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A CLASSIFICATION OF ISOMORPHISM-INVARIANT RANDOM DIGRAPHS

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Abstract. We classify isomorphism-invariant random digraphs (IIRDs) according to where randomness lies, namely, on arcs, vertices, vertices and arcs together as arc random digraphs (ARD), vertex random digraphs (VRD), vertex-arc random digraphs (VARD) as an extension of the classification of isomorphism-invariant random graphs (IIRGs) [1], and introduce randomness in direction (together with arcs, vertices, etc.) also which in turn yield direction random digraphs (DRDs) and its variants, respectively. We demonstrate that for the number of vertices $n \geq 4$, ARDs and VRDs are mutually exclusive and are both proper subsets of VARDs, and also demonstrate the existence of VARDs which are neither ARDs nor VRDs, and the existence of IIRDs that are not VARDs (e.g., random nearest neighbor digraphs(RNNDs)). We demonstrate that to obtain a DRD as an IIRD, one has to start with an IIRG and insert directions randomly. Depending on the type of IIRG, we obtain direction-edge random digraphs (DERDs), direction-vertex random digraphs (DVRDs), and direction-vertex-edge random digraphs (DVERDs), and demonstrate that DERDs and DVRDs have an overlap but are mutually exclusive for $n \geq 4$, and both are proper subsets of DVERDs which is a proper subset of DRDs and also the complement of DRDs in IIRDs is nonempty (e.g., RNNDs). We also study the relation of DRDs with VARDs, VRDs, and ARDs and show that for $n \geq 4$, the intersection of DERDs and VARDs is ARDs; we provide some results and open problems and conjectures. For example, the relation of DVRDs and DVERDs with the VARDs (hence with ARDs and VRDs) are still open problems for $n \geq 4$. We also show positive dependence between the arcs of a VARD whose tails are same which implies the asymptotic distribution of the arc density of VRDs and ARDs has nonnegative variance.

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1. Introduction

A *directed graph* (or *digraph*) D consists of a nonempty finite set $V(D)$ of elements called *vertices* and a finite set $A(D)$ of ordered pairs of distinct vertices called *arcs* (or *directed edges*). We will often denote D as $D = (V, A)$.

For an arc (u, v), the vertex u is called the *tail* and the vertex v is called the *head*. The head and tail of an arc are called the *end-vertices*. The above definition of a digraph implies that we allow a digraph to have arcs with the same end-vertices (for example, both (u, v) and (v, u) may be in A). In this paper we only consider simple digraphs. That is, we do not allow *parallel* (i.e., *multiple*) arcs, i.e., pairs of arcs with the same tail and the same head, or *loops* (i.e., arcs whose heads and tails coincide). When parallel arcs and loops are admissible, we speak of *directed pseudographs*; directed pseudographs without loops are *directed multigraphs*. For more information about graphs and digraphs see, e.g., [7].

For a positive integer n, let $[n] := \{1, 2, \ldots, n\}$, $\mathcal{D}_n$ denote the set of all digraphs with vertex set $[n]$, and $2^{\mathcal{D}_n}$ denote the set of all subsets of $\mathcal{D}_n$. A *random digraph* is a probability space $(\mathcal{D}_n, 2^{\mathcal{D}_n}, P)$, and we write $\mathbf{D} = (\mathcal{D}_n, P)$ where P is a probability measure. We call a random digraph as *degenerate* if all the probability mass is on one digraph. We can also think of $\mathbf{D}$ as the outcome of an experiment of picking a digraph from $\mathcal{D}_n$ with distribution P. For every $D \in \mathcal{D}_n$, we write $P(\{D\})$ as $P(D)$ for convenience in notation. Also, for a measure space $(\Omega, \mathcal{F}, \mu)$, let $\mathcal{F}^n$ and μ^n denote the usual product σ-algebra and product measure, respectively. For the set of real numbers, we consider the Borel σ-algebra, and throughout this paper we suppress the σ-algebra notation as long as there is no necessity nor ambiguity.

Example 1.1 (Uniform Random Digraph Model)**.** *For positive integers n and m with $n \geq 2$ and $0 < m < n(n-1)$, $\mathbf{D}(n, m)$ is the random digraph such that*

$$P(D) = \begin{cases} \frac{1}{\binom{n(n-1)}{m}}, & \text{if } |A(D)| = m \\ 0, & \text{otherwise} \end{cases}$$

for every $D \in \mathcal{D}_n$. In other words, $\mathbf{D}(n, m)$ picks a digraph uniformly at random among the ones with vertex set $[n]$ and having exactly m arcs. Note that there are $\binom{n(n-1)}{m}$ such digraphs, and m is not chosen to be 0 or $n(n-1)$ to obtain a nondegenerate random digraph. $\mathbf{D}(n, m)$ is the digraph counterpart of the Erdős–Rényi random graph $\mathbf{G}(n, m)$ ([10]). For some asymptotic properties of uniform random digraphs, see [18] and [13].

A digraph D_1 is *isomorphic* to a digraph D_2 (or D_1 and D_2 are *isomorphic*) if there is a bijection $f : V(D_1) \to V(D_2)$ such that $(u, v) \in A(D_1)$ if and only if $(f(u), f(v)) \in A(D_2)$.

Definition 1.2 (Isomorphism Invariance)**.** *Let* $\mathbf{D} = (\mathcal{D}_n, P)$ *be a random digraph. We say that* $\mathbf{D}$ *is* isomorphism-invariant *if* $P(D_1) = P(D_2)$ *whenever* D_1 *and* D_2 *are isomorphic digraphs in* $\mathcal{D}_n$*. The random digraphs which are isomorphism-invariant are called* isomorphism-invariant random digraphs (IIRDs)*.*

Throughout the article, we only consider nondegenerate IIRDs. Our work is inspired by the isomorphism-invariant random graph (IIRG) classification of [1], where the authors consider randomness in the defining units of graphs, namely, vertices and edges, and exploit the symmetry in the binary relation defining the edges between vertices. They consider edge-random graphs (ERGs), vertex-random graphs (VRGs), and vertex-edge random graphs (VERGs) and demonstrate ERGs and VRGs are mutually exclusive for $n \geq 4$ (excluding degenerate graphs), and they are both proper subsets of VERGs for $n \geq 6$. However, the study of IIRDs is not only a straightforward extension of IIRGs, because the binary relation is not symmetric for digraphs and there is an additional defining unit, the *direction of the arcs*. First, as digraph counterparts of IIRGs, we introduce arc random digraphs (ARDs), vertex random digraphs (VARDs), and vertex-arc random digraphs (VARDs), and study the inclusion/exclusion relations between them. We demonstrate that, for $n \geq 4$, ARDs and VRDs are mutually exclusive and their union is a proper subset of VARDs, i.e., we show that there is no IIRD that is both ARD and VRD and the existence of VARDs that are neither ARDs nor VRDs, although any VARD can be arbitrarily closely approximated by VRDs. Furthermore, we show that the complement of VARDs in IIRDs is nonempty, and provide random nearest neighbor digraphs (RNNDs) as an example. See Figure 1 for a Venn diagram representation of VARDs. Along this line, we point out the similarities and differences between VARDs and VERGs until we introduce randomness in the direction. Notice that we reduced the lower bound for the inclusion/exclusion relations for VERGs; the lower bound was 6 for VERGs ([1]) and it is 4 for VARDs. Similar to the results of [1], we demonstrate that ARDs and VRDs are mutually exclusive and they are both proper subsets of VARDs for $n \geq 4$, thereby reducing the lower bound from 6 to 4.

Digraphs have an additional component compared to graphs, namely, the direction of the arcs, hence one can also attach randomness to the direction as well. Direction random digraphs (DRDs) are defined by adding random directions to pairs of vertices which are joined by an edge in a given graph. However, we show that DRDs are isomorphism-invariant only if the graph we start with (which is called the underlying graph) is isomorphism invariant. And depending on where the randomness lies in the underlying graph, adding randomness on the direction gives rise to direction-edge random digraphs (DERDs), direction-vertex random digraphs (DVRDs), and direction-vertex-edge random digraphs (DVERDs). For example, in DERDs,

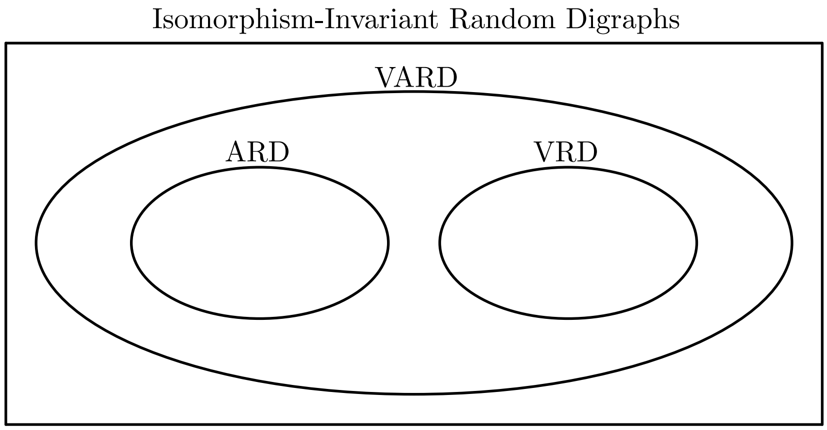

FIGURE 1. Venn diagram of vertex-arc random digraphs for $n \geq 4$. Our results in this paper show that ARD $\cap$ VRD $= \emptyset$ and all the four regions in the figure are nonempty for $n \geq 4$.

the randomness lies in both edges and the directions, i.e., DERDs are obtained from ERGs by randomly adding directions to their edges. Our results on the inclusion/exclusion relations between DRDs indicate that DERDs and DVRDs only overlap for $n \leq 3$ but otherwise are mutually exclusive, and both are proper subsets of DVERDs which is a proper subset of DRDs. These results are illustrated in Figure 2.

We also study the relations between DRDs and VARDs and show that intersection of DERDs and VARDs yield ARDs for $n \geq 4$; for $n = 3$, any DERD is a VARD, any DERD is a DVRD, and we conjecture that any DERD is a VRD as well; for $n = 2$, any IIRD is a DERD, DVRD and VRD. See Figure 3 for a Venn diagram representation of VARDs and DERDs. The relations between DVRDs and VARDs and also relation between DVERDs and VARDs (i.e., identifying the types of digraphs in their intersection) are still open problems. DERDs have two probabilities associated with them, the edge probability p_e and direction probability p_d. We find the probabilities p_e and p_d that guarantees that the DERD is also an ARD with arc probability p_a.

We also show the existence of positive dependence for two arcs having the same tail in a VARD (also for two edges sharing a vertex in a VERG), which has implications for the arc and edge densities of VARDs and VERGs, respectively. We show that the arc (edge) density of VARDs (VERGs) is a U-statistic, and as $n \to \infty$, the arc (edge) density of ARDs (ERGs) converges in law to a constant (i.e., its asymptotic distribution is degenerate), and arc (edge) density of VRDs (VRGs) converges in law to a normal distribution provided its asymptotic variance is positive. Positive dependence guarantees that the asymptotic distribution of arc (edge) density of VRDs (VRGs) is

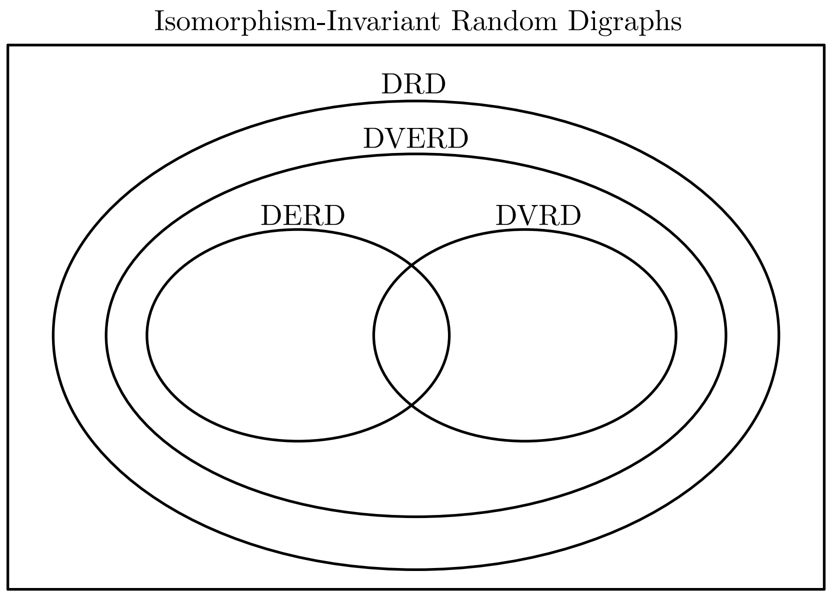

Figure 2. Venn diagram of direction random digraphs (DRDs). The results of the paper imply that all the six regions in the figure are nonempty. In particular, the region DERD$\cap$DVRD only consists of DRDs with $n \leq 3$ and DRDs generated by $\mathbf{G}(n, p_e = 1)$.

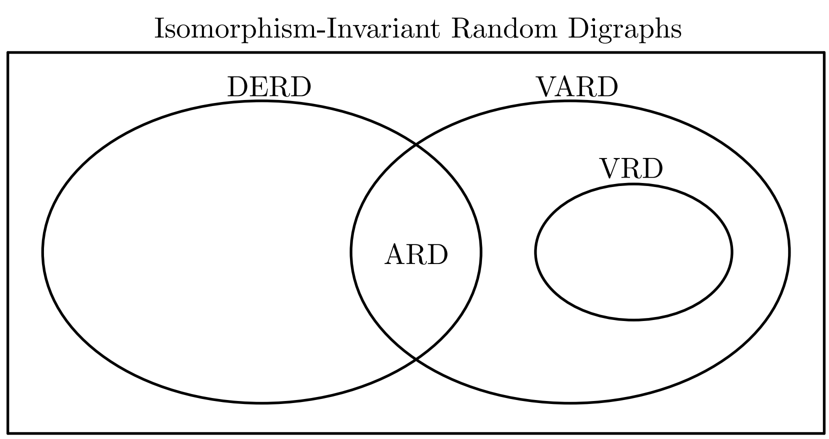

Figure 3. Venn diagram of DERDs and VARDs for $n \geq 4$. The results in this paper indicate that all the five regions in the figure are nonempty. In addition, the intersection of the classes DERDs and VARDs is the family of ARDs, i.e., DERD $\cap$ VARD = ARD.

nonnegative, so the asymptotic distribution of arc (edge) density of VRDs (VRGs) is degenerate if the asymptotic variance is zero or it is the normal distribution. In particular, arc density of proximity catch digraphs (PCDs), which are also VRDs, have asymptotically normal distribution whose variance equals the covariance between having the arc (u, v) and having the arc (u, v') (see [4] and references therein for arc density of PCDs). Hence, positive dependence guarantees that the asymptotic distribution is valid (i.e., has positive variance) or degenerate (i.e., has zero variance) and the covariance cannot be negative. Similarly, the underlying graphs of PCDs are VRGs ([5]), and positive dependence guarantees that the variance of the distribution of its edge density is nonnegative. We also study the RNNDs and show that they are IIRDs, but do not belong to VARDs or DRDs, which is serves as an example that complements of VARDs and DRDs in IIRDs is nonempty.

In Section 3, we introduce the arc random digraphs (ARDs), vertex random digraphs (VRDs) and vertex-arc random digraphs (VARDs). In Section 4, for $n \geq 4$, we prove that there is no random digraph which is both an ARD and a VRD, and there exist VARDs which are neither ARDs nor VRDs. We introduce the direction random digraphs (DRDs), direction-edge random digraphs (DERDs), direction-vertex random digraphs (DVRDs), and direction-vertex-edge random digraphs (DVERDs) in Section 5. We examine the relations of DERDs with ARDs and VARDs in Section 6. In particular, we show that ARDs are the only random digraphs which are both DERD and VARD for $n \geq 4$, and any DERD with $n \leq 3$ is a VARD. We discuss positive dependence and its relation to the distribution of arc and edge densities of VARDs and VERGs, respectively, in Section 7. Discussion and conclusions are provided in Section 8. A list of abbreviations used in the article is provided in Table 1.

2. Preliminaries

We first summarize IIRGs introduced by [1]. The reasons to include this summary are two-fold: (i) The IIRG classification of [1] as ERGs, VRGs, and VERGs provides a nice foundation for classification of IIRDs as ARDs, VRDs, and VARDs. (ii) Adding the direction randomness to ERGs, VRGs, and VERGs give rise to more interesting DRDs, namely, DERDs, DVRDs, and DVERDs.

A *graph* G is a finite nonempty set $V(G)$ of elements called *vertices* together with a set $E(G)$ of unordered pairs of vertices of G called *edges*. An edge $\{u, v\}$ is denoted by uv for convenience in the text. Let $\mathcal{G}_n$ denote the set of all graphs with $V(G) = [n]$ and $2^{\mathcal{G}_n}$ be the set of all subsets of $\mathcal{G}_n$. A *random graph* is a probability space $(\mathcal{G}_n, 2^{\mathcal{G}_n}, P)$, and we write $\mathbf{G} = (\mathcal{G}_n, P)$ where P is a probability measure. We write $P(G)$ instead of $P(\{G\})$ for convenience in notation.

IIRD:	Isomorphism-Invariant Random Digraph	(p. 44)
IIRG:	Isomorphism-Invariant Random Graph	(p. 49)
ERG:	Edge Random Graph	(p. 49)
VRG:	Vertex Random Graph	(p. 49)
VERG:	Vertex-Edge Random Graph	(p. 50)
ARD:	Arc Random Digraph	(p. 51)
GARD:	Generalized Arc Random Digraph	(p. 51)
VRD:	Vertex Random Digraph	(p. 52)
VARD:	Vertex-Arc Random Digraph	(p. 53)
DRD:	Direction Random Digraph	(p. 61)
DERD:	Direction-Edge Random Digraph	(p. 62)
DVRD:	Direction-Vertex Random Digraph	(p. 62)
DVERD:	Direction-Vertex-Edge Random Digraph	(p. 62)
RNND:	Random Nearest Neighbor Digraph	(p. 58)

TABLE 1. A list of abbreviations used in the article together with the page numbers where they are formally defined.

The random graph model was first introduced by [12] and [10]. The model of Gilbert corresponds to ERG $\mathbf{G}(n, p_e)$ in [1] in which each edge is inserted, independent of others, with probability p_e between vertices i and j among a given set of (nonrandom) vertices. The model introduced by Erdős and Rényi is the uniform random graph $\mathbf{G}(n, m)$ which picks a graph with vertex set $[n]$ uniformly at random among the ones with exactly m edges. However, in literature, both of these models are usually called the Erdős–Rényi model as [10] developed the theory.

A graph G_1 is *isomorphic* to a graph G_2 (or G_1 and G_2 are *isomorphic*) if there exists a bijection $f : V(G_1) \to V(G_2)$ such that $uv \in E(G_1)$ if and only if $f(u)f(v) \in E(G_2)$. We say that the random graph $\mathbf{G} = (\mathcal{G}_n, P)$ is *isomorphism-invariant* if $P(G_1) = P(G_2)$ whenever G_1 is isomorphic to G_2. The random graphs which are isomorphism-invariant are called *isomorphism-invariant random graphs* (IIRGs).

Definition 2.1. *An* edge random graph (ERG) *is a random graph* $\mathbf{G}(n, p_e) = (\mathcal{G}_n, P)$ *where* $p_e \in [0, 1]$ *and*

$$P(G) = p_e^{|E(G)|}(1 - p_e)^{\binom{n}{2} - |E(G)|} \text{ for every } G \in \mathcal{G}_n.$$

Let Ω be a set, $\mathbf{x} = (x_1, \ldots, x_n) \in \Omega^n$ and $\phi_e : \Omega \times \Omega \to \{0, 1\}$ be a symmetric function. Then the $(\mathbf{x}, \phi_e)$*-graph*, denoted $G(\mathbf{x}, \phi_e)$, is defined to be the graph, G, with vertex set $[n]$ such that for every $i, j \in [n]$ with $i \neq j$ we have $ij \in E(G)$ if and only if $\phi_e(x_i, x_j) = 1$.

Definition 2.2. *Let* $(\Omega, \mathcal{F}, \mu)$ *be a probability space and* $\phi_e : \Omega \times \Omega \to \{0, 1\}$ *be a symmetric measurable function. The* vertex random graph (VRG),

$\mathbf{G}(n, \Omega, \mu, \phi_e)$, *is the random graph* $(\mathcal{G}_n, P)$ *satisfying*

$$P(G) = \int \mathbf{1}_{\{G(\mathbf{x},\phi_e)=G\}} d(\mu\mathbf{x}) \text{ for every } G \in \mathcal{G}_n,$$

where $d(\mu\mathbf{x})$ *is short-hand for the product integrator* $d(\mu^n(\mathbf{x})) = d(\mu x_1) \cdots d(\mu x_n)$.

In words, ERGs are generated as follows: A fixed set of n vertices is given and edges are inserted randomly and independently with probability p_e between pairs of vertices. On the other hand, for VRGs, n vertices are randomly generated from μ and edges are inserted deterministically between pairs of vertices. Notice that in a VRG the randomness lies in the structure attached to the vertices, and once these random structures have been assigned to the vertices, all the edges are uniquely determined.

Definition 2.3. *Let* $(\Omega, \mathcal{F}, \mu)$ *be a probability space and* $\phi_e : \Omega \times \Omega \to [0, 1]$ *be a symmetric measurable function. The* vertex-edge random graph (VERG), $\mathbf{G}(n, \Omega, \mu, \phi_e)$, *is the random graph* $(\mathcal{G}_n, P)$ *with*

$$P(G) = \int P_{\mathbf{x}}(G) d(\mu\mathbf{x}), \text{ for every } G \in \mathcal{G}_n,$$

where for given $\mathbf{x} = (x_1, \ldots, x_n)$ *and* G

$$P_{\mathbf{x}}(G) = \prod_{ij \in E(G)} \phi_e(x_i, x_j) \times \prod_{ij \notin E(G)} (1 - \phi_e(x_i, x_j)).$$

In words, a VERG is generated as follows: a random sample of size n is drawn with distribution μ from Ω, say $\mathbf{X} = (X_1, \ldots, X_n)$. Then conditional on $\mathbf{X}$, independently for each pair of distinct vertices i and j, the edge ij is inserted with probability $\phi_e(X_i, X_j)$.

Observe that the same notation $\mathbf{G}(n, \Omega, \mu, \phi_e)$ is used for both VRGs and VERGs. However, this causes no confusion, since ϕ_e takes values in $\{0, 1\}$ for VRGs and in $[0, 1]$ for VERGs. In other words, VRGs form a special case of VERGs with ϕ_e taking values only in $\{0, 1\}$. Therefore, every VRG is a VERG. In addition, it is easy to see that letting ϕ_e be identically equal to p gives that every ERG is a VERG.

Let $\mathbf{G_1} = (\mathcal{G}_n, P_1)$ and $\mathbf{G_2} = (\mathcal{G}_n, P_2)$ be random graphs. The *total variation distance* between $\mathbf{G_1}$ and $\mathbf{G_2}$ is defined to be

$$d_{\mathrm{TV}}(\mathbf{G_1}, \mathbf{G_2}) = \frac{1}{2} \sum_{G \in \mathcal{G}_n} |P_1(G) - P_2(G)|.$$

Similarly, for any two random digraphs $\mathbf{D_1} = (\mathcal{D}_n, P_1)$ and $\mathbf{D_2} = (\mathcal{D}_n, P_2)$, the *total variation distance* between $\mathbf{D_1}$ and $\mathbf{D_2}$ is defined to be

$$d_{\mathrm{TV}}(\mathbf{D_1}, \mathbf{D_2}) = \frac{1}{2} \sum_{D \in \mathcal{D}_n} |P_1(D) - P_2(D)|.$$

3. ARDs, VRDs and VARDs

3.1. **Arc random digraphs.** One of the most commonly studied random digraphs is the binomial (or Bernoulli) random digraph model, $\mathbf{D}(n, p_a)$, in which each of the $n(n-1)$ possible arcs is included independently with probability p_a. Such random digraphs give rise to arc random digraphs.

Definition 3.1. *An* arc random digraph (ARD) *is a random digraph* $\mathbf{D}(n, p_a) = (\mathcal{D}_n, P)$ *where* $0 < p_a < 1$ *and*

$$P(D) = p_a^{|A(D)|}(1-p_a)^{n(n-1)-|A(D)|} \quad \textit{for every } D \in \mathcal{D}_n.$$

Notice that *ARDs are the digraph counterparts of random graphs* $\mathbf{G}(n, p_e)$*, i.e., of ERGs.* That is each arc is inserted, independent of others, with probability p_a from vertex i to j among a given set of (nonrandom) vertices. For some asymptotic properties of $\mathbf{D}(n, p_a)$ see [15], [19], and [16].

Definition 3.2. *Let* $\mathrm{p_a} : [n] \times [n] \to [0, 1]$ *be a function (that is not necessarily symmetric in its arguments). The* generalized arc random digraph (GARD)*,* $\mathbf{D}(n, \mathrm{p_a})$*, is the random digraph* $(\mathcal{D}_n, P)$ *with*

$$P(D) = \prod_{(i,j)\in A(D)} \mathrm{p_a}(i, j) \times \prod_{(i,j)\notin A(D)} (1 - \mathrm{p_a}(i, j)) \quad \textit{for every } D \in \mathcal{D}_n.$$

In other words, in a GARD each arc appears independently of others and the arc (i, j) occurs with probability $\mathrm{p_a}(i, j)$. Note that an ARD is special case of a GARD with a constant $\mathrm{p_a}$, i.e., $\mathrm{p_a}(i, j) = p_a$ for all i, j. As the classical random digraph model $\mathbf{D}(n, p_a)$ may not fit real life networks, inhomogeneous models like GARDs are of interest for such scenarios (see, e.g., [3]).

Clearly, any ARD is isomorphism-invariant. The following proposition implies that a GARD is isomorphism-invariant if and only if it is an ARD.

Proposition 3.3. *Let* $\mathbf{D}$ *be an isomorphism-invariant GARD. Then* $\mathbf{D} = \mathbf{D}(n, p_a)$ *for some* p_a*, i.e.,* $\mathrm{p_a}(i, j) = p_a$ *for all* i, j *and* $\mathbf{D}$ *is an ARD.*

Proof. We show that $\mathrm{p_a}(i, j) = \mathrm{p_a}(k, l) = p_a$ for any two ordered pairs (i, j) and (k, l). First note that

$$\mathrm{p_a}(i, j) = P((i, j) \in A(\mathbf{D})) = \sum_{D:(i,j)\in A(D)} P(D). \tag{3.1}$$

Fix a permutation on $[n]$ which maps i to k and j to l. Observe that this permutation induces a one-to-one correspondence between the sets $\{D \in \mathcal{D}_n : (i, j) \in A(D)\}$ and $\{D' \in \mathcal{D}_n : (k, l) \in A(D')\}$ such that matched digraphs are isomorphic. As $\mathbf{D}$ is isomorphism-invariant, this correspondence implies

$$\sum_{D:(i,j)\in A(D)} P(D) = \sum_{D':(k,l)\in A(D')} P(D'). \tag{3.2}$$

Hence, the result follows by (3.1) and (3.2). □

3.2. **Vertex Random Digraphs.** Let Ω be a set, $\mathbf{x} = (x_1, \ldots, x_n) \in \Omega^n$ and $\phi_a : \Omega \times \Omega \to \{0, 1\}$ be a function. Then the $(\mathbf{x}, \phi_a)$-*digraph*, denoted $D(\mathbf{x}, \phi_a)$, is defined to be the digraph, D, with vertex set $[n]$ such that for all $i, j \in [n]$ with $i \neq j$ we have

$$(i, j) \in A(D) \text{ if and only if } \phi_a(x_i, x_j) = 1.$$

Clearly, every digraph D with $V(D) = [n]$ is an $(\mathbf{x}, \phi_a)$-digraph for some choice of Ω, $\mathbf{x}$, and ϕ_a. More specifically, choose $\mathbf{x}$ to be the identity function on $\Omega = [n]$ and define $\phi_a(i, j) = \mathbf{1}_{\{(i,j) \in A(D)\}}$ where $\mathbf{1}_{\{.\}}$ is the indicator function.

Definition 3.4. *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and $\phi_a : \Omega \times \Omega \to \{0, 1\}$ be a measurable function. The* vertex random digraph (VRD)*, $\mathbf{D}(n, \Omega, \mu, \phi_a)$, is the random digraph $(\mathcal{D}_n, P)$ with*

$$P(D) = \int \mathbf{1}_{\{D(\mathbf{x}, \phi_a) = D\}} d(\mu\mathbf{x}) \textit{ for every } D \in \mathcal{D}_n.$$

So, in a VRG, n vertices are drawn i.i.d. from μ and then arcs are inserted between vertices based on a binary relation in a deterministic fashion. Note that in a VRD the randomness resides in the structure attached to the vertices, as in VRGs, and when these random structures are assigned to the vertices, all the arcs are uniquely determined. That is, *VRDs constitute the digraph counterpart for VRGs.*

Example 3.5 (Proximity catch digraphs (PCDs) [6])**.** *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. The* proximity map *$N(\cdot)$ is a function from Ω to $\mathcal{F}$. The* proximity region *associated with $x \in \Omega$, denoted $N(x)$, is the image of $x \in \Omega$ under $N(\cdot)$. The points in $N(x)$ are thought of as being "closer" to $x \in \Omega$ than the points in $\Omega \setminus N(x)$. For a given $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ the* proximity catch digraph *is the digraph with the vertex set $V = [n]$ and the arc set $A = \{(i, j) : x_j \in N(x_i)\}$. In other words, we insert the arc (i, j) if and only if x_j is in the proximity region of x_i. Note that for a given $N(\cdot)$, a random PCD is a VRD, $\mathbf{D}(n, \Omega, \mu, \phi_a)$, with $\phi_a(x_i, x_j) = \mathbf{1}_{\{x_j \in N(x_i)\}}$. For instance, one can take $\Omega = \mathbb{R}$, $N(x) = (x - r, x + r)$ and $\phi_a(x, y) = \mathbf{1}_{\{x - r \leq y \leq x + r\}}$ for some $r > 0$.*

Example 3.6 (Random intersection digraphs [2])**.** *Let n and m be positive integers, and μ be a distribution on $2^{[m]} \times 2^{[m]}$ (ordered pairs of subsets of $[m]$). Given two collections of subsets $S_1, \ldots, S_n$ and $T_1, \ldots, T_n$ of the set $[m]$, define the intersection digraph with vertex set $[n]$ such that the arc (i, j) is present in the digraph whenever $S_i \cap T_j$ is nonempty for $i \neq j$. $\mathbf{D}(n, m, \mu)$ is the random intersection digraph generated by independent and identically distributed pairs of random subsets (S_i, T_i) under μ, $1 \leq i \leq n$. Note that $\mathbf{D}(n, m, \mu)$ is a VRD with $\Omega = 2^{[m]} \times 2^{[m]}$ and $\phi_a((S, T), (S', T')) = \mathbf{1}_{\{S \cap T' \neq \emptyset\}}$.*

By letting $\Omega = [0, 1]$, μ be the uniform distribution over $[0, 1]$, and $\phi_a(x, y) = \mathbf{1}_{\{x \leq p_a\}}$, we see that every $\mathbf{D}(2, p_a)$ is a VRD.

Recall that in a VRD, $\mathbf{D}(n, \Omega, \mu, \phi_a)$, ϕ_a is not required to be symmetric. However, if ϕ_a is a symmetric function, whenever we see the arc (i, j) in $A(\mathbf{D})$, we see the arc (j, i) as well. In this case, for every $D \in \mathcal{D}_n$ in which there exists $(i, j) \in A(D)$ with $(j, i) \notin A(D)$, we have $P(D) = 0$. On the other hand, in an ARD, $\mathbf{D}(n, p_a)$, we have $P(D) > 0$ for every $D \in \mathcal{D}_n$. Therefore, whenever ϕ_a is symmetric and nonconstant μ^2-a.s., $\mathbf{D}(n, \Omega, \mu, \phi_a)$ is not an ARD. For instance, one can take $\Omega = \mathbb{R}^d$, μ to be an a.e. continuous distribution and $\phi_a(x, y) = \mathbf{1}_{\{||x-y||_d \leq r\}}$, where $|| \cdot ||_d$ is the usual Euclidean norm in $\mathbb{R}^d$ and r is a fixed positive real number. Notice that these random digraphs are random PCDs in which $N(x)$ is the closed ball with radius r and center x. If we consider symmetric arcs as one edge only, this type of random digraph reduces to what are called *random geometric graphs*. For more information on random geometric graphs, see [21].

3.3. **Vertex-arc random digraphs.** We now generalize the random digraphs introduced in the previous two subsections by combining the structures where the randomness lies.

Definition 3.7. *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and $\phi_a : \Omega \times \Omega \to [0, 1]$ be a measurable function. The* vertex-arc random digraph (VARD)*, $\mathbf{D}(n, \Omega, \mu, \phi_a)$, is the random digraph $(\mathcal{D}_n, P)$ with*

$$P(D) = \int P_{\mathbf{x}}(D) d(\mu\mathbf{x}), \text{ for every } D \in \mathcal{D}_n,$$

where for given $\mathbf{x} = (x_1, \ldots, x_n)$ and $D = (V, A)$

$$P_{\mathbf{x}}(D) = \prod_{(i,j)\in A} \phi_a(x_i, x_j) \times \prod_{(i,j)\notin A} (1 - \phi_a(x_i, x_j)).$$

The construction of a VARD is very similar to that of VERG as *VARDs are the digraph counterpart of VERGs.* A random sample of size n is drawn with distribution μ from Ω, say $\mathbf{X} = (X_1, \ldots, X_n)$, and then conditional on $\mathbf{X}$, independently for each ordered pair of distinct vertices i and j, the arc (i, j) is inserted with probability $\phi_a(X_i, X_j)$.

Note that we use the same notation $\mathbf{D}(n, \Omega, \mu, \phi_a)$ for both VRDs and VARDs. But, since ϕ_a only takes values 0 or 1 for VRDs and in $[0, 1]$ for VARDs, this causes no confusion. Particularly, VRDs form a special case of VARDs with ϕ_a taking values only in $\{0, 1\}$. Therefore, every VRD is a VARD. Moreover, it is easy to verify that letting ϕ_a be identically equal to p_a gives that every ARD is a VARD.

Proposition 3.8. *Every VARD is isomorphism-invariant.*

Proof. Let $\mathbf{D}(n, \Omega, \mu, \phi_a)$ be a VARD and $D, D' \in \mathcal{D}_n$ be isomorphic digraphs. Then there exists a permutation σ on $[n]$ such that

$$(i, j) \in A(D) \ \Leftrightarrow \ (\sigma(i), \sigma(j)) \in A(D').$$

Let σ^{-1} be the inverse of σ and $\mathbf{y} = (y_1, \ldots, y_n)$ such that $y_i = x_{\sigma^{-1}(i)}$ for all $1 \le i \le n$, i.e., $x_i = y_{\sigma(i)}$ for all $1 \le i \le n$. Then note that

$$\begin{aligned}
P_{\mathbf{x}}(D) &= \prod_{(i,j)\in A(D)} \phi_a(x_i, x_j) \times \prod_{(i,j)\notin A(D)} (1 - \phi_a(x_i, x_j)) \\
&= \prod_{(i,j)\in A(D)} \phi_a(y_{\sigma(i)}, y_{\sigma(j)}) \times \prod_{(i,j)\notin A(D)} (1 - \phi_a(y_{\sigma(i)}, y_{\sigma(j)})) \\
&= \prod_{(\sigma(i),\sigma(j))\in A(D')} \phi_a(y_{\sigma(i)}, y_{\sigma(j)}) \times \prod_{(\sigma(i),\sigma(j))\notin A(D')} (1 - \phi_a(y_{\sigma(i)}, y_{\sigma(j)})) \\
&= \prod_{(i,j)\in A(D')} \phi_a(y_i, y_j) \times \prod_{(i,j)\notin A(D')} (1 - \phi_a(y_i, y_j)) \\
&= P_{\mathbf{y}}(D'). \qquad (3.3)
\end{aligned}$$

As $\mathbf{y}$ is a permutation of $\mathbf{x}$, Fubini's theorem and (3.3) imply that

$$P(D) = \int P_{\mathbf{x}}(D)\mu(d\mathbf{x}) = \int P_{\mathbf{y}}(D')\mu(d\mathbf{y}). \qquad (3.4)$$

Furthermore, the change of variables that maps y_i to x_i in the integrand above results

$$\int P_{\mathbf{y}}(D')\mu(d\mathbf{y}) = \int P_{\mathbf{x}}(D')\mu(d\mathbf{x}) = P(D'), \qquad (3.5)$$

since the mapping is a permutation and the Jacobian of a permutation matrix is ± 1. Thus, the results in (3.4) and (3.5) together imply that $P(D) = P(D')$, and so the desired result follows. □

As a corollary, we easily see that any VRD is isomorphism-invariant since every VRD is a VARD, and the same holds for ARDs as well.

4. Inclusion/exclusion relations between ARDs, VRDs and VARDs

In the previous section, we have shown that every ARD and VRD is a VARD, and every VARD is isomorphism-invariant. In this section we prove that, for $n \ge 4$, there exists no random digraph which is both ARD and VRD, and the union of the classes ARDs and VRDs is not the entire class of VARDs.

The following theorem implies that the families ARDs and VRDs are disjoint for $n \ge 4$.

Theorem 4.1. *If an ARD, $\mathbf{D}(n, p_a)$, with $n \ge 4$ is represented as a VARD, $\mathbf{D}(n, \Omega, \mu, \phi_a)$, then $\phi_a(x, y) = p_a$ μ^2-a.s.*

Proof. Suppose that an ARD, $\mathbf{D}(n, p_a)$, with $n \ge 4$ is represented as a VARD, $\mathbf{D}(n, \Omega, \mu, \phi_a)$. For the proof of the theorem, we borrow some tools from functional analysis which are presented in the proof of Theorem 4.2 in

[1]. Let $h : \Omega \times \Omega \to [0, 1]$ be a symmetric measurable function and T be the integral operator with kernel h on the space $L^2(\Omega, \mu)$ of μ-square-integrable functions on Ω:

$$(Tg)(x) = \int h(x, y)g(y)d(\mu y).$$

Since h is bounded and μ is a finite measure, the kernel h is in $L^2(\mu \times \mu)$. Integral operators with such kernels are Hilbert–Schmidt operators and are thus compact operators. Moreover, as h is symmetric, the integral operator T is self-adjoint, which implies that $L^2(\Omega, \mu)$ has an orthonormal basis $(\psi_i)_{i\geq 1}$ of eigenfunctions for T such that $T\psi_i = \lambda_i\psi_i$ for not necessarily distinct real eigenvalues λ_i with $\lambda_i \to 0$ as $i \to \infty$ (see Chapter VI in [23]). We may assume that λ_1 is the largest eigenvalue. Then we have

$$h(x, y) = \sum_{i\geq 1} \lambda_i\psi_i(x)\psi_i(y) \ \mu^2\text{-a.s.}$$

with the sum converging in L^2. As ψ_i's are orthonormal, it follows that

$$\begin{aligned} &\mathbf{E}(h(X_1, X_2)h(X_2, X_3)h(X_3, X_4)h(X_4, X_1)) \\ &= \iiiint h(x_1, x_2)h(x_2, x_3)h(x_3, x_4)h(x_4, x_1) \\ &\quad \times d(\mu x_1)d(\mu x_2)d(\mu x_3)d(\mu x_4) \\ &= \sum_{i\geq 1} \lambda_i^4. \end{aligned} \tag{4.1}$$

In the proof of Theorem 4.2 of [1], it was natural to take $h(x, y)$ to be the edge probability $\phi_e(x, y)$ in VERGs, however, the arc probability $\phi_a(x, y)$ for VARDs is not symmetric in its arguments, hence a direct extension by taking $h(x, y) = \phi_a(x, y)$ would cause many difficulties (see Remark 1). Instead, to tackle the asymmetry issue, we take $h(x, y)$ to be symmetric functions of its arguments as $\phi_a(x, y)\phi_a(y, x)$ or $(1 - \phi_a(x, y))1 - \phi_a(y, x))$ (see below).

Now let E_1 be the event that both $(1, 2)$ and $(2, 1)$ are in $A(\mathbf{D})$. As $\mathbf{D}$ is an ARD, $\mathbf{D}(n, p_a)$, it is easy to see that $P(E_1) = p_a^2$. On the other hand, since $\mathbf{D}$ is represented as a VARD, $\mathbf{D}(n, \Omega, \mu, \phi_a)$, we have

$$P(E_1) = \mathbf{E}(\phi_a(X_1, X_2)\phi_a(X_2, X_1)).$$

Thus, letting $h(x, y) = \phi_a(x, y)\phi_a(y, x)$ gives $p_a^2 = \mathbf{E}(h(X_1, X_2))$. As

$$\mathbf{E}(h(X_1, X_2)) = \iint h(x, y)d(\mu x)d(\mu y) = \langle T\mathbf{1}, \mathbf{1}\rangle \leq \lambda_1,$$

we get $p_a^2 \leq \lambda_1$, where $\mathbf{1}$ is the function with constant value 1.

Let E_2 be the event that $(1, 2), (2, 1), (2, 3), (3, 2), (3, 4), (4, 3), (4, 1), (1, 4)$ are all in $A(\mathbf{D})$. Since $\mathbf{D}$ is an ARD, $\mathbf{D}(n, p_a)$, it is easy to see that

$$P(E_2) = p_a^8. \tag{4.2}$$

By the representation of $\mathbf{D}$ as a VARD, $\mathbf{D}(n, \Omega, \mu, \phi_a)$, we also have

$$P(E_2) = \mathbf{E}(h(X_1, X_2)h(X_2, X_3)h(X_3, X_4)h(X_4, X_1)). \tag{4.3}$$

Now combining the results in (4.1), (4.2), and (4.3) gives

$$p_a^8 = \sum_{i\geq 1} \lambda_i^4. \tag{4.4}$$

Since $p_a^2 \leq \lambda_1$, we have $p_a^8 \leq \lambda_1^4$ and thus, by (4.4) we obtain that $\lambda_1 = p_a^2$ and $\lambda_i = 0$ for every $i \geq 2$, that is $h(x,y) = p_a^2 \psi_1(x)\psi_1(y)$. But then we have

$$\begin{aligned} p_a^2 \int \psi_1^2(x) d(\mu x) = p_a^2 = \mathbf{E}(h(X_1, X_2)) \\ &= p_a^2 \iint \psi_1(x)\psi_1(y) d(\mu x) d(\mu y) \\ &= p_a^2 \left(\int \psi_1(x) d(\mu x) \right)^2, \end{aligned}$$

which implies that

$$\int \psi_1^2(x) d(\mu x) = \left(\int \psi_1(x) d(\mu x) \right)^2, \tag{4.5}$$

since $p_a \neq 0$. As the equality in equation (4.5) is the equality in the Cauchy–Schwarz inequality for ψ_1 and $\mathbf{1}$, we see that ψ_1 is constant μ-a.s. Since $\int \psi_1^2(x) d(\mu x) = 1$, we get $\psi_1 = 1$ μ-a.s. or $\psi_1 = -1$ μ-a.s., and therefore $h(x,y) = p_a^2$ μ^2-a.s., that is

$$\phi_a(x,y)\phi_a(y,x) = p_a^2 \ \mu^2\text{-a.s.} \tag{4.6}$$

Next, let E_1' be the event that neither of the arcs $(1,2)$ and $(2,1)$ is in $A(\mathbf{D})$, and E_2' be the event that none of the arcs $(1,2), (2,1), (2,3), (3,2), (3,4)$, $(4,3)$,$(4,1)$,$(1,4)$ is in $A(\mathbf{D})$. Choosing $h(x,y)$ to be $(1-\phi_a(x,y))(1-\phi_a(y,x))$ allows us to follow the same arguments above for E_1 and E_2 replaced with E_1' and E_2', respectively, and with $1-p_a$ taking place of p_a. Therefore, we obtain that

$$(1-\phi_a(x,y))(1-\phi_a(y,x)) = (1-p_a)^2 \ \mu^2\text{-a.s.} \tag{4.7}$$

Finally, the equations in (4.6) and (4.7) give the desired result. □

Remark. The function h in the proof of Theorem 4.1 is taken to be symmetric. Otherwise, if we had taken the kernel h to be ϕ_a, the operator T need not be self-adjoint, the eigenvalues would be complex and the eigenfunctions would not be orthogonal and hence the succeeding arguments in the proof would not be true. So, h being symmetric is a crucial condition for the proof. Such an issue does not come up for VERGs, as ϕ_e is symmetric, hence one can take $h = \phi_e$ and obtain $\phi_e = p_e$ μ^2-a.s. (as in the proof of Theorem 4.2. in [1]). However, in our case, ϕ_a is not symmetric as we are dealing with digraphs. Hence we tackle this hurdle by taking $h(x,y)$ to be $\phi_a(x,y)\phi_a(y,x)$ and $(1-\phi_a(x,y))(1-\phi_a(y,x))$, respectively, and obtain the desired result. □

As any VRD is a VARD with ϕ_a taking values in $\{0,1\}$, by Theorem 4.1 we have the following corollary.

Corollary 4.2. *Any ARD,* $\mathbf{D}(n,p_a)$*, with* $n \geq 4$ *is not a VRD.*

4.1. **All regions in Figure 1 are nonempty.** Clearly, ARDs and VRDs are nonempty. We next show that union of ARD and VRD families do not constitute the entire class of VARDs when $n \geq 4$.

Theorem 4.3. *There exist VARDs with* $n \geq 4$ *which are neither a VRD nor an ARD.*

Proof. Let $0 < \alpha < \beta < 1$ be real numbers. Consider a VARD, $\mathbf{D}(n,\Omega,\mu,\psi_a)$, with $n \geq 4$ such that $\psi_a(x,y) \in \{\alpha,\beta\}$ and $\psi_a(x,y) \neq \psi_a(y,x)$ for any $x \neq y$. Equivalently, we have

$$\psi_a(x,y)\psi_a(y,x) = \alpha\beta \tag{4.8}$$

and

$$(1-\psi_a(x,y))(1-\psi_a(y,x)) = (1-\alpha)(1-\beta) \tag{4.9}$$

for $x \neq y$. For example, one can take $\Omega = \mathbb{R}$, μ to be a continuous distribution and $\psi_a(x,y) = \alpha\mathbf{1}_{\{x\leq y\}} + \beta\mathbf{1}_{\{y<x\}}$.

Now suppose that it has another VARD representation $\mathbf{D}(n,\Omega',\nu,\phi_a)$. We claim that ϕ_a satisfies the same properties of ψ_a given in (4.8) and (4.9) ν^2-a.s. Recall that in the proof of Theorem 4.1, the properties of an ARD, $\mathbf{D}(n,p_a)$, that we used are

$$P(E_2) = p_a^8 = (p_a^2)^4 = (P(E_1))^4 \tag{4.10}$$

and

$$P(E_2') = (1-p_a)^8 = ((1-p_a)^2)^4 = (P(E_1'))^4. \tag{4.11}$$

Notice that the equalities in (4.10) and (4.11) hold for $\mathbf{D}(n,\Omega,\mu,\psi_a)$ when p_a^2 and $(1-p_a)^2$ are replaced with $\alpha\beta$ and $(1-\alpha)(1-\beta)$, respectively. That is, for $\mathbf{D}(n,\Omega,\mu,\psi_a)$ we have

$$P(E_2) = (\alpha\beta)^4 = (P(E_1))^4 \text{ and } P(E_2') = ((1-\alpha)(1-\beta))^4 = (P(E_1'))^4.$$

Therefore, following the same arguments in the proof of Theorem 4.1 we obtain

$$\phi_a(x,y)\phi_a(y,x) = \alpha\beta$$

and

$$(1-\phi_a(x,y))(1-\phi_a(y,x)) = (1-\alpha)(1-\beta)\ \nu^2\text{-a.s.}$$

and hence the claim follows. Then, by the choice of α,β, and ψ_a, we see that $\mathbf{D}(n,\Omega,\mu,\psi_a)$ has neither an ARD nor a VRD representation. □

Recall that Corollary 4.2 and Theorem 4.3 imply that for $n \geq 4$, we have $\text{ARD} \cap \text{VRD} = \emptyset$ and $\text{VARD}\backslash(\text{ARD} \cup \text{VRD}) \neq \emptyset$, respectively.

Remark (Approximation to VARDs by VRDs). However, any VARD can be arbitrarily closely approximated by VRDs. That is, for any VARD, $\mathbf{D}$, and $\epsilon > 0$, there exists a VRD, $\mathbf{D}'$, such that $d_{\text{TV}}(\mathbf{D}, \mathbf{D}') < \epsilon$. This result is a straightforward extension of approximation of VERGs by VRGs which follows by the idea in the proof of Theorem 3.3 in [1]. Let $\mathbf{D}$ be a VARD, $\mathbf{D}(n, \Omega, \mu, \phi_a)$. Let M be a positive integer satisfying $n^2/\epsilon < M$, and $\Omega' = \Omega \times [0,1]^M \times [M]$. Endow Ω' with the product measure of its factors, i.e., independently pick $x \in \Omega$ with respect to μ, an M-tuple $f \in [0,1]^M$ uniformly, and $a \in [M]$ uniformly. Denote this measure by ν. We next define $\psi_a : \Omega' \times \Omega' \to \{0,1\}$ as follows: For every $y_1 = (x_1, f_1, a_1), y_2 = (x_2, f_2, a_2) \in \Omega'$, let $\psi_a(y_1, y_2)$ be the indicator of the event $\phi_a(x_1, x_2) \geq f_1(a_2)$. Consider the random digraph $\mathbf{D}'$ with VRD representation $\mathbf{D}(n, \Omega', \nu, \psi_a)$. By following the same arguments in the proof of Theorem 3.3 in [1], one can easily obtain that $d_{\text{TV}}(\mathbf{D}, \mathbf{D}') < n^2/M < \epsilon$. □

The complement of VARDs in the set of IIRDs in Figure 1 is also nonempty (similarly, the complement of VERGs in the classification of IIRGs in [1] is nonempty). Next we provide an important digraph family as an example for IIRDs that are not VARDs.

4.2. Random nearest neighbor digraphs are IIRD but not VARD.

Random nearest neighbor digraphs (RNNDs) are one of the most commonly studied random digraphs (e.g., see [11], [9], [8], and [22]). Let $n \geq 3$, $k \geq 1$, and $d \geq 1$ be integers with $k < n - 1$. Let μ be a probability distribution over $\mathbb{R}^d$ with density function f that is assumed to be continuous almost everywhere with respect to the Lebesgue measure. Let $|\cdot|$ denote a fixed norm on $\mathbb{R}^d$ and $X = (X_1, \dots, X_n)$ be an i.i.d. vector in $\mathbb{R}^d$ drawn from μ.

For given $\mathbf{x} = (x_1, \dots, x_n)$, the set of k *nearest neighbors* (kNNs) of x_i is the closest k points to x_i among the points $\{x_1, \dots, x_n\} \setminus \{x_i\}$ with respect to the given norm $|\cdot|$ and denoted as $k\text{NN}_{\mathbf{x}}(x_i)$. As the occurrence of a tie is an event with zero probability for points from an a.e. continuous f, we may assume that $k\text{NN}_{\mathbf{x}}(x_i)$ is well-defined for each i with probability 1. The *k nearest neighbor digraph* of $\mathbf{x}$ is the digraph with vertex set $V = [n]$ and the arc set $A = \{(i,j) : x_j \in k\text{NN}_{\mathbf{x}}(x_i)\}$, (i.e., the arc (i,j) is inserted if and only if x_j is one of the kNNs of x_i) and denoted as $k\text{NND}(\mathbf{x})$.

Definition 4.4. *The* random nearest neighbor digraph (RNND) *is the random digraph* $\mathbf{D}(n, [k], d, \mu, |\cdot|)$ *with*

$$P(D) = \int \mathbf{1}_{\{kNND(\mathbf{x})=D\}} d(\mu\mathbf{x}) \text{ for every } D \in \mathcal{D}_n.$$

Notice that we picked k to be less than $n-1$, because otherwise, we obtain a degenerate random digraph.

Proposition 4.5. *Every RNND is isomorphism-invariant.*

Proof. Let $\mathbf{D}(n, [k], d, \mu, |\cdot|)$ be a RNND and $D, D' \in \mathcal{D}_n$ be isomorphic digraphs. Then there exists a permutation σ on $[n]$ such that

$$(i, j) \in A(D) \;\Leftrightarrow\; (\sigma(i), \sigma(j)) \in A(D').$$

Let σ^{-1} be the inverse of σ and $\mathbf{y} = (y_1, \ldots, y_n)$ such that $y_i = x_{\sigma^{-1}(i)}$ for all $1 \le i \le n$, i.e., $y_{\sigma(i)} = x_i$. Then it is easy to see that

$$k\mathrm{NND}(\mathbf{x}) = D \;\Leftrightarrow\; k\mathrm{NND}(\mathbf{y}) = D'.$$

The rest of the proof is similar to that of Proposition 3.8. □

As in VRDs, in the construction of a RNND, once $\mathbf{x}$ is fixed, then the arcs are uniquely determined. However, in a VRD, by definition, inserting the arc (i, j) only depends on x_i and x_j whereas in a RNND it depends on all the data points. The following proposition implies that a RNND is not a VRD.

Proposition 4.6. *A RNND is not a VARD.*

Proof. We show that no RNND has a VARD representation. In any VARD, $\mathbf{D}(n, \Omega, \mu, \phi_a)$, with $n \ge 3$ we have

$$\begin{aligned}
P(\{(1,2),(1,3)\} \subset A(\mathbf{D})) &= \iiint \phi_a(x_1, x_2)\phi_a(x_1, x_3) d(\mu x_1) d(\mu x_2) d(\mu x_3) \\
&= \int \left(\int \phi_a(x_1, x_2) d(\mu x_2) \right) \\
&\quad \times \left(\int \phi_a(x_1, x_3) d(\mu x_3) \right) d(\mu x_1) \\
&= \int \left(\int \phi_a(x_1, x_2) d(\mu x_2) \right)^2 d(\mu x_1) \\
&\ge \left(\iint \phi_a(x_1, x_2) d(\mu x_1) d(\mu x_2) \right)^2 \\
&= (P((1,2) \in A(\mathbf{D})))^2 \qquad (4.12)
\end{aligned}$$

by Fubini's theorem and the Cauchy–Schwarz inequality applied to the constant function $\mathbf{1}$ and $\int \phi_a(x_1, x_2) d(\mu x_2)$.

On the other hand, in a RNND, we have

$$\begin{aligned}
P(\{(1,2),(1,3)\} \subset A(\mathbf{D})) = \frac{k(k-1)}{(n-1)(n-2)} &< \left(\frac{k}{n-1} \right)^2 \\
&= (P((1,2) \in A(\mathbf{D})))^2 \qquad (4.13)
\end{aligned}$$

by symmetry, and hence the result follows by (4.12) and (4.13). □

Note that Proposition 4.6 implies that the region VARD^c in Figure 1 is nonempty.

The *underlying graph* of a digraph D, denoted $U(D)$, is the graph obtained by replacing each arc of D with an edge, disallowing multiple edges between two vertices ([7]).

Definition 4.7. *The* underlying random graph *of a random digraph* $\mathbf{D} = (\mathcal{D}_n, P_{\mathbf{D}})$ *is the random graph* $\mathbf{G} = (\mathcal{G}_n, P_{\mathbf{G}})$ *such that*

$$P_{\mathbf{G}}(G) = \sum_{U(D)=G} P_{\mathbf{D}}(D) \text{ for every } G \in \mathcal{G}_n.$$

For example, the underlying random graph of an ARD, $\mathbf{D}(n, p_a)$, is an ERG, namely, $\mathbf{G}(n, p_e)$ with $p_e = 2p_a - p_a^2$. Moreover, notice also that the underlying random graph of a VARD, $\mathbf{D}(n, \Omega, \mu, \phi_a)$, is the VERG, $\mathbf{G}(n, \Omega, \mu, \phi_u)$, where $\phi_u(x, y) = \phi_a(x, y) + \phi_a(y, x) - \phi_a(x, y)\phi_a(y, x)$. In particular, the underlying random graph of a VRD is a VRG.

Remark (Underlying graph of a RNND is not a VERG). For any set of points in $\mathbb{R}^d$, the number of points sharing a common kNN is bounded above by a constant which is independent of the number of points in the set (see, [25]). That is, there exists a number c which only depends on d, k, and the norm $|\cdot|$ such that in any kNND a vertex is the head of at most c arcs. Therefore, a vertex of the underlying graph of a kNND is incident to at most $c + k$ edges. Hence, if $\mathbf{G}$ is the underlying random graph of a RNND with $n \geq c+k+2$, then we have $P(\{\{1, 2\}, \{1, 3\}, \ldots, \{1, n\}\} \subset E(\mathbf{G})) = 0$ which implies that $\mathbf{G}$ is not a VERG since that probability is always positive in a nondegenerate VERG (see the inequality in (7)). □

Remark (Other NN type random digraphs). One can also generate NN type random digraphs different from RNNDs. For instance, in the construction of a RNND, insert the arc (i, j) if and only if x_j is the kth NN of x_i (i.e., insert only the one to its kth NN instead of putting arcs from each point to its all kNNs). We can generalize RNNDs to $\mathbf{D}(n, S_k, d, \mu, |\cdot|)$ where S_k is a nonempty subset of $[k]$ and we insert the arc (i, j) if and only if x_j is the sth NN of x_i for some $s \in S_k$. Then the results for RNNDs in this section are also valid for any $\mathbf{D}(n, S_k, d, \mu, |\cdot|)$, i.e., every $\mathbf{D}(n, S_k, d, \mu, |\cdot|)$ is isomorphism-invariant, has no VARD representation, and for large n, has an underlying random graph which is not a VERG. □

For $n = 3$, the only possible value of k is 1. In this case, the pair with the minimum distance are NNs of each other and the NN of the remaining point is one of the points in this pair. Thus, by symmetry we have $P(A(\mathbf{D}) = \{(i, j), (j, i), (k, i)\}) = 1/6$ for every pairwise distinct $i, j, k \in \{1, 2, 3\}$, and therefore any RNND, $\mathbf{D}(3, 1, d, \mu, |\cdot|)$, is a uniform distribution over six digraphs independent of d, μ and $|\cdot|$. Also, note that the underlying random graph of $\mathbf{D}(3, 1, d, \mu, |\cdot|)$ is always $\mathbf{G}(3, 2)$.

Remark (RNND and $\mathbf{D}(n, nk)$ are not the same). Observe that any RNND and $\mathbf{D}(n, nk)$ have the same number of arcs. However, these two random digraphs are different. Because, for the event $E = \{\{(1, 2), \ldots, (1, k + 2)\} \subset A(\mathbf{D})\}$, we have $P(E) = 0$ in a RNND, since each vertex is tail of exactly k arcs, whereas $P(E) > 0$ in $\mathbf{D}(n, nk)$ since $nk \geq k+1$. Note that the number of edges in the underlying graph of a kNND is nk minus the number of symmetric arcs. It is easy to see that for $n > 3$, there exist kNNDs with different number of symmetric arcs, and therefore the underlying random graph of a RNND with $n > 3$ is not a $\mathbf{G}(n, m)$. □

5. Direction Random Digraphs

One can also obtain IIRDs by first generating an IIRG and then assigning directions randomly to each edge. Hence the IIRG classification of [1] is also useful in the generation process of DRDs. Along this line, we first generate an IIRG, $\mathbf{G} = (\mathcal{G}_n, P_{\mathbf{G}})$, and then for each edge $ij \in E(\mathbf{G})$, independent of other edges, pick a one sided or two sided direction randomly between i and j. For a given direction probability $1/2 \leq p_d < 1$, we put only the arc (i, j) with probability $1 - p_d$, only the arc (j, i) with probability $1 - p_d$, and both of the arcs with probability $2p_d - 1$ (i.e., there is a symmetric arc between vertices i and j with probability $2p_d - 1$ and the arc (i, j) avoiding the reverse arc is inserted with probability $1 - p_d$ and the same holds for arc (j, i)). Observe that the arc (i, j) is put with probability $1 - p_d + 2p_d - 1 = p_d$ (i.e., arc (i, j) exists as the only arc between i and j and also when there is a symmetric arc between vertices i and j). Also, note that we omit the case $p_d = 1$ as it removes randomness in direction.

For a digraph $D \in \mathcal{D}_n$, let $n_a(D) = |A(D)|$ and $n_e(D) = |E(U(D))|$ (i.e., the number of edges of the underlying graph of D). Also, let $n_s(D)$ denote the number of pairs of vertices i and j such that both (i, j) and (j, i) are in $A(D)$ (i.e., the number of symmetric arcs in D), and $n_{as}(D)$ denote the number of arcs (i, j) in $A(D)$ with $(j, i) \notin A(D)$. We write n_a, n_e, n_s, and n_{as}, respectively, dropping the digraph D in the notation for brevity. Note that $n_e = n_s + n_{as}$ and $n_a = 2n_s + n_{as}$, hence $n_a = n_e + n_s$.

Definition 5.1. *Let* $\mathbf{G} = (\mathcal{G}_n, P_{\mathbf{G}})$ *be an IIRG and* $1/2 \leq p_d < 1$. *A* direction random digraph (DRD) *is a random digraph* $\mathbf{D} = (\mathcal{D}_n, P)$ *with*

$$P(D) = P_{\mathbf{G}}(U(D))(1 - p_d)^{n_{as}}(2p_d - 1)^{n_s} \text{ for every } D \in \mathcal{D}_n,$$

and we say that $\mathbf{D}$ *is* generated by $\mathbf{G}$ *with* direction probability p_d.

A natural question is why not start with any nonrandom graph and insert directions randomly to the edges to obtain DRDs. There is a simple answer to the question. Unfortunately, if directions are randomly inserted to the edges of a (fixed) graph, the resulting random digraph is not isomorphism-invariant unless we start with an *empty graph* (the graph with no edges) or a *complete graph* (the graph with all possible edges). Notice that the underlying random graph of a DRD generated by $\mathbf{G}$ is $\mathbf{G}$ itself. Observe that if the digraphs D_1 and D_2 are isomorphic, then so are the (underlying) graphs $U(D_1)$ and $U(D_2)$, and we also have $n_s(D_1) = n_s(D_2)$ and $n_{as}(D_1) = n_{as}(D_2)$. Thus, a DRD is isomorphism-invariant only if it is generated by an IIRG. Moreover, notice also that we may consider a (fixed) graph as a degenerate random graph. Also, it is easy to see that the empty graph and the complete graph with vertex set $[n]$ are the only graphs in $\mathcal{G}_n$ which are isomorphic to no other graph in $\mathcal{G}_n$, and therefore these two graphs are the only isomorphism-invariant degenerate random graphs.

5.1. **DERDs, DVRDs and DVERDs.** We next provide three classes of DRDs which are generated by ERGs, VRGs or VERGs.

Definition 5.2. *The DRD generated by an ERG,* $\mathbf{G}(n, p_e)$*, with direction probability* p_d *is called a* direction-edge random digraph (DERD) *and denoted as* $\mathbf{D}(n, p_e, p_d)$*.*

Notice that letting p_d be 1/2 avoids symmetric arcs, and hence in this case, after generating an ERG, each edge is independently oriented in one of the two directions with equal probability (e.g., see the model in [24]). For example, letting $p_e = 1$ and $p_d = 1/2$ gives a *random tournament* in which each edge of a complete graph is independently oriented in one direction with equal probability. For more information about tournaments, see [20].

Definition 5.3. *A DRD generated by a VRG is called* direction-vertex random digraph (DVRD)*. A* direction-vertex-edge random digraph (DVERD) *is a DRD generated by a VERG.*

Notice that the underlying random graphs of a DERD, a DVRD and a DVERD are an ERG, a VRG, and a VERG, respectively. Clearly any ERG or VRG is a VERG, and hence every DERD and DVRD has a DVERD representation. In addition, the results in [1] imply the following: A nondegenerate DRD which is both a DERD and a DVRD either has $n \leq 3$ or is generated by an ERG, $\mathbf{G}(n, p_e)$, with $p_e = 1$. For every $n \geq 6$, there exist DVERDs which are neither DERDs nor DVRDs. Moreover, for $n \geq 3$, there exist DRDs which are not among DVERDs, and for $n \leq 3$, any DVERD is also a DVRD.

Remark (Approximation to DVERDs by DVRDs). Let $\mathbf{D} = (\mathcal{D}_n, P_{\mathbf{D}})$ be a DVERD generated by a VERG, $\mathbf{G} = (\mathcal{G}_n, P_{\mathbf{G}})$, with direction probability p_d. By Theorem 3.3 in [1], for any $\epsilon > 0$ there exists a VRG, $\mathbf{G}' = (\mathcal{G}_n, P_{\mathbf{G}'})$, satisfying $d_{\text{TV}}(\mathbf{G}, \mathbf{G}') < \epsilon$. Let $\mathbf{D}' = (\mathcal{D}_n, P_{\mathbf{D}'})$ be the DVRD generated by $\mathbf{G}'$ with the same direction probability p_d. Then, it is easy to see that

$$\sum_{U(D)=G} |P_{\mathbf{D}}(D) - P_{\mathbf{D}'}(D)| = |P_{\mathbf{G}}(G) - P_{\mathbf{G}'}(G)|$$

for every $G \in \mathcal{G}_n$, and therefore we get $d_{\text{TV}}(\mathbf{D}, \mathbf{D}') = d_{\text{TV}}(\mathbf{G}, \mathbf{G}')$ which implies that the total variation distance between $\mathbf{D}$ and $\mathbf{D}'$ is less than ϵ. □

6. Inclusion/exclusion relations of DERDs with respect to VARDs

In this section, for $n \geq 4$, we show that a random digraph is both a DERD and a VARD if and only if it is an ARD, and any DERD with $n \leq 3$ is also a VARD.

Proposition 6.1. *A DERD,* $\mathbf{D}(n, p_e, p_d)$*, is an ARD,* $\mathbf{D}(n, p_a)$*, if and only if*

$$p_d = \frac{1}{1 + \sqrt{1 - p_e}} \text{ and } p_a = 1 - \sqrt{1 - p_e}.$$

Proof. Suppose that $\mathbf{D}(n, p_e, p_d)$ is an ARD $\mathbf{D}(n, p_a)$. Then we have $p_e p_d = p_a$ since both are equal to $P((1, 2) \in A(\mathbf{D}))$. Similarly, we have $p_e(2p_d - 1) = p_a^2$ as both are equal to $P(\{(1, 2), (2, 1)\} \subset A(\mathbf{D}))$. Solving these two equations gives $p_d = (1 \pm \sqrt{1 - p_e})/p_e$. If $p_e = 1$, then definitely $p_d = 1$. Otherwise, $(1 - \sqrt{1 - p_e})/p_e < 1 < (1 + \sqrt{1 - p_e})/p_e$ and hence $p_d = (1 - \sqrt{1 - p_e})/p_e$. Note that $(1 - \sqrt{1 - p_e})/p_e = 1/(1 + \sqrt{1 - p_e})$ and so $1/2 \le p_d \le 1$. Finally, $p_a = p_e p_d = 1 - \sqrt{1 - p_e}$.

We next show that whenever $p_d = 1/(1 + \sqrt{1 - p_e})$ and $p_a = 1 - \sqrt{1 - p_e}$, $\mathbf{D}(n, p_e, p_d)$ is $\mathbf{D}(n, p_a)$. Note that in that case, $1 - p_d = p_a(1 - p_a)/p_e$, $2p_d - 1 = p_a^2/p_e$ and $1 - p_e = (1 - p_a)^2$. Therefore, for a given $D \in \mathcal{D}_n$ we have

$$\begin{aligned}
P(D) &= p_e^{n_e}(1 - p_e)^{\binom{n}{2} - n_e}(1 - p_d)^{n_{as}}(2p_d - 1)^{n_s} \\
&= p_e^{n_e}(1 - p_e)^{\frac{(n(n-1) - 2n_e)}{2}} \frac{p_a^{n_{as}}(1 - p_a)^{n_{as}}}{p_e^{n_{as}}} \frac{p_a^{2n_s}}{p_e^{n_s}} \\
&= p_e^{n_e - n_{as} - n_s} p_a^{n_{as} + 2n_s}(1 - p_a)^{n(n-1) - 2n_e + n_{as}} \\
&= p_a^{n_a}(1 - p_a)^{n(n-1) - n_a},
\end{aligned}$$

since $n_e = n_{as} + n_s$ and $n_a = n_{as} + 2n_s$. Thus, the desired result follows. □

In fact, for $n \ge 4$, the family of ARDs is the intersection of the classes DERDs and VARDs.

Theorem 6.2. *If a DERD,* $\mathbf{D}(n, p_e, p_d)$*, with* $n \ge 4$ *has a VARD representation* $\mathbf{D}(n, \Omega, \mu, \phi_a)$*, then* $p_d = 1/(1 + \sqrt{1 - p_e})$ *and* $\phi_a(x, y) = p_e p_d$ μ^2*-a.s.*

Proof. Suppose $\mathbf{D}(n, p_e, p_d)$ has a VARD representation $\mathbf{D}(n, \Omega, \mu, \phi_a)$. Note that, as in any ARD, the events $(i, j) \in A(\mathbf{D})$ and $(k, l) \in A(\mathbf{D})$ are independent in a DERD whenever $\{i, j\} \ne \{k, l\}$. Therefore, one can apply the method used in the proof of Theorem 4.1 and obtain

$$\phi_a(x, y)\phi_a(y, x) = p_e(2p_d - 1) \ \mu^2\text{-a.s.} \tag{6.1}$$

and

$$(1 - \phi_a(x, y))(1 - \phi_a(y, x)) = 1 - p_e \ \mu^2\text{-a.s.} \tag{6.2}$$

Solving the equations in (6.1) and (6.2) yields

$$\phi_a(x, y) + \phi_a(y, x) = 2p_e p_d \ \mu^2\text{-a.s.} \tag{6.3}$$

and

$$\phi_a(x, y) = p_e p_d \pm \sqrt{(1 - \sqrt{1 - p_e} - p_e p_d)(1 + \sqrt{1 - p_e} - p_e p_d)} \tag{6.4}$$

μ^2-a.s. If $p_d > 1/(1+\sqrt{1-p_e}) = (1-\sqrt{1-p_e})/p_e$, then the numbers in the right-hand side of (6.4) have imaginary parts, and hence we get a contradiction, since ϕ_a takes only real values.

Suppose $p_d \le 1/(1+\sqrt{1-p_e})$. By the result in (4.12), recall that in any VARD, $\mathbf{D}(n,\Omega,\mu,\phi_a)$, with $n \ge 3$ we have

$$P(\{(1,2),(1,3)\} \subset A(\mathbf{D})) \ge \left(P((1,2) \in A(\mathbf{D}))\right)^2. \tag{6.5}$$

On the other hand, in any DERD, $\mathbf{D}(n,p_e,p_d)$, with $n \ge 3$ we have

$$P(\{(1,2),(1,3)\} \subset A(\mathbf{D})) = (p_e p_d)^2 = \left(P((1,2) \in A(\mathbf{D}))\right)^2.$$

Therefore, we have the equality in the Cauchy–Schwarz inequality in (4.12). Thus, $\int \phi_a(x_1,x_2)d(\mu x_2) = c$ μ-a.s. for some constant c. Since

$$p_e p_d = P((1,2) \in A(\mathbf{D})) = \iint \phi_a(x_1,x_2)d(\mu x_2)d(\mu x_1) = \int c\; d(\mu x_1) = c,$$

we obtain $c = p_e p_d$, that is,

$$\int \phi_a(x,y)d(\mu y) = p_e p_d \quad \mu\text{-a.s.} \tag{6.6}$$

Similarly, in a VARD, $\mathbf{D}(n,\Omega,\mu,\phi_a)$, with $n \ge 4$ we have

$$\begin{aligned}
&P(\{(1,2),(2,3),(1,4),(4,3)\} \subset A(\mathbf{D})) \\
&= \iiiint \phi_a(x_1,x_2)\phi_a(x_2,x_3)\phi_a(x_1,x_4)\phi_a(x_4,x_3) \\
&\quad \times d(\mu x_1)d(\mu x_2)d(\mu x_3)d(\mu x_4) \\
&= \iint \left(\int \phi_a(x_1,x_2)\phi_a(x_2,x_3)d(\mu x_2)\right) \\
&\quad \times \left(\int \phi_a(x_1,x_4)\phi_a(x_4,x_3)d(\mu x_4)\right) d(\mu x_1)d(\mu x_3) \\
&= \iint \left(\int \phi_a(x_1,x_2)\phi_a(x_2,x_3)d(\mu x_2)\right)^2 d(\mu x_1)d(\mu x_3) \\
&\ge \left(\iiint \phi_a(x_1,x_2)\phi_a(x_2,x_3)d(\mu x_1)d(\mu x_2)d(\mu x_3)\right)^2 \\
&= \left(P(\{(1,2),(2,3)\} \subset A(\mathbf{D}))\right)^2 \qquad (6.7)
\end{aligned}$$

by Fubini's theorem and the Cauchy–Schwarz inequality applied to the constant function $\mathbf{1}$ and $\int \phi_a(x_1,x_2)\phi_a(x_2,x_3)d(\mu x_2)$. Since in a DERD, $\mathbf{D}(n,p_e,p_d)$, with $n \ge 4$ we have

$$\begin{aligned}
P(\{(1,2),(2,3),(1,4),(4,3)\} \subset A(\mathbf{D})) &= (p_e p_d)^4 \\
&= \left(P(\{(1,2),(2,3)\} \subset A(\mathbf{D}))\right)^2,
\end{aligned}$$

we obtain $\int \phi_a(x_1,x_2)\phi_a(x_2,x_3)d(\mu x_2)$ is constant μ^2-a.s. by the equality in the Cauchy–Schwarz inequality in (6.7). By the equality in (6.7), one can easily verify that

$$\int \phi_a(x,y)\phi_a(y,z)d(\mu y) = (p_e p_d)^2 \quad \mu^2\text{-a.s.} \tag{6.8}$$

Let $s(x,y) = i(\phi_a(x,y) - p_e p_d)$. Combining the results in (6.3), (6.4), (6.6), and (6.8) gives

$$s(x,y) = i(\phi_a(x,y) - p_e p_d) = -i(\phi_a(y,x) - p_e p_d) = \overline{s(y,x)} \quad \mu^2\text{-a.s.} \tag{6.9}$$

and

$$\int s(x,y)s(y,z)d(\mu y) = 0 \quad \mu^2\text{-a.s.} \tag{6.10}$$

Let T be the integral operator with kernel s on the space $L^2(\Omega,\mu)$

$$(Tg)(x) = \int s(x,y)g(y)d(\mu y).$$

Since s is bounded and μ is a finite measure, the kernel s is in $L^2(\mu\times\mu)$. Moreover, the integral operator T is compact and self-adjoint by (6.9), which implies that $L^2(\Omega,\mu)$ has an orthonormal basis $(\psi_i)_{i\geq 1}$ of eigenfunctions for T such that $T\psi_i = \lambda_i\psi_i$ for not necessarily distinct eigenvalues λ_i, and

$$s(x,y) = \sum_{i\geq 1} \lambda_i \psi_i(x)\overline{\psi_i(y)} \ \mu^2\text{-a.s.} \tag{6.11}$$

with the sum converging in L^2 ([23]). Since ψ_i's are orthonormal, equations (6.10) and (6.11) imply

$$\sum_{i\geq 1} \lambda_i^2 \psi_i(x)\overline{\psi_i(z)} = 0 \ \mu^2\text{-a.s.} \tag{6.12}$$

Therefore, for any $m \geq 1$, by multiplying equation (6.12) by $\psi_m(z)\overline{\psi_m(x)}$ and integrating over x and z we obtain $\lambda_m^2 = 0$, i.e., $\lambda_m = 0$ for each m. Thus, $s(x,y) = 0$ μ^2-a.s. and hence $\phi_a(x,y) = p_e p_d$ μ^2-a.s. which implies $p_d = 1/(1+\sqrt{1-p_e})$. □

Remark. Notice that Theorem 6.2 and Proposition 6.1 together imply Theorem 4.1. However, we provide the proof of Theorem 4.1 to keep the proof of Theorem 6.2 shorter and also to point out similarities and differences of our techniques with those used in [1]. □

However, for $n \leq 3$, any DERD has a VARD representation.

Theorem 6.3. *Any DERD, $\mathbf{D}(n, p_e, p_d)$, with $n \leq 3$ is also a VARD.*

Proof. Let $\oplus$ and $\ominus$ denote addition and subtraction modulo 1, respectively. In other words, for real numbers $0 \leq x, y < 1$,

$$x \oplus y = \begin{cases} x+y, & \text{if } x+y < 1 \\ x+y-1, & \text{if } x+y \geq 1 \end{cases}$$

and

$$x \ominus y = \begin{cases} x - y, & \text{if } x - y \geq 0 \\ x - y + 1, & \text{if } x - y < 0. \end{cases}$$

If U_1, U_2, U_3 are independent uniform random variables over $[0, 1)$, then so are $U_1 \oplus U_2, U_2 \oplus U_3$, and $U_3 \oplus U_1$ (see Lemma 4.5 in [1]). Therefore, $\mathbf{G}(3, p_e)$ can be represented as a VRG $\mathbf{G}(3, [0, 1), \nu, f)$ where ν is the uniform distribution on $[0, 1)$ and $f(x, y) = \mathbf{1}_{\{x \oplus y \leq p_e\}}$ ([1]).

Let $g(x, y) = \mathbf{1}_{\{x \ominus y \leq 1/2\}} + (2p_d - 1)\mathbf{1}_{\{x \ominus y > 1/2\}}$ for every $0 \leq x, y < 1$. We claim that $\mathbf{D}(3, p_e, p_d)$ is a VARD, $\mathbf{D}(3, \Omega, \mu, \phi_a)$, where $\Omega = [0, 1) \times [0, 1)$, μ is the product of two uniform distributions on $[0, 1)$ and $\phi_a((u_1, u_1'), (u_2, u_2')) = f(u_1, u_2)g(u_1', u_2')$. First note that

$$g(x, y) + g(y, x) = 2p_d \quad \text{and} \quad g(x, y)g(y, x) = 2p_d - 1, \tag{6.13}$$

for every $0 \leq x, y < 1$. As f is a symmetric indicator function, the equations in (6.13) imply

$$\phi_a((u_1, u_1'), (u_2, u_2'))\phi_a((u_2, u_2'), (u_1, u_1')) = f(u_1, u_2)(2p_d - 1), \tag{6.14}$$

$$\begin{aligned} &\phi_a((u_1, u_1'), (u_2, u_2'))(1 - \phi_a((u_2, u_2'), (u_1, u_1'))) \\ &= f(u_1, u_2)(g(u_1', u_2') - (2p_d - 1)) \end{aligned} \tag{6.15}$$

and

$$\begin{aligned} &(1 - \phi_a((u_1, u_1'), (u_2, u_2')))(1 - \phi_a((u_2, u_2'), (u_1, u_1'))) \\ &= 1 - f(u_1, u_2). \end{aligned} \tag{6.16}$$

We next focus on the function g. It is easy to see that

$$\int_0^1 g(x, y)\ dy = \int_0^1 g(x, y)\ dx = \frac{1}{2}1 + \frac{1}{2}(2p_d - 1) = p_d, \tag{6.17}$$

for every $0 \leq x, y < 1$.

Consider the circle obtained by identifying the end points of the interval $[0, 1]$ such that $1/4$ is on the arc that starts from 0 and ends at $1/2$ along the clockwise direction. Then, $x \ominus y$ is equal to the length of the arc of this circle which starts from x and ends at y along the counterclockwise direction. Notice that $g(x, y)g(y, z)$ and $g(x, y)g(y, z)g(z, x)$ depends on the ordering of x, y, z along counterclockwise direction and whether the points x, y, z form an acute or obtuse triangle.

If x, y, z form an acute triangle, there are basically two cases for the ordering, x, y, z or x, z, y. In the first case, $g(x, y)g(y, z) = (2p_d - 1)^2$, and in the latter case $g(x, y)g(y, z) = 1^2 = 1$.

If x, y, z form an obtuse triangle, all six permutations of x, y, z (x, y, z; x, z, y; y, x, z; y, z, x; z, x, y; z, y, x) are possible with the point at the middle corresponding to the obtuse angle. Then, we have $g(x, y)g(y, z) = (2p_d - 1)^2, (2p_d - 1), (2p_d - 1), (2p_d - 1), (2p_d - 1), 1$, respectively. Moreover, it is easy to show that three uniformly at random points on the circle form an acute triangle with probability 1/4. Therefore, we obtain

$$\begin{aligned}\int_{[0,1)^3} g(x,y)g(y,z)\ dxdydz &= \frac{1}{4}\cdot\frac{1}{2}((2p_d-1)^2+1)\\ &\quad+\frac{3}{4}\cdot\frac{1}{6}((2p_d-1)^2+4(2p_d-1)+1)\\ &= p_d^2. \end{aligned} \tag{6.18}$$

Similarly, we have

$$\begin{aligned}\int_{[0,1)^3} g(x,y)g(y,z)g(z,x)\ dxdydz &= \frac{1}{8}((2p_d-1)^3+1)\\ &\quad+\frac{1}{8}(3(2p_d-1)^2+3(2p_d-1))\\ &= p_d^3. \end{aligned} \tag{6.19}$$

By using the results in (6.13)–(6.19), one can easily verify that

$$\int P_{\mathbf{x}}(D)d(\mu\mathbf{x}) = p_e^{n_e}(1-p_e)^{3-n_e}(1-p_d)^{n_{as}}(2p_d-1)^{n_s},$$

for every $D\in\mathcal{D}_3$, and hence the desired result follows. Furthermore, the same setting works for $n=2$ as well. □

For $n=2$ the families DERDs and VRDs coincide with IIRDs. Let D_1, D_2, D_3, and D_4 be the digraphs with vertex set [2] which has only the arc (1,2), only the arc (2,1), both of the arcs, and none of the arcs, respectively. Note that to obtain an isomorphism-invariant random digraph, a necessary and sufficient condition is $P(D_1)=P(D_2)$. Let $\mathbf{D}$ be the random digraph with $P(D_1)=P(D_2)=p_1$, $P(D_3)=p_2$, and $P(D_4)=1-2p_1-p_2$. First observe that $\mathbf{D}$ is an ARD if and only if $\sqrt{p_2}(1-\sqrt{p_2})=p_1$. Letting $p_e=2p_1+p_2$ and $p_d=(p_1+p_2)/(2p_1+p_2)$ gives that $\mathbf{D}$ is a DERD $\mathbf{D}(2,p_e,p_d)$. With the same p_e and p_d, let $\phi_a((u_1,u_1'),(u_2,u_2'))=\mathbf{1}_{\{u_1\oplus u_1'\le p_e\}}\mathbf{1}_{\{u_2\ominus u_2'\le p_d\}}$. Then, it is easy to see that $\mathbf{D}$ is also VRD $\mathbf{D}(2,\Omega,\mu,\phi_a)$, where $\Omega=[0,1)\times[0,1)$, μ is the uniform distribution on $[0,1)^2$. Therefore, when $n=2$, any isomorphism-invariant random digraph is both a DERD and a VRD (hence also a VARD).

Proposition 6.4. *A RNND is not a DRD.*

Proof. We show that there is no RNND which is also a DRD by contradiction. Suppose that a RNND, $\mathbf{D}=(\mathcal{D}_n,P)$, is a DRD generated by the random graph $\mathbf{G}=(\mathcal{G}_n,P_{\mathbf{G}})$ and with direction probability p_d. Let G be a graph in $\mathcal{G}_n$ with $P_{\mathbf{G}}(G)>0$, and D be a digraph in $\mathcal{D}_n$ such that $U(D)=G$, $n_s(D)=0$, and contains a vertex which is the tail of no arc. In other words, D is a digraph whose underlying graph is G, contains no symmetric arcs, and there exists a vertex v in $V(D)$ such that v is the head of every arc incident to v. Then, as $\mathbf{D}$ is a DRD, we have

$$P(D) = P_{\mathbf{G}}(G)(1-p_d)^{n_{as}} > 0, \tag{6.20}$$

since $P_{\mathbf{G}}(G) > 0$ and $p_d < 1$. On the other hand, for any given $\mathbf{x} = (x_1, \ldots, x_n)$, every vertex is the tail of exactly k arcs in kNND($\mathbf{x}$). Therefore, we obtain $P(D) = 0$ which contradicts with (6.20). □

Note that Proposition 6.4 implies that the region DRD^c in Figure 2 is nonempty.

6.1. **Open Problems for $n \geq 3$.** We have shown that DERDs and VRDs coincide for $n = 2$. However, starting with $n = 3$, things start to get more complicated. The function ϕ_a constructed in the proof of Theorem 6.3 is binary (only takes the values 0 or 1) if and only if $p_d = 1/2$. Hence, by Theorem 6.3 we see that any $\mathbf{D}(3, p_e, 1/2)$ has a VRD representation. But, by Proposition 6.1, $\mathbf{D}(3, p_e, 1/2)$ is an ARD only if $p_e = 0$ which gives a degenerate random digraph, and hence $\mathbf{D}(3, p_e, 1/2)$ does not yield a nondegenerate ARD. Below we provide a list of open problems regarding the relation of DVERDs and VARDs and their subfamilies.

- Other than the degenerate ones mentioned above, is there any DERD, $\mathbf{D}(3, p_e, p_d)$, with $p_d > 1/2$ which is also a VRD?
- Is there an ARD with $n = 3$ which has a VRD representation?
- Identifying IIRDs that are in the intersection of DVRDs and VARDs (that is, which IIRDs have both DVRD and VARD representations?)
- Identifying IIRDs that are in the intersection of DVERDs and VARDs

Furthermore, we have the below conjecture for $n = 3$.

Conjecture. *Any DERD with $n = 3$ is also a VRD.*

7. Positive dependence between arcs of VARDs and its relation with their arc Density

Recall that by the inequality in equation (4.12), for any VARD, $\mathbf{D}$, we have the positive dependence

$$\begin{aligned} P(\{(1,2),(1,3)\} \subset A(\mathbf{D})) &\geq P((1,2) \in A(\mathbf{D}))P((1,3) \in A(\mathbf{D})) \\ &= P((1,2) \in A(\mathbf{D}))^2. \end{aligned} \tag{7.1}$$

Furthermore, the inequality in (7.1) can be generalized by Hölder's inequality as follows

$$\begin{aligned} P(\{(1,2),\ldots,(1,m)\} \subset A(\mathbf{D})) &\geq \prod_{i=2}^{m} P((1,i) \in A(\mathbf{D})) \\ &= P((1,2) \in A(\mathbf{D}))^{m-1} \end{aligned} \tag{7.2}$$

where $2 \leq m \leq n$, and notice that equality holds for every ARD. Similarly, we have the same inequality in (7.2) for any DVERD as well and note that equality holds for every DERD. However, there are random digraphs other than DERDs satisfying equality in (7.2) for each m. For example, consider the VRD, $\mathbf{D}(n,[0,1),\mu,\phi_a)$, where μ is the uniform distribution over $[0,1)$ and $\phi_a(x,y) = \mathbf{1}_{\{x\ominus y\geq 3/8\}}\mathbf{1}_{\{y\ominus x\geq 3/8\}}$. Clearly, in this case, we have $P((1,i)\in A(\mathbf{D})) = 1/4$ for each i and $P(\{(1,2),\ldots,(1,m)\}\subset A(\mathbf{D})) = (1/4)^{m-1}$.

Also, it is easy to verify that $\mathbf{D}(n,[0,1),\mu,\phi_a)$ has no DERD representation, since $P(\{(1,2),(1,3),(2,3)\}\subset A(\mathbf{D})) = 0$. In the same manner, one can easily obtain similar results for random graphs. In other words, for any VERG, $\mathbf{G}(n,\Omega,\mu,\phi_e)$, and $2\leq m\leq n$, we have

$$P(\{\{1,2\},\{1,3\},\ldots,\{1,m\}\}\subset E(\mathbf{G}))\geq P(\{1,2\}\in E(\mathbf{G}))^{m-1},$$

and equality holds for every ERG. The underlying random graph of the VRD, $\mathbf{D}(n,[0,1),\mu,\phi_a)$, described above is an example for random graphs with no ERG representation which attains equality in (7) for every m.

This positive dependence for VARDs and VERGs has an interesting implication for their arc and edge densities, respectively. The *arc density* (*edge density*) of a digraph $D=(V,A)$ (graph $G=(V,E)$) of order $|V|=n$, denoted $\rho_a(D)$ ($\rho_e(G)$), is defined as

$$\rho_a(D) = \frac{|A|}{n(n-1)} \quad \left(\rho_e(E) = \frac{|E|}{\binom{n}{2}}\right)$$

([14]).

Thus $\rho_a(D)$ ($\rho_e(G)$) represents the ratio of the number of arcs (edges) in the digraph D (graph G) to the number of arcs (edges) in the complete symmetric digraph (complete graph) of order n, which is $n(n-1)$ (resp. $n(n-1)/2$).

Since the vertices $X_1,\cdots,X_n \overset{iid}{\sim} \mu$, the arc density of the VARD, D, is a U-statistic,

$$\rho_a(D) = \frac{1}{n(n-1)}\sum_{i<j}\sum h_a(X_i,X_j) \tag{7.3}$$

where $h_a(X_i,X_j) = \mathbf{1}_{\{(X_i,X_j)\in A\}} + \mathbf{1}_{\{(X_j,X_i)\in A\}}$. We denote $h_a(X_i,X_j)$ as h_a^{ij} for brevity of notation. Although the digraph is asymmetric, h_a^{ij} is defined as the number of arcs in D between vertices X_i and X_j, in order to produce a symmetric kernel with finite variance ([17]). Similarly, the edge density of the VERG, G, is a U-statistic,

$$\rho_e(G) = \frac{2}{n(n-1)}\sum_{i<j}\sum h_e(X_i,X_j) \tag{7.4}$$

where $h_e(X_i,X_j) = \mathbf{1}_{\{X_iX_j\in E\}}$. We denote $h_e(X_i,X_j)$ as h_e^{ij} for brevity of notation. Since the graph is symmetric, h_a^{ij} is already a symmetric kernel with finite variance.

The random variable $\rho_a(D)$ ($\rho_e(E)$) depends on n, μ, and ϕ_a (ϕ_e). Then the expectation $\mathbf{E}[\rho_a(D)]$ is

$$\mathbf{E}[\rho_a(D)] = \frac{1}{n(n-1)} \sum\sum_{i<j} (P((X_i, X_j) \in A) + P((X_j, X_i) \in A)), \tag{7.5}$$

since $\mathbf{E}[h_a(X_i, X_j)] = P((X_i, X_j) \in A) + P((X_j, X_i) \in A)$. Similarly, the expectation $\mathbf{E}[\rho_e(G)]$ is

$$\mathbf{E}[\rho_e(G)] = \frac{2}{n(n-1)} \sum\sum_{i<j} P(X_i X_j \in E). \tag{7.6}$$

In particular, for VRDs, $P((X_i, X_j) \in A) = P((X_1, X_2) \in A) =: \pi_a$, since only vertices are random and they are i.i.d. from μ. Then $\mathbf{E}[\rho_a(D)] = \pi_a$. Similarly, for ARDs, $P((X_i, X_j) \in A) = P((X_1, X_2) \in A) = p_a$, since only arcs are random and independently inserted with probability p_a. Then $\mathbf{E}[\rho_a(D)] = p_a$.

The variance $\mathbf{Var}[\rho_a(D)]$ for VRDs and ARDs simplifies to

$$\mathbf{Var}[\rho_a(D)] = \frac{1}{2n(n-1)} \mathbf{Var}[h_a^{12}] + \frac{n-2}{n(n-1)} \mathbf{Cov}[h_a^{12}, h_a^{13}] \leq 1/4. \tag{7.7}$$

A central limit theorem for U-statistics ([17]) yields

$$\sqrt{n}\big(\rho_a(D) - \mathbf{E}[\rho_a(D)]\big) \xrightarrow{\mathcal{L}} N\big(0, \mathbf{Cov}[h_a^{12}, h_a^{13}]\big) \tag{7.8}$$

provided $\mathbf{Cov}[h_a^{12}, h_a^{13}] > 0$. Recall that $\mathbf{Cov}[h_a^{12}, h_a^{13}] = \mathbf{E}[h_a^{12}\, h_a^{13}] - \mathbf{E}[h_a^{12}]\,\mathbf{E}[h_a^{13}]$. Then for ARDs, $\mathbf{Cov}[h_a^{12}, h_a^{13}] = 0$, since h_a^{12} and h_a^{13} are independent because arcs are independently inserted with probability p_a for each ordered pair of vertices. Thus, the asymptotic distribution of arc density of ARDs is degenerate with $\rho_a(D) \xrightarrow{L} p_a$ as $n \to \infty$. We also have

$$\begin{aligned}
\mathbf{E}[h_a^{12}\, h_a^{13}] = \mathbf{E}[&(I((X_1, X_2) \in A) + I((X_2, X_1) \in A))\,(I((X_1, X_3) \in A) \\
&+ I((X_3, X_1) \in A)] \\
&= P(\{(X_1, X_2), (X_1, X_3)\} \subset A) + P(\{(X_1, X_2), (X_3, X_1)\} \subset A) \\
&\quad + P(\{(X_2, X_1), (X_1, X_3)\} \subset A) + P(\{(X_2, X_1), (X_3, X_1)\} \subset A).
\end{aligned}$$

So $\mathbf{Cov}[h_a^{12}, h_a^{13}]$ is equal to

$$\begin{aligned}
&P(\{(X_1, X_2), (X_1, X_3)\} \subset A) + P(\{(X_1, X_2), (X_3, X_1)\} \subset A) \\
&+ P(\{(X_2, X_1), (X_1, X_3)\} \subset A) + P(\{(X_2, X_1), (X_3, X_1)\} \subset A) \\
&- [2\, P((X_1, X_2) \in A)]^2.
\end{aligned}$$

By the positive dependence shown in equation (4.12), we have

$$P(\{(X_1, X_2), (X_1, X_3)\} \subset A) \geq (P((X_1, X_2) \in A))^2,$$

and similar to the derivation of positive dependence in equation (4.12), by relabeling the vertices, we can show that

$$P(\{(X_1, X_2), (X_3, X_1)\} \subset A) \geq (P((X_1, X_2) \in A))^2,$$

$$P(\{(X_2, X_1), (X_1, X_3)\} \subset A) \geq (P((X_1, X_2) \in A))^2,$$

and

$$P(\{(X_2, X_1), (X_3, X_1)\} \subset A) \geq (P((X_1, X_2) \in A))^2.$$

So by positive dependence, we show that $\mathbf{Cov}[h_a^{12}12, h_a^{12}13] \geq 0$, and hence the asymptotic variance of arc density for VRDs is nonnegative, so the asymptotic distribution of the arc density for VRDs is a normal distribution or it is degenerate with zero variance. For example, $\mathbf{Cov}[h_a^{12}, h_a^{13}] = 0$ if the VRD is the complete digraph or the empty digraph. This result guarantees the asymptotic distribution of PCDs is a valid distribution as PCDs are VRDs.

Similarly, for VRGs, $P(X_iX_j \in E) = P(X_1X_2 \in E) =: \pi_e$, since only vertices are random and they are iid from μ, so $\mathbf{E}[\rho_e(G)] = \pi_e$. Similarly, for ERGs, $P(X_iX_j \in E) = P(X_1X_2 \in E) = p_e$, since only edges are random and independently inserted with probability p_e. Then $\mathbf{E}[\rho_e(G)] = p_e$.

The variance $\mathbf{Var}[\rho_e(G)]$ for VRGs and ERGs simplifies to the one in equation (7.7) with a's replaced with e's and the CLT result follows provided $\mathbf{Cov}[h_e^{12}, h_e^{13}] > 0$. For ERGs, $\mathbf{Cov}[h_e^{12}, h_e^{13}] = 0$ since h_e^{12} and h_e^{13} are independent. Thus, the asymptotic distribution of edge density of ERGs is degenerate with $\rho_e(G) \xrightarrow{L} p_e$ as $n \to \infty$.

Recall that $h_e^{ij} = I\{X_iX_j \in E\}$. Then

$$\begin{aligned}
\mathbf{E}[h_e^{12}\, h_e^{13}] &= \mathbf{E}[I\{X_1X_2 \in E\}I\{X_1X_3 \in E\}] \\
&= \mathbf{E}[I\{\{X_1X_2, X_1X_3\} \subset E\}] \\
&= P(\{X_1X_2, X_1X_3\} \subset E).
\end{aligned}$$

So $\mathbf{Cov}[h_e^{12}, h_e^{13}] = P(\{X_1X_2, X_1X_3\} \subset E) - [P(X_1X_2 \in E)]^2$. By the positive dependence in edges of VRGs (which can be shown as in equation (4.12)), we have $P(\{X_1X_2, X_1X_3\} \subset E) \geq (P(X_1X_2 \in E))^2$. So we have $\mathbf{Cov}[h_e^{12}, h_e^{13}] \geq 0$, and hence the asymptotic variance of edge density for VRGs is nonnegative, so the asymptotic distribution of the edge density for VRGs is a normal distribution or it is degenerate with zero variance. For example, $\mathbf{Cov}[h_e^{12}, h_e^{13}] = 0$ if the VRG is the complete graph or the empty graph. This result guarantees the asymptotic distribution of the underlying graphs of the PCDs is a valid distribution as the underlying graphs of PCDs are VRGs ([5]).

8. Discussion and Conclusions

In this paper, we present four families, namely, arc random digraphs (ARDs), vertex random digraphs (VRDs), vertex-arc random digraphs (VAR Ds), and direction random digraphs (DRDs) of isomorphism-invariant random digraphs (IIRDs) based on where the randomness resides. The first three of these classes can be viewed as extensions of the isomorphism-invariant random graph (IIRG) classification of [1] to digraphs. We also introduce randomness in direction (together with arcs, vertices, etc.) to obtain the family of DRDs which includes direction-edge random digraphs

(DERDs), direction-vertex random digraphs (DVRDs), and direction-vertex-edge random digraphs (DVERDs). We demonstrate that to obtain a DRD as an IIRD, one has to start with an IIRG and insert directions randomly.

The main results of this paper are illustrated in Figures 1–3. For $n \geq 4$, we show that there is no random digraph that is both an ARD and a VRD (which is the digraph counterpart of the result in [1], that there is no nondegenerate random graph which is both an ERG and a VRG for $n \geq 4$). It is shown in [1] that for every $n \geq 6$, there exist VERGs which neither belong to ERGs nor VRGs. We reduce the lower bound for n from 6 to 4 by using the asymmetric structure of the function ϕ_a, and obtain the digraph counterpart of their result, i.e., there exist VARDs which have no ARD or VRD representation for $n \geq 4$. We also show the existence of IIRDs that are not VARDs, and provide random nearest neighbor digraphs(RNNDs) as an example. However, for DRDs we have the same lower bound for n; that is, for $n \geq 6$, there exist DVERDs which neither belong to DERDs or DVRDs. We show that there are IIRDs which are not DRDs and as an example provide RNNDs again. Moreover, the underlying graph of a RNND with large n serves as an example of IIRG which is not a VERG. We also show that for $n \geq 4$, ARDs are the only random digraphs which have both DERD and VARD representations. The method we use for the latter result is not applicable to the intersection of the families DVRDs and VARDs (and also to the intersection of DVERDs and VARDs), since we lose the independence of the edges in a VRG. Therefore, identifying all random digraphs with both DVRD and VARD representations is a challenging problem, and remains open. We also provide other open problems, e.g., identifying the digraphs in the intersection of DVERDs and VARDs and their subfamilies.

For $n = 2$, we show that any isomorphism-invariant random digraph has DERD, DVRD and VRD representations. But things get more complicated for $n = 3$ or larger. For $n = 3$, we show that any DERD has a VARD representation and any DERD whose edge probability is $1/2$ is also a VRD. However, the question of whether there is other DERDs with VRD representation is open, and we conjecture that any DERD is a VRD as well for $n = 3$. Yet, when $n = 3$, every DVERD is a DVRD.

We also show positive dependence between the arcs of a VARD that share the same tail. This positive dependence guarantees the variance of the asymptotic distribution of the arc density of VRDs and ARDs is nonnegative. Hence the asymptotic distribution of the arc density of VRDs is normal if the asymptotic variance is positive, otherwise it is degenerate and asymptotic distribution of ARDs is degenerate. In particular, PCDs are a member of the VRD family, and so their arc density either converges in distribution to its expectation (i.e., is degenerate) or to normal distribution. Arc density of PCDs have been applied for testing multi-class spatial patterns exploiting their asymptotic normality ([4]). These results hold for VRGs and ERGs (and thus for the underlying graphs of VRDs and ARDs) as well. Similar to arc density of PCDs, edge density of the underlying graphs of PCDs converges in distribution to its expected value or normal distribution.

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