag{4.1}$$

where $t_i \in \{0, 1\}$, $i = 1, \ldots, k$, and the summation is over all induced subgraphs Λ of Γ^c whose vertex set $\{v_{i_1}, \ldots, v_{i_s}\}$ corresponds to $\{i_j | t_{i_j} = 0\}$.

Proof. [1] Let $\Gamma^* = \Gamma_{[x_1,\ldots,x_k]}$. Denoting an all-ones matrix by $\mathbf{J}$, it is easy to check that, the matrix $\mathbf{J} + \mathbf{L}_{\Gamma^*}$ associated with Γ^* has the following block-matrix structure:

$$\mathbf{J} + \mathbf{L}_{\Gamma^*} = (\mathbf{D}_{ij})_{1 \leqslant i,j \leqslant k}, \tag{4.2}$$

[1] The idea of this proof is borrowed from [15, Theorem 4.1].

where $\mathbf{D}_{ij}$ is a matrix of size $x_i \times x_j$ with

$$D_{ij} = \begin{cases} m_i\mathbf{I} & \text{if } i = j, \\ 0 & \text{if } i \neq j, v_i \sim v_j \text{ in } \Gamma, \\ \mathbf{J} & \text{otherwise.} \end{cases}$$

We need only to evaluate $\det(\mathbf{J} + \mathbf{L}_{\Gamma^*})$, because $\kappa(\Gamma^*) = \det(\mathbf{J} + \mathbf{L}_{\Gamma^*})/n^2$. In what follows, D denotes the determinant of the matrix on the right hand side of (4.2). In order to compute this determinant, we apply the following row and column operations: For $h = 0, 1, 2, \ldots, k-1$, we subtract column j from columns $j + r$ where

$$\begin{cases} j = 1 + \sum\limits_{l=1}^{h} x_l, \\ r = 1, 2, \ldots, x_{h+1} - 1. \end{cases}$$

Then, we add rows $i + s$ to row i where

$$\begin{cases} i = 1 + \sum\limits_{l=1}^{h} x_l, \\ s = 1, 2, \ldots, x_{h+1} - 1. \end{cases}$$

(Note that, when $m > n$, we adopt the convention that $\sum_{i=m}^{n} x_i = 0$.) Using the above operations, it is easy to see that

$$D = \det\left(\mathbf{M}_{ij}\right)_{1\leqslant i,j\leqslant k},$$

where $\mathbf{M}_{ij}$ is a matrix of size $x_i \times x_j$ with

$$\mathbf{M}_{ij} = \begin{cases} m_i\mathbf{I} & \text{if } i = j, \\ 0 & \text{if } i \neq j, v_i \sim v_i \text{ in } \Gamma, \\ x_i\mathbf{E}_{1,1} + \mathbf{E}_{2,1} + \cdots + \mathbf{E}_{x_i,1} & \text{otherwise,} \end{cases}$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{E}_{i,j}$ denotes the square matrix having 1 in the (i, j) position and 0 elsewhere.

Therefore, taking out the common factors and developing the determinant along the columns j, $j \neq 1 + \sum_{l=1}^{h} x_l$, $h = 0, 1, 2, \ldots, k-1$, one gets

$$D = \Phi^{-1} \prod_{i=1}^{k} {m_i}^{x_i} \cdot \det\left(c_{ij}\right)_{1\leqslant i,j\leqslant k}, \tag{4.3}$$

where

$$c_{ij} = \begin{cases} \lambda_i & \text{if } i = j, \\ 0 & \text{if } i \neq j, v_i \sim v_j \text{ in } \Gamma, \\ 1 & \text{otherwise.} \end{cases}$$

As the reader might have noticed, the matrix $(c_{ij})_{1\leqslant i,j\leqslant k} - \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_k)$ is exactly the adjacency matrix of the graph Γ^c. Consequently, we get

$$\det\left(\begin{bmatrix} \lambda_1 & c_{12} & \ldots & c_{1k} \\ c_{21} & \lambda_2 & \ldots & c_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ c_{k1} & c_{k2} & \ldots & \lambda_k \end{bmatrix}\right) = \Psi + \sum_{\Lambda} \det \mathbf{A}_{\Gamma^c}(\Lambda)\lambda_1^{t_1}\lambda_2^{t_2}\cdots\lambda_k^{t_k},$$

where $t_i \in \{0,1\}$, $i = 1,2,\ldots,k$, and the summation is over all induced subgraphs Λ of Γ^c whose vertex set $\{v_{i_1},\ldots,v_{i_s}\}$ corresponds to $\{i_j | t_{i_j} = 0\}$. This is substituted in (4.3):

$$D = \Psi^{-1}\prod_{i=1}^{k} m_i^{x_i} \cdot \Big(\Psi + \sum_{\Lambda} \det \mathbf{A}_{\Gamma^c}(\Lambda)\lambda_1^{t_1}\lambda_2^{t_2}\cdots\lambda_k^{t_k}\Big).$$

□

Remark 4.2. In this remark, we describe an alternate approach to the computation of $\kappa(\Gamma^*)$. Note that $\mathbf{L}_{\Gamma^*}$ can be written as a $k\times k$ block matrix, where, for distinct $i,j = 1,\ldots,k$, the (i,j) off-diagonal block is either $-\mathbf{J}$ or a zero block according to whether v_i is adjacent to v_j, or not, and where the jth diagonal block is $m_j\mathbf{I} - \mathbf{J}, j = 1,\ldots,k$. From this block structure, and applying the technique of equitable partitions (see [3]), it follows readily that the product of the nonzero eigenvalues of $\mathbf{L}_{\Gamma^*}$ is equal to $\alpha_2\cdots\alpha_k\left(\prod_{j=1}^{k}(m_j)^{x_j-1}\right)$, where $0,\alpha_2,\alpha_3,\ldots,\alpha_k$ are the eigenvalues of the $k\times k$ matrix $\mathbf{S}$ whose entries are given by

$$s_{pq} = \begin{cases} 0 & \text{if } v_p \neq v_q \text{ and } v_p \text{ is not adjacent to } v_q \text{ in } \Gamma, \\ -x_q & \text{if } v_p \text{ is adjacent to } v_q \text{ in } \Gamma, \\ -\sum_{j\neq p} s_{pj} & \text{if } p = q. \end{cases}$$

(Observe that $\mathbf{S}$ is singular since every row sums to 0.) In order to complete the computation of the complexity of L_{Γ^*}, we need to find the product $\alpha_2\cdots\alpha_k$. For each $j = 1,\ldots,k$, let $\mathbf{S}_{(j)}$ denote the principal submatrix of $\mathbf{S}$ formed by deleting row j and column j. Then $\alpha_2\cdots\alpha_k = \sum_{j=1}^{k}\det(\mathbf{S}_{(j)})$.

To compute the quantities $\det(\mathbf{S}_{(j)}), j = 1,\ldots,k$, we consider a weighted directed graph W on vertices $1,\ldots,k$, whose construction we now describe. Begin with the graph Γ on vertices $v_1, v_2,\ldots,v_k$. For each edge $\{v_i,v_j\}$ of Γ, W contains the arcs $i\to j$ and $j\to i$; the weight of the arc $i\to j$ in W is $w(i,j) = x_j$, and the weight of the arc $j\to i$ is $w(j,i) = x_i$; if there is no edge between v_i and v_j in Γ, then W contains neither an arc from i to j nor an arc from j to i. Fix an index j with $1\leqslant j\leqslant k$. We can find $\det(\mathbf{S}_{(j)})$ from a generalization of the matrix tree theorem as follows (see [6]). Let τ_j be the set of all spanning directed subgraphs of W such that:

(a) the underlying spanning subgraph is a tree;
(b) in the spanning directed subgraph of W, for each vertex $i\neq j$, there is a directed path from i to j.

For each directed graph $\tau \in \tau_j$, let the weight of τ, $\sigma(\tau)$, be the product of the weights of the arcs in τ (these arc weights are inherited from W). Then $\det(\mathbf{S}_{(j)}) = \sum_{\tau\in\tau_j} \sigma(\tau)$. So, we have $\alpha_2 \cdots \alpha_k = \sum_{j=1}^{k} \sum_{\tau\in\tau_j} \sigma(\tau)$. Consequently, we obtain the following formula for $\kappa(\Gamma^*)$:

$$\kappa(\Gamma^*) = \frac{1}{n}\left[\prod_{j=1}^{k} (m_j)^{x_j-1}\right]\left[\sum_{j=1}^{k}\sum_{\tau\in\tau_j} \sigma(\tau)\right].$$

Next, we present some applications of the preceding results.

Application 1: clique-replaced trees. Here we consider the particular case that our graph Γ is a tree on vertices $v_1, \ldots, v_k$. For this special case, we apply the technique of Remark 4.2 in order to obtain the complexity of the graph $\Gamma_{[x_1,\ldots,x_k]}$. For each $j = 1, \ldots, k$, let δ_j denote the degree of vertex v_j.

Observe that for each $l = 1, \ldots, k$, τ_l contains a single directed tree τ with all arcs oriented towards vertex l of the weighted graph W. For each vertex $j \neq l$ of W, there are $\delta_j - 1$ arcs in τ for which j is the tail, and the weight of each such arc is x_j; similarly in τ there are δ_l arcs for which l is the tail, and the weight of each such arc is x_l. We thus find that $\sigma(\tau) = x_l \prod_{j=1}^{k} x_j^{\delta_j - 1}$. Consequently for the matrix $\mathbf{S}$ of Remark 4.2, we have

$$\sum_{l=1}^{k} \det(\mathbf{S}_{(l)}) = \left(\prod_{j=1}^{k} x_j^{\delta_j-1}\right) \sum_{l=1}^{k} x_l.$$

It now follows that

$$\kappa\left(\Gamma_{[x_1,\ldots,x_k]}\right) = \left(\prod_{j=1}^{k} m_j^{x_j-1}\right)\left(\prod_{j=1}^{k} x_j^{\delta_j-1}\right).$$

Application 2: Cayley's theorem. In the case when $\Gamma = K_t$ and $x_j = x, j = 1, \ldots, t$, we have $\Gamma_{[x_1,\ldots,x_t]} = K_{tx}$. Moreover, in the situation of Theorem 4.1 we have: $n_i = \cdots = n_t = tx - 1$, $m_i = \cdots = m_t = tx$, $\lambda_i = \cdots = \lambda_t = t$ and $\Psi = t^t$. Substitution into (4.1) yields

$$\kappa(K_{tx}) = \left(\prod_{i=1}^{t} (tx)^x\right)(t^t + 0)/(t^t(tx)^2) = (tx)^{tx-2},$$

which is equivalent to Cayley's result.

Application 3: complexity of $\mathcal{P}(\mathbb{Z}_n)$. Given a natural number n, the *divisor graph* $D(n)$ of n is the graph with vertex set $\pi_d(n) = \{d_1, \ldots, d_k\}$, the set of all divisors of n, in which two distinct divisors d_i and d_j are adjacent if and only if $d_i|d_j$ or $d_j|d_i$. Let $d_1 > d_2 > \cdots > d_k$ (evidently $d_1 = n$ and $d_k = 1$). This shows that (see also [13, Theorem 2.2]):

$$\mathcal{P}(\mathbb{Z}_n) = D(n)_{[\phi(d_1),\ldots,\phi(d_k)]}. \tag{4.4}$$

In what follows, we put $\Gamma = D(n)$, $n_i = \phi(d_i) - 1 + \sum_{d_j \in N_\Gamma(d_i)} \phi(d_j)$, $m_i = n_i + 1$, $\lambda_i = m_i/\phi(d_i)$, $i = 1, \dots, k$, and $\Phi = \prod_{i=2}^{k-1} \lambda_i$, $\Psi = \prod_{i=1}^{k} \lambda_i$. By using Theorem 4.1, we have the following alternate proof of a result in [15].

Corollary 4.3 ([15, Theorem 4.1]). *Let $d_1 > d_2 > \cdots > d_k$ be the divisors of a positive integer n. With the notation as above, we have*

$$\kappa(\mathbb{Z}_n) = \prod_{i=1}^{k} m_i^{\phi(d_i)} \Big(\Phi + \sum_{\Lambda} \det \mathbf{A}_{\Gamma^c}(\Lambda) \lambda_2^{t_2} \lambda_3^{t_3} \cdots \lambda_{k-1}^{t_{k-1}} \Big) / (\Phi n^2),$$

where $t_i \in \{0, 1\}$, $2 \leqslant i \leqslant k-1$, and the summation is over all induced subgraphs Λ of $\Gamma^c \setminus \{d_1, d_k\}$ whose vertex set $\{d_{i_1}, \dots, d_{i_s}\}$ corresponds to $\{i_j | t_{i_j} = 0\}$.

Proof. Using (4.4) and Theorem 4.1, we obtain

$$\kappa(\mathbb{Z}_n) = \prod_{i=1}^{k} m_i^{\phi(d_i)} \Big(\Psi + \sum_{\Lambda'} \det \mathbf{A}_{\Gamma^c}(\Lambda') \lambda_1^{t_1} \lambda_2^{t_2} \cdots \lambda_k^{t_k} \Big) / (\Psi n^2), \tag{4.5}$$

where $t_i \in \{0, 1\}$, $i = 1, \dots, k$, and the summation is over all induced subgraphs Λ' of Γ^c whose vertex set $\{d_{i_1}, \dots, d_{i_s}\}$ corresponds to $\{i_j | t_{i_j} = 0\}$. Since $\deg_\Gamma(d_1) = \deg_\Gamma(d_k) = k - 1$, we obtain $\deg_{\Gamma^c}(d_1) = \deg_{\Gamma^c}(d_k) = 0$. Thus, if an induced subgraph Λ' of Γ^c contains d_1 or d_k, then $\det \mathbf{A}_{\Gamma^c}(\Lambda') = 0$, while if it does not contain d_1 and d_k, then $\lambda_1 \lambda_k$ divides the sum

$$\sum_{\Lambda'} \det \mathbf{A}_{\Gamma^c}(\Lambda) \lambda_1^{t_1} \lambda_2^{t_2} \cdots \lambda_k^{t_k}.$$

Hence, we can write

$$\sum_{\Lambda'} \det \mathbf{A}_{\Gamma^c}(\Lambda) \lambda_1^{t_1} \lambda_2^{t_2} \cdots \lambda_k^{t_k} = \lambda_1 \lambda_k \sum_{\Lambda} \det \mathbf{A}_{\Gamma^c}(\Lambda) \lambda_2^{t_2} \cdots \lambda_{k-1}^{t_{k-1}}$$

where the Λ run over all induced subgraphs of $\Gamma^c \setminus \{d_1, d_k\}$ whose vertex set $\{d_{i_1}, \dots, d_{i_s}\}$ corresponds to $\{i_j | t_{i_j} = 0\}$. Substituting this into (4.5) and simplifying now yields the result. □

5. Computing the Complexity $\kappa(G)$

In this section we consider the problem of finding the complexity of power graphs associated with certain finite groups.

5.1. **The simple groups** $L_2(q)$**.** Let $q = p^n \geqslant 4$ for a prime p and some $n \in \mathbb{N}$. We are going to find an explicit formula for $\kappa(L_2(q))$. Before we start, we need some well-known facts about the simple groups $G = L_2(q)$, $q \geqslant 4$, which are proven in [8]:

(a) $|G| = q(q-1)(q+1)/k$ and $\mu(G) = \{p, (q-1)/k, (q+1)/k\}$, where $k = \gcd(q-1, 2)$.

(b) Suppose that P is a Sylow p-subgroup of G. Then P is an elementary abelian p-group of order q, which is a TI-subgroup, and $|N_G(P)| = q(q-1)/k$.
(c) Let $A \subset G$ be a cyclic subgroup of order $(q-1)/k$. Then A is a TI-subgroup and the normalizer $N_G(A)$ is a dihedral group of order $2(q-1)/k$.
(d) Let $B \subset G$ be a cyclic subgroup of order $(q+1)/k$. Then B is a TI-subgroup and the normalizer $N_G(B)$ is a dihedral group of order $2(q+1)/k$.

We recall that a subgroup $H \leqslant G$ is a *TI-subgroup* (trivial intersection subgroup) if for every $g \in G$, either $H^g = H$ or $H \cap H^g = \{1\}$.

Theorem 5.1. *Let $q = p^n$, with p prime and $n \in \mathbb{N}$, let $G = L_2(q)$. Then we have:*

$$\kappa(G) = p^{\frac{(q^2-1)(p-2)}{p-1}} \cdot \kappa\left(\mathbb{Z}_{\frac{q-1}{k}}\right)^{q(q+1)/2} \cdot \kappa\left(\mathbb{Z}_{\frac{q+1}{k}}\right)^{q(q-1)/2},$$

where $k = \gcd(q-1, 2)$, except exactly in the cases $(p, n) = (2, 1)$, $(3, 1)$. In particular, we have

(1) $A_5 \cong L_2(5) \cong L_2(4)$ *and* $\kappa(A_5) = 3^{10} \cdot 5^{18}$ *(see* [15]*).*
(2) $L_3(2) \cong L_2(7)$ *and* $\kappa(L_3(2)) = 2^{84} \cdot 3^{28} \cdot 7^{40}$*.*
(3) $A_6 \cong L_2(9)$ *and* $\kappa(A_6) = 2^{180} \cdot 3^{40} \cdot 5^{108}$*.*

Proof. Let $q = p^n$, with p prime and $n \in \mathbb{N}$, and $(p, n) \neq (2, 1), (3, 1)$. As already mentioned, G contains abelian subgroups P, A and B, of orders q, $(q-1)/k$ and $(q+1)/k$, respectively, every distinct pair of their conjugates intersects trivially, and every element of G is a conjugate of an element in $P \cup A \cup B$. Let

$$G = N_P u_1 \cup \cdots \cup N_P u_r = N_A v_1 \cup \cdots \cup N_A v_s = N_B w_1 \cup \cdots \cup N_B w_t,$$

be coset decompositions of G by $N_P = N_G(P)$, $N_A = N_G(A)$ and $N_B = N_G(B)$, where $r = [G : N_P] = q + 1$, $s = [G : N_A] = q(q+1)/2$, and $t = [G : N_B] = (q-1)q/2$. Then, we have

$$G = P^{u_1} \cup \cdots \cup P^{u_r} \cup A^{v_1} \cup \cdots \cup A^{v_s} \cup B^{w_1} \cup \cdots \cup B^{w_t}. \tag{5.1}$$

Applying Theorem 3.4 (b) in [15] to (5.1), we obtain

$$\kappa(G) = \kappa_G(P)^r \cdot \kappa_G(A)^s \cdot \kappa_G(B)^t = \kappa(\mathbb{E}_q)^r \cdot \kappa\left(\mathbb{Z}_{\frac{q-1}{k}}\right)^s \cdot \kappa\left(\mathbb{Z}_{\frac{q+1}{k}}\right)^t,$$

and so by Corollary 3.4, we get

$$\kappa(G) = \left(p^{\frac{q-1}{p-1}(p-2)}\right)^r \cdot \kappa\left(\mathbb{Z}_{\frac{q-1}{k}}\right)^s \cdot \kappa\left(\mathbb{Z}_{\frac{q+1}{k}}\right)^t.$$

The result follows. ∎

5.2. **Extra-special p-groups of order p^3.** In the sequel, P will be a p-group, with p prime. We recall below some facts about extra-special groups and other necessary information. We begin with the definition of the extra-special groups. A p-group P is called *extra-special* if $Z(P) = [P, P] = \Phi(P) \cong \mathbb{Z}_p$, where $\Phi(P)$ is the Frattini subgroup of P. If P is an extra-special p-group, then the order of P is p^{2n+1} for some positive integer n. The smallest nonabelian extra-special groups are of order p^3. When $p = 2$, there are, up to isomorphism, two extra-special 2-groups of order 8, namely, D_8 and Q_8. The exponent of both of these groups is $p^2 = 4$. Furthermore, from [15, Table 1], we have $\kappa(D_8) = 2^4$ and $\kappa(Q_8) = 2^{11}$.

For each odd prime p, up to isomorphism, there are just two nonisomorphic extra-special p-groups of order p^3. The first one has exponent p, which is called the *Heisenberg group* and denoted by H_p. In fact, H_p as a subgroup of $\mathrm{GL}(3, p)$ can be presented in the following way:

$$H_p = \left\{ \left(\begin{array}{ccc} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{array} \right) \;\middle|\; x, y, z \in \mathrm{GF}(p) \right\}.$$

The other one has exponent p^2, which is denoted by A_p, and contains transformations $x \mapsto ax + b$ from $\mathbb{Z}_{p^2}$ to $\mathbb{Z}_{p^2}$, where $a \equiv 1 \pmod p$ and $b \in \mathbb{Z}_{p^2}$.

The groups H_p and A_p are usually presented as:

$$H_p = \langle x, y, z \mid x^p = y^p = z^p = 1, [x, y] = z, [x, z] = [y, z] = 1 \rangle,$$

and

$$A_p = \langle x, y \mid x^p = y^{p^2} = 1, y^x = y^{p+1} \rangle.$$

Theorem 5.2. *Let p be an odd prime. Then, we have:*

(a) $\kappa(H_p) = p^{(p-2)(p^2+p+1)}$.

(b) $\kappa(A_p) = p^{2p^3-p-5}$.

Proof. (a) Clearly, we have

$$H_p = \bigcup_{j=1}^{p^2+p+1} C_j,$$

where $C_j \subset H_p$ is a subgroup of order p, and $C_i \cap C_j = 1$ for $i \neq j$. Now, by [15, Theorem 3.4 (b)], we obtain

$$\kappa(H_p) = \prod_{j=1}^{p^2+p+1} \kappa(C_j) = \prod_{j=1}^{p^2+p+1} p^{p-2} = p^{(p-2)(p^2+p+1)},$$

as desired.

(b) In this case, we have

$$A_p = \bigcup_{j=1}^{p+1} B_j,$$

where $B_j \subset A_p$ is a subgroup of order p^2, and $B_i \cap B_j = Z(A_p)$ for $i \neq j$. Therefore, the power graph of A_p has the following form:

$$\mathcal{P}(A_p) = K_p \vee \left[(p+1)K_{p^2-p}\right].$$

It follows by Lemma 2.1 that the eigenvalues of the Laplacian matrix $\mathbf{L}_{\mathcal{P}(A_p)}$ are:

$$p^3,\ \underbrace{p^2, p^2, \ldots, p^2}_{p-1},\ \underbrace{p^2, p^2, \ldots, p^2}_{p^3-2p-1},\ \underbrace{p, p, \ldots, p}_{p},\ 0.$$

Using (2.1), we get $\kappa(A_p) = p^{2p^3-p-4}$, as required. □

5.3. **Frobenius groups.** Suppose $1 \subset H \subset G$ and $H \cap H^g = 1$ whenever $g \in G \setminus H$. Then H is a *Frobenius complement* in G. A group which contains a Frobenius complement is called a *Frobenius group*. A famous theorem of Frobenius asserts that in a Frobenius group G with a Frobenius complement H, the set

$$F = \left(G \setminus \bigcup_{g \in G} H^g\right) \cup \{1\},$$

is a normal subgroup of G and $G = FH$, $F \cap H = 1$. We call F the *Frobenius kernel* of G.

Theorem 5.3. *Let G be a Frobenius group, H a Frobenius complement and F the Frobenius kernel corresponding with H. Then, we have:*

$$\kappa(G) = \kappa_G(F)\kappa_G(H)^{|F|}.$$

In particular, if G is a nonabelian group of order pq, where $p < q$ are primes, then $\kappa(G) = q^{q-2}p^{(p-2)q}$.

Proof. Let G be a Frobenius group, let H be its Frobenius complement and F its Frobenius kernel. Then G can be written as the union of its subgroups:

$$G = F \cup \bigcup_{g \in F} H^g.$$

Again, it follows from [15, Theorem 3.4 (b)] that

$$\kappa(G) = \kappa_G(F) \prod_{g \in F} \kappa_G(H^g) = \kappa_G(F)\kappa_G(H)^{|F|},$$

as required. □

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Department of Mathematics, University of Manitoba, Winnipeg, MB, Canada
E-mail address: Stephen.Kirkland@umanitoba.ca

Faculty of Mathematics, K. N. Toosi University of Technology, Tehran, Iran, and Department of Mathematical Sciences, Kent State University, Kent, OH, USA
E-mail address: moghadam@kntu.ac.ir and amoghadd@kent.edu

Department of Mathematics, Florida State University, Tallahassee, FL, USA
E-mail address: navidsalehy@math.fsu.edu

Department of Mathematics, Florida State University, Tallahassee, FL, USA
E-mail address: nimasalehy@math.fsu.edu

Faculty of Mathematics, K. N. Toosi University of Technology, Tehran, Iran
E-mail address: zohoorattar@mail.kntu.ac.ir