



## HAMILTON CYCLES IN BIDIRECTED COMPLETE GRAPHS

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ABSTRACT. Zaslavsky observed that the topics of directed cycles in directed graphs and alternating cycles in edge 2-colored graphs have a common generalization in the study of coherent cycles in bidirected graphs. There are classical theorems by Camion, Harary and Moser, Häggkvist and Manoussakis, and Saad which relate strong connectivity and Hamiltonicity in directed “complete” graphs and edge 2-colored “complete” graphs. We prove two analogues to these theorems for bidirected “complete” signed graphs.

### 1. INTRODUCTION

There are many similarities between the two topics of directed paths and directed cycles within directed graphs and alternating paths and alternating cycles within edge-2-colored graphs. One viewpoint for these similarities is that these topics have a common generalization as coherent paths and coherent cycles in bidirected graphs. This common generalization was first noted by Zaslavsky [21, p.26].

A *bidirected graph* is a graph  $B$  in which each edge is given two directional arrows, one at each end of the edge. (Bidirected graphs were introduced by Edmonds and Johnson [7].) Often a bidirection  $\beta$  on a graph  $G$  is thought of as a function  $\beta: V(G) \times E(G) \rightarrow \{-1, 0, +1\}$  where  $\beta(v, e) = +1$  means the directional arrow of  $e$  at  $v$  is pointed towards  $v$ ,  $\beta(v, e) = -1$  means the directional arrow of  $e$  at  $v$  is pointed away from  $v$ , and  $\beta(v, e) = 0$  means that  $e$  is not incident to  $v$ . There are three types of bidirected edges: *directed*, *introverted*, and *extroverted* (see Figure 1).

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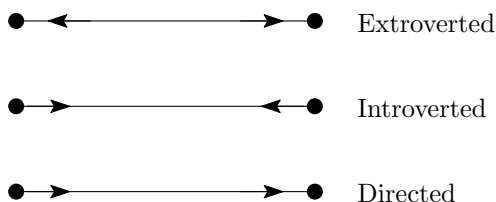


FIGURE 1. There are three types of edges in a bidirected graph.

Directed edges in a bidirected graph correspond to directed edges in a directed graph and introverted and extroverted edges correspond to the two different colors in an edge-2-colored graph. A vertex  $v$  in a bidirected graph is called a *source* when all bidirectional arrows at  $v$  are pointed away from  $v$ , a *sink* when all bidirectional arrows at  $v$  are pointed towards  $v$ , and *singular* when it is either a source or a sink. A *directed cycle* in a directed graph is a cycle in the underlying graph that has no singular vertex, an *alternating cycle* in an edge-2-colored graph is a cycle in the underlying graph that has no singular vertex, and so a *coherent cycle* in a bidirected graph is defined to be a cycle in the underlying graph without a singular vertex. Thus a coherent cycle in a bidirected graph generalizes both alternating cycles in edge-2-colored graphs and directed cycles in directed graphs. Directed paths, alternating paths, and coherent paths are similarly defined. Coherent paths then come in the same three types as bidirected edges: extroverted, introverted, and directed.

Say that a bidirected graph  $B$  is *Hamiltonian* if there is a Hamilton cycle in the underlying graph which is coherent. There are classical theorems dealing with conditions under which various forms of directed “complete” graphs and edge-2-colored “complete” graphs are Hamiltonian. These include Theorems 3.1 by Camion [3], Theorem 3.2 by Harary and Moser [12], Theorem 4.1 by Häggkvist, R. and Manoussakis, and Theorem 4.2 by Saad [18]. In this paper we will prove some analogues to these theorems for various forms of bidirected “complete” signed graphs. Signed graphs provide a convenient setting for the study of bidirected graphs. This will be described in section 2. Most of the work in this paper was developed as part of the master’s thesis of the second author [16].

## 2. PRELIMINARIES

**2.1. Ordinary Graphs.** All graphs in this paper will be loopless but not necessarily simple. Our graph-theory terminology is mostly standard and should cause no confusion to the reader familiar with the subject. We will, however, review some pertinent terms here. A *Hamilton cycle* in a graph  $G$  is a cycle passing through all of the vertices. A *2-factor* in a graph  $G$  spanning 2-regular subgraph, i.e., a vertex-disjoint union of cycles covering all of the vertices of  $G$ . Hence a 2-factor is a Hamilton cycle if and only if

it has one component. It is worth noting that the problem of determining whether or not an arbitrary graph  $G$  has a Hamilton cycle is NP-hard while the problem of determining whether or not  $G$  has a 2-factor is polynomial. Meijer, Rodriguez, and Rappaport outline a polynomial algorithm for deciding if a graph  $G$  has a 2-factor in [15]. The basic idea is as follows: take an input graph  $G$  of minimum degree 2. Create a new graph  $\widehat{G}$  from  $G$  by replacing each vertex  $v$  in  $G$  with a graph  $G_v$  consisting of a copy of  $K_{2,d(v)}$  along with pendant edges attached to the degree-2 vertices of the  $K_{2,d(v)}$ . The edges of  $G$  incident to  $v$  are then attached to the degree-1 vertices of  $G_v$ . Now the 2-factors of  $G$  correspond to complete matchings in  $\widehat{G}$ . Deciding if a graph has a complete matching is calculable in polynomial time. The first such polynomial-time algorithm was given by Edmonds [6].

**2.2. Signed Graphs.** A *signed graph* is a pair  $(G, \sigma)$  in which  $G$  is a graph (which is often referred to as the *underlying graph*) and  $\sigma$  is a labeling of the edges of  $G$  with elements of the multiplicative group  $\{+, -\}$ . When drawing signed graphs, positive edges are drawn as solid curve segments and negative edges as dashed curve segments. A cycle or path  $C$  in  $G$  is called *positive* when the product of signs on its edges is positive; otherwise the cycle or path is called *negative*. A signed graph  $(G, \sigma)$  is said to be *simple* when every cycle of length two is negative; that is, there are no two edges with the same sign that are parallel in  $G$ . Thus  $G$  need not be a simple graph for  $(G, \sigma)$  to be a simple signed graph.

If  $G$  is an ordinary graph, then  $G$  can be thought of as a signed graph in which each edge is positive. We sometimes denote this by  $+G$ . By  $-G$  we mean the signed graph with underlying graph  $G$  in which each edge is negative. By  $\pm G$  we mean the signed graph obtained from  $G$  by doubling each edge and for each such edge pair labeling one positive and one negative. As a shorthand notation, we write  $\{u, v\}^\epsilon$  to mean an edge with endpoints  $u$  and  $v$  having sign  $\epsilon$ .

Given a signed graph  $(G, \sigma)$  a *switching function* is a function

$$\eta: V(G) \rightarrow \{+, -\}.$$

The function  $\sigma^\eta$  is the labeling of the edges of  $G$  defined by

$$\sigma^\eta(e) = \eta(u)\sigma(e)\eta(v)$$

in which  $u$  and  $v$  are the endpoints of edge  $e$ . Notice that the edges of  $G$  that change signs under the switching function  $\eta$  are those edges in the edge cut  $[\eta^{-1}(-), \eta^{-1}(+)]$ . As such, a brief way to refer to the action of  $\eta$  is that it *switches* on the vertices of  $\eta^{-1}(-)$ . Given a graph  $G$ , this switching action defines an equivalence relation on the set of signed graphs with underlying graph  $G$ . Two signed graphs on the same underlying graph are called *switching equivalent* when they are in the same equivalence class; furthermore, they are switching equivalent if and only if they have the same set of negative cycles (confer Zaslavsky [20]).

**2.3. Bidirected Signed Graphs.** A *bidirection* on a signed graph  $(G, \sigma)$  is a function

$$\beta: V(G) \times E(G) \rightarrow \{-1, 0, +1\}$$

where for each edge  $e = \{x, y\}^{\sigma(e)}$  of  $G$ ,  $\beta(x, e)\beta(y, e) = -\sigma(e)$  and  $\beta(v, f) = 0$  when  $(v, f)$  is not a vertex-edge incidence in  $G$ . Again,  $\beta(v, e) = +1$  means the directional arrow of  $e$  at  $v$  points towards  $v$  and  $\beta(v, e) = -1$  means the directional arrow of  $e$  at  $v$  points away from  $v$ . Note that negative edges are either introverted or extroverted while positive edges are directed. Also, a coherent cycle is necessarily positive with the negative edges of that positive cycle in an alternating introverted-extroverted pattern around the cycle; conversely, a positive cycle in a signed graph has exactly two different coherent bidirections. We call the triple  $(G, \sigma, \beta)$  a *bidirected signed graph*. A bidirected signed graph is *Hamiltonian* when it has a coherent Hamilton cycle.

If  $\eta$  is a switching function on bidirected signed graph  $(G, \sigma, \beta)$ , then  $\beta^\eta$  is defined by  $\beta^\eta(v, e) = \eta(v)\beta(v, e)$ ; that is, the arrows incident to vertices in  $\eta^{-1}(-)$  are reversed while arrows incident to vertices in  $\eta^{-1}(+)$  remain the same. Thus  $\beta^\eta$  is a bidirection on signed graph  $(G, \sigma^\eta)$  if and only if  $\beta$  is a bidirection on signed graph  $(G, \sigma)$  and a path or cycle of  $G$  is coherent under  $\beta$  if and only if it is coherent under  $\beta^\eta$ .

Given a bidirected signed graph  $B = (G, \sigma, \beta)$ , construct its *auxiliary graph*  $\text{Aux}(B)$  as follows. For each vertex  $v$  in  $B$  add two vertices  $v_+$  and  $v_-$  to  $\text{Aux}(B)$ . For each edge  $e = \{u, v\}^{\sigma(e)}$  in  $B$ , add edge  $\{u_{\beta(u,e)}, v_{\beta(v,e)}\}$  to  $\text{Aux}(B)$ . Proposition 2.1 is immediate and tells us that the task of either finding a bidirected 2-factor in a bidirected graph  $B$  or certifying that  $B$  does not have a bidirected 2-factor can be computed in polynomial time.

**Proposition 2.1.** *A subset  $X$  of the edge set of a bidirected graph  $B$  forms a coherent 2-factor if and only if  $X$  form a complete matching in  $\text{Aux}(B)$ .*

### 3. STRONG CONNECTIVITY

A directed graph  $B$  is said to be *strongly connected* when for any two vertices  $u$  and  $v$  in  $B$ , there is a directed  $uv$ -path and a directed  $vu$ -path. Robbins showed [17] that a graph  $G$  has a strongly-connected orientation if and only if  $G$  is 2-edge-connected. Also, if  $G$  is Hamiltonian, then  $G$  must be 2-connected. Consider the following theorems.

**Theorem 3.1** (Camion [3]). *If  $D$  is a directed  $K_n$  for  $n \geq 3$  (i.e.,  $D$  is a tournament), then  $D$  is Hamiltonian if and only if  $D$  is strongly connected.*

**Theorem 3.2** (Harary and Moser [12]). *If  $D$  is a directed  $K_n$  for  $n \geq 3$ , then  $D$  has a directed cycle of length  $k$  for every  $k \in \{3, \dots, n\}$  if and only if  $D$  is strongly connected.*

Consider the following predicate statement  $P_0(G)$  in which  $G$  is a 2-connected graph. Theorem 3.1 tells us that  $P_0(K_n)$  is true for  $n \geq 3$  and

certainly  $P_0(C_n)$  is true where  $C_n$  is a cycle with  $n \geq 3$ . Grötschel and Harary [9] proved that there is no other 2-connected simple graph  $G$  for which  $P_0(G)$  is true.

*$P_0(G) = \text{Any directed graph } D \text{ with underlying graph } G \text{ is Hamiltonian if and only if } D \text{ is strongly connected.}$*

Saad [18] defines an edge-2-colored graph  $B$  to be *strongly connected* when for any two vertices  $u$  and  $v$ , there are two alternating  $uv$ -paths  $P_1$  and  $P_2$  in which the edges of  $P_1$  and  $P_2$  incident to  $x \in \{u, v\}$  differ in color. Strong connectivity in bidirected graphs is defined to generalize both of these concepts in directed graphs and edge-2-colored graphs. A bidirected graph  $B$  is said to be *strongly connected* when for any two vertices  $u$  and  $v$ , there are two coherent  $uv$ -paths  $P_1$  and  $P_2$  in which the arrows of  $P_1$  and  $P_2$  at  $x \in \{u, v\}$  are different. While an ordinary graph  $G$  has a strongly connected orientation if and only if  $G$  is edge 2-connected, it seems harder to characterize when a signed graph  $(G, \sigma)$  has a strongly connected bidirection. We now consider the following predicate statement in which  $(G, \sigma)$  is a 2-connected simple signed graph which has a strongly connected bidirection. Theorem 3.3 gives us that  $P_1(\pm K_n)$  is true for any  $n \geq 3$ . It is easily seen that  $P_1(\pm C_n)$  is true as well. It would be interesting to know whether or not there are any other such signed graphs satisfying  $P_1(G, \sigma)$ .

*$P_1(G, \sigma) = \text{Any bidirected signed graph } B \text{ with underlying signed graph } (G, \sigma) \text{ is Hamiltonian if and only if } B \text{ is strongly connected.}$*

**Theorem 3.3.** *If  $B$  is a bidirected  $\pm K_n$  for some  $n \geq 3$ , then  $B$  has a coherent cycle of length  $k$  for every  $k \in \{3, \dots, n\}$  if and only if  $B$  is strongly connected.*

*Proof.* Certainly, the existence of a coherent cycle of length  $n$  in  $B$  implies that  $B$  is strongly connected. Now assume that  $B$  is strongly connected. First, we show that  $B$  contains some coherent cycle. Given any two vertices  $u$  and  $v$  in  $B$ , strong connectivity implies that there are two coherent  $uv$ -paths  $P_1$  and  $P_2$  whose arrows at  $u$  and  $v$  differ. Without loss of generality, we may assume that  $P_1$  has length at least two and now apply switching so that every edge in  $P_1$  is directed from  $u$  to  $v$ . Now, if  $P_1$  and  $P_2$  share no internal vertex in common, then  $P_1 \cup P_2$  forms a coherent cycle. If  $P_2$  shares an internal vertex in common with  $P_1$ , then  $P_2$  has length at least two. Thus neither the edge  $\{u, v\}^+$  nor the edge  $\{u, v\}^-$  is in  $P_1 \cup P_2$ . Since the arrows of  $P_1$  and  $P_2$  differ at both  $u$  and  $v$  and since every edge in  $P_1$  is directed from  $u$  to  $v$ , either  $P_1 \cup \{u, v\}^+$  and  $P_2 \cup \{u, v\}^+$  is a coherent cycle.

Second, we show that  $B$  must contain a coherent cycle of length three. Let  $C$  be a coherent cycle of length at least four. By switching, assume that all edges of  $C$  are directed. Now take any chord  $\{u, v\}^+$  of  $C$  in  $B$ . There is now a coherent cycle  $C'$  in  $C \cup \{u, v\}^+$  of length strictly less than  $C$ . Thus a coherent triangle in  $B$  can be found.

Now, inductively, assume that  $B$  has coherent cycles of lengths  $3, \dots, m$ . If  $m = n$ , then we are done. Assuming that  $m \leq n - 1$ , we will show that  $B$  contains a coherent cycle of length  $m + 1$ . Let  $C$  be a coherent cycle of length  $m$  in  $B$  and apply switching so that every edge in  $C$  is directed. Consider a vertex  $v$  not in  $C$ . If two positive  $vC$ -edges have different directions, then the coherent cycle  $C$  may be extended to a coherent cycle of length  $m + 1$ . Similarly, if two negative  $vC$ -edges have different bidirections (that is, one introverted and one extroverted), then again  $C$  can be extended to a coherent cycle of length  $m + 1$ . So now assume that for every vertex  $v \notin C$ , all positive  $vC$ -edges directed the same and all negative  $vC$ -edges bidirected the same.

Now there are four possible patterns for the  $vC$ -edges: all arrows at  $v$  are into  $v$ , all arrows at  $v$  are out from  $v$ , all arrows at  $C$  are into  $C$  and all arrows at  $C$  are out from  $C$ . Let  $I, O, I_C$ , and  $O_C$  be the blocks of the partition of  $V(B) - V(C)$  based on these four patterns. By applying switching at the vertices in  $I$ , we may assume that  $I = \emptyset$ . Now, if there is  $v \in I_C$  and  $v' \in O_C$ , then using either of the two  $vv'$ -edges we can extend  $C$  to a coherent cycle of length  $m + 1$ . Now switch if necessary all of the vertices of  $C$  and we may assume that  $I = \emptyset$  and  $I_C = \emptyset$ .

If there is  $v \in O$  and  $v' \in O_C$ , then we can use  $v$  and  $v'$  to extend  $C$  to a coherent cycle of length  $m + 1$  unless the  $\{v, v'\}^+$ - and  $\{v, v'\}^-$ -edges both have outward arrows at  $v$ . Also, if there are distinct vertices  $v, v' \in O$ , then  $v$  and  $v'$  may be used to extend  $C$  to a coherent cycle of length  $m + 1$  unless the  $\{v, v'\}^-$ -edge is introverted. So assume that all of the negative edges with both endpoints in  $O$  are introverted and all of the  $OO_C$ -edges have out arrows at  $O$ . So now if  $O \neq \emptyset$ , then there is no coherent path from any vertex  $v \in O$  to a vertex  $v' \in C$  which has an in-arrow at  $v$ , a contradiction of strong connectivity. Thus  $O = \emptyset$  and  $O_C \neq \emptyset$ . Now if there is a negative edge in  $B$  with both endpoints on  $C$  that is extroverted, then there is a coherent path  $P$  of length  $m - 1$  that is extroverted as indicated in Figure 2.

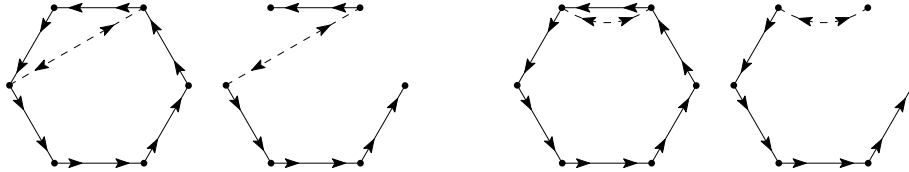


FIGURE 2. An extroverted chord of a directed cycle of length  $m$  yields an extroverted path of length  $m - 1$ .

Using the path  $P$  and any vertex from  $O_C$  we get a coherent cycle of length  $m + 1$ . So we can now assume that all of the negative edges with both endpoints on  $C$  are introverted. Now every coherent path from a vertex

$c \in C$  to a vertex  $v \in O_C = V(B) - V(C)$  must have its arrow at  $c$  directed away from  $c$ , a contradiction of strong connectivity.  $\square$

#### 4. STRONG CONNECTIVITY AND A COHERENT 2-FACTOR

For general bidirected graphs, strong connectivity alone is not sufficient to assure Hamiltonicity. Adding in the existence of a coherent 2-factor along with strong connectivity does yield interesting results.

**Theorem 4.1** (Häggkvist and Manoussakis [11]). *If  $B$  is a directed  $K_{n,n}$ , then  $B$  is Hamiltonian if and only if  $B$  is strongly connected and contains a directed 2-factor.*

**Theorem 4.2** (Saad [18]). *If  $B$  is an edge 2-colored  $K_n$ , then  $B$  is Hamiltonian if and only if  $B$  is strongly connected and contains an alternating 2-factor.*

Again, the main result of [9] tells us that the 2-factor in Theorem 4.1 is needed as a sufficient condition. To show that the alternating 2-factor is needed as a sufficient condition in Theorem 4.2, consider the following example. Take any  $K_n$  with  $n \geq 5$  and  $a + b = n$  with  $a > b \geq 2$ . Color the edges in vertex-disjoint  $K_a$ - and  $K_b$ -subgraphs blue and the remaining edges in the  $K_{a,b}$ -subgraph red. There can be no alternating 2-factor and so there is no alternating Hamilton cycle; however, this edge 2-colored  $K_n$  is strongly connected. We are thus motivated to search for 2-connected simple signed graphs  $(G, \sigma)$  that have a strongly connected bidirection and which satisfy the following predicate.

$P_2(G, \sigma) =$  Any bidirected signed graph  $B$  with underlying signed graph  $(G, \sigma)$  is Hamiltonian if and only if  $B$  is strongly connected and contains a coherent 2-factor.

Theorem 4.3 is a simple corollary of Theorem 4.1, but Theorem 4.4 requires a full proof.

**Theorem 4.3.** *If  $B$  is an edge 2-colored  $K_{n,n}$ , then  $B$  is Hamiltonian if and only if  $B$  is strongly connected and contains an alternating 2-factor.*

*Proof.* Given the vertex bipartition  $X, Y$  for  $B$  with underlying graph  $K_{n,n}$ , if we switch on the vertices of  $X$ , then  $+K_{n,n}$  becomes  $-K_{n,n}$  while coherence of paths and cycles and strong connectivity are preserved. The result is thus implied by Theorem 4.1.  $\square$

**Theorem 4.4.** *If  $B$  is a bidirected  $\pm K_{n,n}$ , then  $B$  is Hamiltonian if and only if  $B$  is strongly connected and contains a coherent 2-factor.*

We will complete this paper with the formal proof of Theorem 4.4. We make some comments and observations beforehand. Theorems 3.1 and 4.2 give us that  $P_2(K_n, \sigma)$  is true for any  $\sigma$  that is switching equivalent to either the all-positive signing or the all-negative signing. We thought that maybe  $P_2(K_n, \sigma)$  is true for any  $\sigma$ ; however, this is not the case. Consider the

following class of examples. For any  $K_n$  with  $n \geq 6$  let  $B$  be the bidirected graph  $B$  on  $(K_n, \sigma)$  constructed as follows. Say that  $B$  has a coherent 2-factor with two cycles  $C_1$  and  $C_2$ . By switching we may assume that the edges of  $C_1$  and  $C_2$  are all positive. Direct one edge from  $C_1$  to  $C_2$  and one edge from  $C_2$  to  $C_1$ . Let all of the other edges of  $B$  be extroverted. The subgraph of  $B_+$  of  $B$  using only the positive edges is strongly connected and so  $B$  is strongly connected; however,  $B$  is not Hamiltonian. This is because a Hamilton cycle in  $B$  must use an equal number of introverted and extroverted edges and so such a cycle cannot use any negative edges and clearly  $B_+$  by itself is not Hamiltonian.

So we are left with the following questions: Is there any signed graph  $(K_n, \sigma)$  aside from those switching equivalent to  $+K_n$  or  $-K_n$  for which  $P_2(K_n, \sigma)$  is true? Is there any ordinary graph  $G$  aside from  $K_n$ ,  $C_n$ , or  $K_{n,n}$  for which  $P_2(G)$  is true? We suspect that the answer is no. As we shall see, the basic strategy in the proof of Theorem 4.4 is as follows. (It is an adaptation of the idea that is used by Saad in [18] and by Bánkfalvi and Bánkfalvi in [2].) In a bidirected graph  $B$ , we call a 4-cycle  $Q$  a *singular quadrilateral* if every vertex of  $Q$  is singular. Now if  $C_1$  and  $C_2$  are two vertex-disjoint coherent cycles in a bidirected graph  $B$  and  $Q$  is a singular quadrilateral with one edge in  $C_1$  and one edge in  $C_2$ , then the edges in the symmetric difference of  $E(Q)$  with  $E(C_1 \cup C_2)$  form a coherent cycle spanning  $V(C_1) \cup V(C_2)$ . Thus we take a coherent 2-factor and hunt for singular quadrilaterals to hook the cycles of the 2-factor together. Note also that a singular quadrilateral must be positive 4-cycle in the underlying signed graph. Graphs and signed graphs other than these mentioned seem to lack a “uniform enough” distribution of positive 4-cycles. Consider, for example, Proposition 4.5.

**Proposition 4.5.** *If every 4-cycle in the signed graph  $(K_n, \sigma)$  is positive, then  $(K_n, \sigma)$  is switching equivalent to  $+K_n$  or  $-K_n$ .*

*Proof.* Extend the sign function  $\sigma$  of a signed graph  $(G, \sigma)$  to all subsets  $X \subseteq E(G)$  by  $\sigma(\emptyset) = +$  and

$$\sigma(X) = \prod_{e \in X} \sigma(e).$$

Now  $\sigma$  yields a linear transformation from the binary cycle space  $Z(G)$  to the multiplicative group  $\{+, -\}$ . The switching equivalence class of a signing on  $G$  is uniquely determined by this linear transformation.

Now say that every 4-cycle in  $(K_n, \sigma)$  is positive. We claim that either every triangle in  $(K_n, \sigma)$  is positive or every triangle in  $(K_n, \sigma)$  is negative. Let  $T$  and  $T'$  be triangles in  $(K_n, \sigma)$ . Evidently there is a sequence of triangles  $T_1, \dots, T_n$  in which  $n \in \{2, 3, 4\}$ ,  $T_1 = T$ ,  $T_n = T'$ , and  $T_i \cup T_{i+1}$  forms a 4-cycle  $Q$  along with a single diagonal edge. As such

$$\sigma(T_i)\sigma(T_{i+1}) = \sigma(T_i + T_{i+1}) = \sigma(Q) = +$$

so  $\sigma(T_i) = \sigma(T_{i+1})$  so  $\sigma(T) = \sigma(T')$  which proves our claim.



Now if  $C$  is a cycle of length  $m$  in  $(K_n, \sigma)$ , then in the binary vector space  $Z(K_n)$ ,  $C$  is the sum of  $m - 2$  triangles. As such  $\sigma(C) = +$  when every triangle in  $(K_n, \sigma)$  is positive and  $\sigma(C) = (-1)^{m-2}$  when every triangle in  $(K_n, \sigma)$  is negative. In the former case  $(K_n, \sigma)$  is switching equivalent to  $+K_n$  and in the latter case  $(K_n, \sigma)$  is switching equivalent to  $-K_n$ .  $\square$

*Proof of Theorem 4.4.* The necessity of these conditions is obvious. Let  $B$  be a bidirected  $\pm K_{n,n}$  that is strongly connected and contains a coherent 2-factor  $F$ . If  $F$  has only one cycle, then we are done and so we assume that  $F$  contains at least two cycles. Apply switching so that the edges of  $F$  are all positive. We will show that there is a coherent 2-factor in  $B$  with fewer cycles than  $F$  and this will complete our proof.

Let  $C$  and  $D$  be two cycles in  $F$ . Consider the collection  $M$  of positive  $CD$ -edges. Claim 4.6 will either yield a completion of our proof or that all positive  $CD$ -edges have the same direction. Also, if  $\eta$  is a switching function with  $\eta^{-1}(-) = V(C)$ , then we can again apply Claim 4.6 to complete our proof or get that all negative  $CD$ -edges in  $B$  (i.e., the positive  $CD$ -edges in  $B^\eta$ ) have the same bidirection.

**Claim 4.6.** *Either there is a coherent cycle in  $B$  with vertex set  $V(C) \cup V(D)$  or all of the edges of  $M$  have the same direction.*

*Proof.* If there is a singular quadrilateral with one edge in  $C$ , one edge in  $D$ , and two edges in  $M$ , then the first result holds. So for the rest of the proof of this claim assume that there is no such singular quadrilateral; furthermore, say that the direction from  $C$  to  $D$  is *down* and the direction from  $D$  to  $C$  is *up*.

Let  $X, Y$  be the bipartition of the vertices of  $\pm K_{n,n}$ . Denote the vertices of  $C$  in cyclic order by  $c_1, \dots, c_l$  in which all edges are directed  $c_i$  to  $c_{i+1}$ . (Addition of subscripts for  $C$  is taken modulo  $l$ .) Denote the vertices of  $d$  in cyclic order by  $d_1, \dots, d_m$  in which all edges are directed  $d_i$  to  $d_{i-1}$ . (Addition of subscripts for  $D$  is taken modulo  $m$ .) Without loss of generality we may assume that  $l \leq m$ ,  $c_{2i+1} \in X$ , and  $d_{2i+1} \in Y$ .

Partition the edges of  $M$  into  $m/2$  matchings  $M_0, M_2, \dots, M_m$  of  $l$ -edges each where

$$M_{2i} = \{ \{c_k, d_{k+2i}\}^+ : k \in \{1, \dots, l\} \}.$$

Since there is no singular quadrilateral, there is no  $1 \leq k \leq l - 1$  for which  $\{c_k, d_{k+2i}\}^+$  is down and  $\{c_{k+1}, d_{k+1+2i}\}^+$  is up. Therefore, for each matching  $M_{2i}$  either: all of its edges are up, all of its edges are down, or there is some  $2 \leq k \leq l$  such that  $\{c_1, d_{1+2i}\}^+, \dots, \{c_{k-1}, d_{k-1+2i}\}^+$  are all up and  $\{c_k, d_{k+2i}\}^+, \dots, \{c_l, d_{l+2i}\}^+$  are all down. In Case 1 say that there is no  $M_{2i}$  whose edges are of this third (i.e., mixed) type. In Case 2 say that there is some  $M_{2i}$  of this mixed type.

CASE 1: If the matchings are all directed upwards or all directed downwards, then we are done. So suppose there is at least one of each. Taking subscripts modulo  $m$  there is  $M_{2i}$  that is directed downwards and  $M_{2i+2}$  that is directed

upwards. There is now a coherent cycle of length  $l + m$  in

$$E(C) \cup E(D) \cup M_{2i} \cup M_{2i+2}$$

as indicated in Figure 3 after rotating  $D$  backwards  $2i$  vertices.

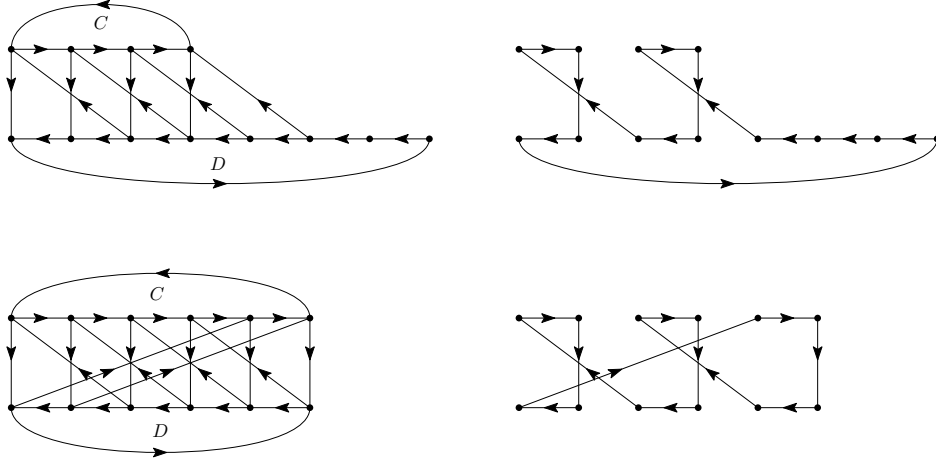


FIGURE 3. Case 1: Hooking together  $C$  and  $D$  using  $M_{2i}$  and  $M_{2i+2}$ . The techniques for  $l < m$  and  $l = m$  are slightly different.

CASE 2: After rotating  $D$  and relabeling its vertices we may assume that  $M_0$  has mixed directions, up to down. The fact that there is no singular quadrilateral linking together  $C$  and  $D$  implies that  $l < m$ . Consider  $M_l$ . Since the last edge of  $M_0$  is down, the first edge of  $M_l$  must be down because there are no singular quadrilaterals for  $C$  and  $D$  (see Figure 4).

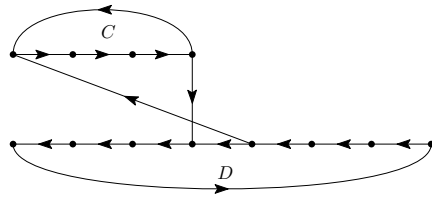


FIGURE 4. Case 2: If the last edge of  $M_{jl}$  is down and the first edge of  $M_{(j+1)l}$  is up, they create a singular quadrilateral for  $C$  and  $D$ . Subscript addition is taken modulo  $m$ .

Thus all of the edges in  $M_l$  are down and inductively we now must have that all of the edges of  $M_r$  are down for each  $r$  in the subgroup  $\langle l \rangle$  of  $\mathbb{Z}_m$ . However, again taking subscripts modulo  $m$ , the last edge of  $M_{-l}$  is down and the first edge of  $M_0$  is up; these form a singular quadrilateral for  $C$  and  $D$ , a contradiction.  $\square$

Now again consider any two cycles  $C$  and  $D$  in  $F$ . Let  $M^+$  be the collection of positive  $CD$ -edges and let  $M^-$  be the collection of negative  $CD$ -edges. Assume that all of the edges in  $M^+$  have the same direction and all of the edges in  $M^-$  have the same bidirection. Now either all of the arrows on the edges of  $M^+ \cup M^-$  at cycle  $C$  are in the same direction or all of the arrows on the edges of  $M^+ \cup M^-$  at cycle  $D$  are in the same direction but not both. Assuming that the arrows of  $M^+ \cup M^-$  at  $D$  are in the same direction, then the arrows of  $M^+$  at  $C$  are in the opposite direction as the arrows of  $M^-$  at  $C$ . In this case, we say that the edges of  $M^+ \cup M^-$  are mixed at  $C$  and unidirectional at  $D$ .

**Claim 4.7.** *If the edges of  $M^+ \cup M^-$  are unidirectional at  $D$  and pointed away from  $D$ , then either: all of the negative edges of  $B$  with both endpoints in  $V(D)$  are introverted or there is a coherent cycle in  $B$  on vertex set  $V(C) \cup V(D)$ . The analogous conclusion holds when the negative edges of  $M^+ \cup M^-$  are unidirectional at  $D$  and pointed towards  $D$ .*

*Proof.* If there is a negative edge  $e$  with both endpoints in  $D$  that is extroverted, then we can find an extroverted path  $P$  in  $D \cup e$  that spans  $V(D)$  as indicated in Figure 2. Now using  $C$ ,  $P$ , one edge from  $M^+$ , and one edge from  $M^-$  we obtain a coherent cycle spanning  $V(C) \cup V(D)$ .  $\square$

Now let  $C_1, \dots, C_t$  be the cycles of  $F$  with  $t \geq 2$ . Assume that  $B$  has no coherent cycle on the union of the vertex sets of any two of the cycles of  $F$ . By Claim 4.6, the  $C_i C_j$ -edges in  $B$  are either unidirectional at  $C_i$  or unidirectional at  $C_j$ . Supposing that the  $C_i C_j$ -edges are unidirectional at  $C_i$  and are pointed towards  $C_i$ , Claim 4.7 now implies that all of the negative edges of  $B$  with both endpoints on  $C_i$  are extroverted. Again by Claim 4.7, if there is another  $C_k$  in which the  $C_i C_k$ -edges are unidirectional at  $C_i$ , then these arrows must also point towards  $C_i$ . Now for each  $C_i \in \{C_1, \dots, C_t\}$ , if there is  $C_j$  for which the  $C_i C_j$ -edges are all unidirectional at  $C_i$ , then apply switching if necessary to  $V(C_i)$  so that the  $C_i C_j$ -edges are unidirectional at  $C_i$  and pointed towards  $C_i$ .

It cannot be that there is  $C_i \in \{C_1, \dots, C_t\}$  such that for every  $C_j \neq C_i$ , the  $C_i C_j$ -edges are unidirectional at  $C_i$ . If this were the case, then there would be no coherent path from any vertex  $u \notin V(C_i)$  to a vertex  $v \in V(C_i)$  whose arrow at  $v$  would point away from  $v$ , a contradiction of strong connectivity. This also implies that  $t \geq 3$ .

Now construct a bidirected  $\pm K_t$  from  $B$ , call it  $\widehat{B}$ , whose vertex set corresponds to  $\{C_1, \dots, C_t\}$  and the two  $C_i C_j$ -edges in  $\widehat{B}$  are bidirected to match the  $C_i C_j$ -edges in  $B$ . As shown in the last paragraph, no vertex of  $\widehat{B}$  is singular and if the two  $C_i C_j$ -edges of  $\widehat{B}$  are unidirectional at  $C_i$ , then they point towards  $C_i$ . So now consider a maximum-length path  $P$  of 2-cycles in  $\widehat{B}$  which are bidirected as shown in Figure 5.

FIGURE 5. A path of 2-cycles in  $\widehat{B}$ .

Since the rightmost vertex of  $P$ , call it  $C$ , is nonsingular in  $\widehat{B}$ , there is a 2-cycle in  $B$  whose arrows at vertex  $C$  are mixed. Since the length of  $P$  is a maximum, the other vertex of this 2-cycle is some interior vertex of  $P$ . Thus  $\widehat{B}$  contains a coherently directed cycle  $\widehat{C}$ . If  $D_1, \dots, D_s$  are the vertices of  $\widehat{C}$ , then the edges of  $\widehat{C}$  can be used to construct a coherent cycle in  $B$  on vertex set  $V(D_1) \cup \dots \cup V(D_s)$ .  $\square$

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