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PERMUTATIONS AVOIDING CONNECTED SUBGRAPHS

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ABSTRACT. There is a permutation of the vertices of a tree for which no proper subtree on at least two vertices is mapped to a subtree, if and only if twice the number of its endpoints is less than or equal to the number of points of the tree; Theorem 4.3. The following more general result follows:

Let $\mathrm{G} = (V(\mathrm{G}), E(\mathrm{G}))$ be a simple graph and let $\boldsymbol{C}(\mathrm{G})$ be the set of subsets $A \subsetneq V(\mathrm{G})$ which induce a connected subgraph of G containing at least two vertices and let $\Pi(\mathrm{G})$ be the set of permutations of $V(\mathrm{G})$ which do not map an element of $\boldsymbol{C}(\mathrm{G})$ to an element of $\boldsymbol{C}(\mathrm{G})$. In the case where G has n vertices and at most $n-1$ edges we give a necessary and sufficient condition on G so that $\Pi(\mathrm{G}) \neq \emptyset$.

1. INTRODUCTION

Investigations into the interaction of the symmetric group on a set S with a relational structure on S, in particular for example a graph or a partial order, have led to important results in a variety of mathematical disciplines. Almost all of those investigations are special cases of the following general setting. Relate elements g of the symmetric group on S to pairs of structure properties φ and ψ so that $\varphi(A)$ implies $\psi(g(A))$ for all subsets A of S. If property φ is equal to property ψ then the elements g of the symmetric group on S preserving property φ form a subgroup of the symmetric group. Which need not be the case if φ is not equal to ψ. Furthermore every subgroup of the symmetric group on S induces a relational structure via the orbits of tuples of elements of S under the action of the group. Such group actions are studied extensively in group theory but are also of considerable importance in model theory. For example Fraïssé limits, see [9] or [10], are essentially determined by their groups of automorphisms. In studies related to the distinguishing number, the relational structure is expanded so that the group of its automorphisms consists only of the identity, see for example [5], [11], [12].

Relating elements g of the symmetric group to structure properties φ so that $\varphi(A)$ implies $\neg\varphi(g(A))$ leads to packing type problems. For example let a graph with V as set of vertices and E as set of edges be given. If φ

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stands for two vertices are adjacent and ψ for two vertices are not adjacent, then an element g of the symmetric group mapping adjacent pairs of vertices to nonadjacent pairs of vertices, is a packing. In this case φ is $\neg\psi$, a situation we wish to understand further. Another example of $\varphi = \neg\psi$ arises from the notion of orthogonality in the theory of clones, an abstract setting for the study of switching circuits. Let P be a partial order on a set V. An *endomorphism* of P is an order-preserving map of V to V. Two partial order relations on the same set are *orthogonal* if their only common endomorphisms, are the identity map and the constant maps. Clones and perpendicular orders are related by the following result:

Theorem 1.1 ([16], [13]). *To every pair of orthogonal orders on a set of at least three elements, there corresponds a pair of clones intersecting in the clone consisting of the projections and the constant maps.*

The first examples of pairs of orthogonal finite orders were given by Demetrovics, Miyakawa, Rosenberg, Simovici and Stojmenović [6]; those orders were in fact bipartite. More examples can be found in [7]. Orthogonal linear orders led to the notion of a simple permutation motivated by the Stanley–Wilf conjecture, now settled by Marcus and Tardös [14]. A. Nozaki, M. Miyakawa, G. Pogosyan and I. G. Rosenberg studied the existence of a linear order orthogonal to a given linear order. They showed that for two linear orders to be orthogonal it is necessary and sufficient that they have no common nontrivial *interval* [15] and also that the proportion of linear orders orthogonal to a given linear order tends to $1/e^2$ as the size of the underlying set tends to infinity. For some of the literature on orthogonal partial orders and related notions on graphs see for example [17], [8], [21], [19].

Let V be a set. A *binary relation* on V is a subset ρ of the Cartesian product $V \times V$, but for convenience we write $x\rho y$ instead of $(x, y) \in \rho$. A map $f : V \to V$ *preserves* ρ if:

$$x\rho y \Rightarrow f(x)\rho f(y)$$

for all $x, y \in V$.

A *binary structure* is a pair $R := (V, (\rho_i)_{i\in I})$ where V is a set and each ρ_i is a binary relation on V.

If $R := (V, (\rho_i)_{i\in I})$ and $R' := (V', (\rho'_i)_{i\in I})$ are two binary structures, a *homomorphism* of R into R' is a map $f : V \to V'$ such that the implication

$$x\rho_i y \Rightarrow f(x)\rho'_i f(y) \tag{1.1}$$

holds for every $x, y \in V$, $i \in I$. If f is one-to-one and implication (1.1) above is a logical equivalence, this is an *embedding*.

Let $R := (V, (\rho_i)_{i\in I})$ be a binary relational structure. An *autonomous* set of R, or an *interval* of R, is a subset A of V so that for all elements a and a' in A and all $v \in V \setminus A$ there exists an isomorphism f of the substructure of R induced by $\{v, a\}$ to the substructure of R induced by $\{v, a'\}$ with $f(v) = v$

and $f(a) = a'$. Hence, if for example R is an oriented graph, then A is an autonomous set for R if for all $A \subseteq V$, all a and a' in A, (v, a) is an edge if and only if (v, a') is an edge and (a, v) is an edge if and only if (a', v) is an edge.

The empty set, the whole set, and the singletons are autonomous and are called *trivial*. We say that R is *prime* if it has no nontrivial autonomous set, it is *semirigid* if the identity map and the constant maps are the only endomorphisms of R and it is *embedding rigid* if the identity map is the only embedding from R to R. Finally, we say that two binary relations ρ and ρ' on a set V are *orthogonal* (or *perpendicular*) if the binary structure (V, ρ, ρ') is semirigid.

Remark 1.2. Semirigidity is often defined for structures made of reflexive relations. Indeed, in that case, all constant functions are endomorphisms and, more generally, any map mapping an autonomous set A on an element $a \in A$ and leaving fixed the complement of A is an endomorphism. Thus, if R is semirigid, R must be prime, and in any case, embedding rigid.

On the other hand if f is a nonconstant endomorphism of a linear order on V and $v \in V$ with $|f^{-1}(v)| \geq 2$, then $f^{-1}(v)$ is a nontrivial autonomous set. Hence we obtained the following basic theorem:

Theorem 1.3. *Let $R = (V, \rho, \rho')$ be a binary relational structures on the set V, where ρ and ρ' are linear orders. Then homomorphisms of R into R are either nontrivial automorphisms or are induced by a nontrivial autonomous set.*

Consider now a linear order on a finite set, say $\{1, 2, \ldots, n\}$. The order need not be the natural order, but of course there is no loss of generality in taking it as the natural order. Then the set of linear orders on $\{1, 2, \ldots, n\}$ can be identified with the set of permutations of $\{1, 2, \ldots, n\}$ with the identity map identified with the natural order. A linear order on a finite set has no nontrivial automorphism. Hence it follows from Theorem 1.3 that a linear order of $\{1, 2, \ldots, n\}$ is orthogonal to the natural order if and only if its associated permutation does not map a nontrivial interval, that is an autonomous set, to an interval. Let G be the graph on $\{1, 2, \ldots, n\}$ for which two numbers are adjacent just in case the numbers are consecutive. It follows that in order to determine the linear orders on $\{1, 2, \ldots, n\}$ which are orthogonal to the natural order, one needs to determine all permutations of $\{1, 2, \ldots, n\}$ which do not map a connected subgraph of G to a connected subgraph of G.

Given a simple graph G on a set of vertices V let $\boldsymbol{C}(\mathrm{G})$ be the set of subsets $A \subsetneq V(\mathrm{G})$ which induce a connected subgraph of G containing at least two vertices. Every permutation π of V which does not map an element of $\boldsymbol{C}(\mathrm{G})$ to an element of $\boldsymbol{C}(\mathrm{G})$ does not, in particular, map an edge to an edge. Hence π is a packing of G. We will show that for almost all graphs with a small number of edges, such as trees, a stronger form of packing holds, that is, we obtain then even for the stronger form of packing results similar to Theorems 1.4 and 1.5 below.

The notion of packing was introduced by Bollobás and Eldridge [2]. The literature on the subject is abundant and the reader is directed to the survey by Woźniak [20]. The following theorem was proved, independently, in [1], [3], [18], and the next theorem in [4]. A permutation π of V is *fixed point free* if $\pi(v) \neq v$ for all $v \in V$.

Theorem 1.4. *Let* $\mathrm{G} = (V(\mathrm{G}), E(\mathrm{G}))$ *be a graph such that* $|E(\mathrm{G})| \leq |V(\mathrm{G})| - 2$*. Then* G *has a fixed point free packing.*

Theorem 1.4 is "best possible" in the sense that there exist graphs with n vertices and $n-1$ edges which are packable, but which cannot be packed without fixed vertices. Consider the disjoint union of a small star and a 3-cycle: i.e., $K_{1,2} \cup C_3$ and $K_{1,3} \cup C_3$.

Theorem 1.5. *Let* $\mathrm{G} = (V(\mathrm{G}), E(\mathrm{G}))$ *be a graph with* $|V(\mathrm{G})| = n$*. If* $|E(\mathrm{G})| \leq n-1$*, then either* G *has a packing or* G *is isomorphic to one of the following graphs:* $K_1 \cup C_3$*,* $K_2 \cup C_3$*,* $K_1 \cup 2\,C_3$*,* $K_1 \cup C_4$*,* $K_{1,n-1}$ *for* $n \geq 2$*, and* $K_{1,n-4} \cup C_3$ *for* $n \geq 8$*.*

A subset $X \subseteq V(\mathrm{G})$ is said to be *connected* if the subgraph of G induced by X is connected. (The empty set and all singleton sets are connected.)

Definition 1.6. *For a graph* G *let* $\boldsymbol{C}(\mathrm{G})$ *be the set of subsets* $A \subsetneq V(\mathrm{G})$ *which induce a connected subgraph of* G *containing at least two vertices. Let* $X \subseteq \boldsymbol{C}(\mathrm{G})$ *and let* $\Pi(X, \mathrm{G})$ *be the set of permutations of* $V(\mathrm{G})$ *which do not map an element of* X *to an element of* X*.*

We pose the following general problem:

Problem 1.7. *Find necessary and sufficient conditions on* G *so that* $\Pi(X, \mathrm{G}) \neq \emptyset$*.*

For instance, Theorem 1.5 answers the problem when $X = E(\mathrm{G})$ and $|E(\mathrm{G})| \leq |V(\mathrm{G})| - 1$. The main purpose of this paper is to prove similar results to Theorems 1.4 and 1.5 in the case where $X = \boldsymbol{C}(\mathrm{G})$. In the context of this paper we will write $\Pi(\mathrm{G})$ for $\Pi(\boldsymbol{C}(\mathrm{G}), \mathrm{G})$. A graph such that $\Pi(\mathrm{G}) \neq \emptyset$ is said to be *strongly packable*. In this case an element of $\Pi(\mathrm{G})$ is called a *strong packing* of G. Note that $\Pi(\mathrm{G}) \neq \emptyset$ if G has at most two vertices.

Our first result is the following:

Theorem 1.8. *Let* $\mathrm{G} = (V(\mathrm{G}), E(\mathrm{G}))$ *be a graph such that* $|E(\mathrm{G})| \leq |V(\mathrm{G})| - 2$*. Then* G *has a fixed point free strong packing.*

Theorem 1.8 is "best possible" in the sense that there exist graphs with n vertices and $n-1$ edges which are strongly packable, but which cannot be strongly packed without fixed vertices. Consider the disjoint union of a small star and a 3-cycle: i.e., $K_{1,2} \cup C_3$ and $K_{1,3} \cup C_3$.

We should mention that the proof of the later theorem is similar to the proof of Theorem 1.4 (see [4]). The proof is provided in Section 2.

Theorem 1.9. *Let* $\mathrm{G} = (V(\mathrm{G}), E(\mathrm{G}))$ *be a graph such that* $|V(\mathrm{G})| = n$ *and* $|E(\mathrm{G})| = n-1$ *and let* $|e(\mathrm{G})|$ *be the number of vertices of* G *with valency* 1.

(i) If G *is connected, then* G *has a strong packing if and only if* $|V(\mathrm{G})| \geq 2|e(\mathrm{G}))|$.

(ii) If G *is disconnected, then* G *has a strong packing if and only if* G *is not isomorphic to any of the following graphs:* $K_2 \cup C_3$, $K_1 \cup 2\,C_3$, $K_1 \cup C_{n-1}$ *for* $n \geq 4$, *and* $K_{1,n-4} \cup C_3$ *for* $n \geq 8$.

In fact, we will prove that a *fixed point free* strong packing exists in the case when G is connected and $|V(\mathrm{G})| \geq 2|e(\mathrm{G}))|$. The proof of statement (i) of the later theorem is included in Section 4. The proof of statement (ii) is included in Section 3.

Throughout, a (p, q) graph is a graph that has p vertices and q edges.

2. Proof of Theorem 1.8

Henceforth , a *tree* T is a finite connected graph without cycles. For a graph $\mathrm{G} = (V(\mathrm{G}), E(\mathrm{G}))$ and $X \subseteq V(\mathrm{G})$ let $\mathrm{G} - X$ be the subgraph of G induced by $V(\mathrm{G}) \setminus X$.

Proof. The proof is by induction on $n = |V(\mathrm{G})| \geq 1$.

The theorem is clearly true for $n = 1$, 2, and 3. We assume that it holds for all $(n, n-2)$ graphs where $n < k$ and $k \geq 4$, and we consider G to be an arbitrary $(k, k-2)$ graph.

First, we suppose that G has an isolated vertex v. Since G has $k-2$ edges, it must possess a vertex u of degree greater than one. Then $\mathrm{G}_1 = \mathrm{G} - \{u, v\}$ is a $(k-2, k-m)$ graph, with $m \geq 4$, so the induction hypothesis guarantees the existence of a strong packing σ of G_1 which is fixed point free. This strong packing can be extended to a strong packing of G by defining $\phi(u) = v$, $\phi(v) = u$, and $\phi(x) = \sigma(x)$ otherwise. It is clear that this extension has no fixed vertices. Let $A \in \boldsymbol{C}(\mathrm{G})$ and suppose that $\phi(A) \in \boldsymbol{C}(\mathrm{G})$. Then $v \notin A \cup \phi(A)$ (because otherwise $A = \{v\}$ or $A = \{u\}$ which is not possible since $|A| > 1$). Hence, $A \cap \{u, v\} = \emptyset = \phi(A) \cap \{u, v\}$. It follows that $\phi(A) = \sigma(A)$. Finally we have that A and $\sigma(A)$ are elements of $\boldsymbol{C}(\mathrm{G}_1)$. Then A is either empty, a singleton, or $V(\mathrm{G}_1)$. The first two cases cannot hold since $1 < |A|$. The case $A = V(\mathrm{G}) \setminus \{u, v\} = V(\mathrm{G}_1)$ is also not possible because G_1 is not connected (this is because $|E(\mathrm{G}_1)| \leq |V(\mathrm{G}_1)| - 2$).

Henceforth, we assume that G has no isolated vertices. Since every cyclic component having r vertices has at least r edges, the components of G must include at least two nontrivial trees T_1 and T_2. If one of these trees, say T_1, is of order two, we write $V(\mathrm{T}_1) = \{v_1, v_2\}$ and consider $\mathrm{G}_2 = \mathrm{G} - \{v_2\}$, which is a $(k-1, k-3)$ graph. The induction hypothesis guarantees a strong packing σ of G_2. We define $\phi : V(\mathrm{G}) \to V(\mathrm{G})$ as follows:

$$\phi(v_1) = v_2,\ \phi(v_2) = \sigma(v_1), \text{ and}$$
$$\phi(v) = \sigma(v) \text{ for all } v \in V(\mathrm{G}_1), \text{ and } v \neq v_1.$$

With this definition, it is easy to see that ϕ is a fixed point free permutation of $V(\mathrm{G})$. Let $A \in \boldsymbol{C}(\mathrm{G})$ and suppose that $\phi(A) \in \boldsymbol{C}(\mathrm{G})$. If $v_1 \in A$ or $v_2 \in A$, then $A = \{v_1, v_2\}$. Hence, $\phi(A) = \{v_2, \sigma(v_1)\}$ which is not connected since $\sigma(v_1) \neq v_1$. Similarly if $v_1 \in \phi(A)$ or $v_2 \in \phi(A)$, then $\phi(A) = \{v_1, v_2\}$. Hence, $A = \{\sigma^{-1}(v_1), v_1\}$ which is not connected since $\sigma^{-1}(v_1) \neq v_2$.

Next we suppose that $A \cap \{v_1, v_2\} = \emptyset = \phi(A) \cap \{v_1, v_2\}$ and therefore $\phi(A) = \sigma(A) \in \boldsymbol{C}(\mathrm{G}_2)$. Since σ is a strong packing of G_2 it follows that A is either empty, a singleton, or $V(\mathrm{G}_2)$. The first two cases cannot hold since $1 < |A|$. The case $A = V(\mathrm{G}_2) = V(\mathrm{G}) \setminus \{v_2\}$ is also not possible because $v_1 \notin A$.

If neither T_1 nor T_2 is of order two, we form the graph $\mathrm{G}_3 = \mathrm{G} - V(\mathrm{T}_1)$. Let $x \in V(\mathrm{T}_1)$ be a vertex of degree at least two and $y \in V(\mathrm{G}_3)$ be also of degree at least two. Then the subgraphs $\mathrm{T}_1 - \{x\}$ and $\mathrm{G}_3 - \{y\}$ both satisfy the induction hypothesis. Let $\sigma : V(\mathrm{T}_1) \setminus \{x\} \to V(\mathrm{T}_1) \setminus \{x\}$ and $\tau : V(\mathrm{G}_3) \setminus \{y\} \to V(\mathrm{G}_3) \setminus \{y\}$ be strong packings of $\mathrm{T}_1 - \{x\}$ and $\mathrm{G}_3 - \{y\}$ respectively. We define $\phi : V(\mathrm{G}) \to V(\mathrm{G})$ as follows:

$$\phi(x) = y, \phi(y) = x,\ \phi(v) = \sigma(v) \text{ for all } v \in V(\mathrm{T}_1) \setminus \{x\} \text{ and}$$
$$\phi(v) = \tau(v) \text{ for all } v \in V(\mathrm{G}_3) \setminus \{y\}.$$

Clearly, this produces a fixed point free permutation of $V(\mathrm{G})$. Let $A \in \boldsymbol{C}(\mathrm{G})$ and suppose that $\phi(A) \in \boldsymbol{C}(\mathrm{G})$. If $x \in A$, then $A \subseteq V(\mathrm{T}_1)$ and hence $\phi(A) = \sigma(A \setminus \{x\}) \cup \{y\}$ which is not connected since $\sigma(A \setminus \{x\}) \subseteq V(\mathrm{T}_1)$ and y is not connected to any vertex of T_1. Similarly, if $y \in A$, then $A \subseteq V(\mathrm{G}_3)$ and hence $\phi(A) = \tau(A \setminus \{y\}) \cup \{x\}$ which is not connected since $\tau(A \setminus \{y\}) \subseteq V(\mathrm{G}_3)$ and x is not connected to any vertex of G_3. Therefore, $A \cap \{x, y\} = \emptyset = \phi(A) \cap \{x, y\}$ and hence $\phi(A) = \sigma(A) \in \boldsymbol{C}(\mathrm{T}_1)$ or $\phi(A) = \tau(A) \in \boldsymbol{C}(\mathrm{G}_3)$. If the first case holds, and since σ is a strong packing of $\mathrm{T}_1 - \{x\}$, then A is either empty, a singleton, or $V(\mathrm{T}_1 - \{x\})$. The first two cases cannot hold since $1 < |A|$. The last case cannot hold too because $\mathrm{T}_1 - \{x\}$ is not connected (this is because x has degree at least two in T_1). If the second case holds, and since τ is a strong packing of $\mathrm{G}_3 - \{y\}$, then A is either empty, a singleton, or $V(\mathrm{G}_3 - \{y\})$. The first two cases cannot hold since $1 < |A|$. The last case cannot hold because $\mathrm{G}_3 - \{y\}$ is not connected, thus completing the proof of the theorem. □

Remark 2.1. Clearly, a strong packing of graph is also a strong packing of any subgraph obtained by deleting edges. Hence, the strong packing of Theorem 1.8 exists for any $(p, p - n)$ graph if $n \geq 2$.

3. Proof of Theorem 1.9: disconnected graphs

Throughout, C_r denotes the cycle on $r \geq 3$ elements. If G and H are graphs with $V(\mathrm{G}) \cap V(\mathrm{H}) = \emptyset$, then $\mathrm{G} \cup \mathrm{H}$ denotes their *disjoint union*, that is, $V(\mathrm{G} \cup \mathrm{H}) = V(\mathrm{G}) \cup V(\mathrm{H})$ and $E(\mathrm{G} \cup \mathrm{H}) = E(\mathrm{G}) \cup E(\mathrm{H})$. Whenever we write $\mathrm{G} \cup \mathrm{H}$ it is implicitly assumed that $V(\mathrm{G}) \cap V(\mathrm{H}) = \emptyset$.

Lemma 3.1. *If* G *and* H *are two disconnected graphs that have strong packings, then* $\mathrm{G} \cup \mathrm{H}$ *has a strong packing.*

Proof. Let α and β be strong packings of G and H respectively. The union of mappings α and β provides a strong packing of $\mathrm{G} \cup \mathrm{H}$, as follows:

$$\phi(v) = \alpha(v), \text{ for } v \in V(\mathrm{G}) \text{ and } \phi(v) = \beta(v), \text{ for } v \in V(\mathrm{H}).$$

Notice that a connected subgraph of $\mathrm{G} \cup \mathrm{H}$ is either in a connected component of G or H. The conclusion of the lemma follows easily since neither G nor H is connected. □

Lemma 3.2. *Let* $\mathrm{G} = \mathrm{H} \cup \mathrm{K}$ *be a graph such that* $|V(\mathrm{H})| \geq 2$ *and there are two distinct vertices* $x, y \in V(\mathrm{H})$ *such that* $\mathrm{H} - \{x, y\}$ *has a strong packing and is either empty, a singleton, or disconnected. Moreover, assume* $\mathrm{K} \cup \{x, y\}$ *has a strong packing* β *such that* $\{\beta(x), \beta(y)\} \subseteq V(\mathrm{K})$*. Then there exists a permutation* ϕ *of* $V(\mathrm{G})$ *such that if* $X \in \boldsymbol{C}(\mathrm{G})$ *and* $\phi(X) \in \boldsymbol{C}(\mathrm{G})$*, then* $X = \{x, y\}$ *or* $\phi(X) = \{x, y\}$*.*

Proof. Let $\alpha \in \Pi(\mathrm{H} - \{x, y\})$. The union of mappings α and β provides the required permutation, as follows:

$$\phi(v) = \alpha(v), \text{ for } v \in V(\mathrm{H} - \{x, y\}) \text{ and } \phi(v) = \beta(v), \text{ for } v \in V(\mathrm{K}) \cup \{x, y\}.$$

Let $A \in \boldsymbol{C}(\mathrm{G})$ and suppose that $\phi(A) \in \boldsymbol{C}(\mathrm{G})$. If $x \in A$ or $y \in A$, then $A \subseteq V(\mathrm{H})$. It follows that $A = \{x\}$ or $A = \{y\}$ or $A = \{x, y\}$ since x and y are the only vertices of H that are mapped under ϕ to vertices of K. The cases $A = \{x\}$ and $A = \{y\}$ are not possible since $1 < |A|$; it follows that $A = \{x, y\}$. Similarly, if $x \in \phi(A)$ or $y \in \phi(A)$, then $\phi(A) = \{x, y\}$. Next we suppose that $A \cap \{x, y\} = \emptyset$ and $\phi(A) \cap \{x, y\} = \emptyset$. It follows that either $A \subseteq V(\mathrm{H} - \{x, y\})$ or $A \subseteq V(\mathrm{K})$. If $A \subseteq V(\mathrm{H} - \{x, y\})$, then A and $\phi(A) = \alpha(A)$ are elements of $\boldsymbol{C}(\mathrm{H} - \{x, y\})$ and hence A is either empty, a singleton or $V(\mathrm{H} - \{x, y\})$ since $\alpha \in \Pi(\mathrm{H} - \{x, y\})$. The first two cases are not possible since $1 < |A|$. The last case is also not possible since by assumption $\mathrm{H} - \{x, y\}$ is either empty, a singleton or disconnected. A contradiction. Else, A and $\phi(A) = \beta(A)$ are elements of $\boldsymbol{C}(\mathrm{K} \cup \{x, y\})$ and hence A is either empty, a singleton, or $V(\mathrm{K} \cup \{x, y\})$ since $\beta \in \Pi(\mathrm{K} \cup \{x, y\})$. The first two cases are not possible since $1 < |A|$. The last case is also not possible since $\mathrm{K} \cup \{x, y\}$ is disconnected. A contradiction. With this contradiction the proof of the lemma is now complete. □

Corollary 3.3. *Let* $\mathrm{G} = \mathrm{H} \cup C_r$ *where* $r \geq 4$ *and* H *is a graph such that* $|V(\mathrm{H})| \geq 2$*, there are two distinct vertices* $x, y \in V(\mathrm{H})$ *such that* $\mathrm{H} - \{x, y\}$ *has a strong packing, and* $\mathrm{H} - \{x, y\}$ *is either empty, a singleton, or disconnected. Then* G *has a strong packing.*

Proof. First we prove that $C_r \cup \{x, y\}$, for $r \geq 4$, has a strong packing. Set $V(C_r) = \{1, 2, ..., r\}$ so that $\{i, i+1\} \in E(C_r)$ for $1 \leq i \leq r$ (with $r+1 = 1$) and let β_r be the permutation of $\{1, 2, ..., r\}$ that maps, in an increasing order, the first $\lfloor r/2 \rfloor$ elements to the even numbers in $\{1, 2, ..., r\}$ and the remaining elements are mapped, in an increasing order, to the odd numbers in $\{1, 2, ..., r\}$. For example:

$$\beta_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}, \beta_5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix}, \beta_6 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 1 & 3 & 5 \end{pmatrix},$$

and

$$\beta_7 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 6 & 1 & 3 & 5 & 7 \end{pmatrix}.$$

Define β to be the permutation of $V(C_r \cup \{x, y\})$ as follows.

- If $r = 4$, then let
  $\beta(1) = x$, $\beta(2) = 4$, $\beta(3) = y$, $\beta(4) = 3$, $\beta(x) = 1$, and $\beta(y) = 3$.
- If $r = 5$, then let
  $\beta(1) = x$, $\beta(2) = 4$, $\beta(3) = y$, $\beta(4) = 2$, $\beta(5) = 5$, $\beta(x) = 1$, and $\beta(y) = 3$.
- If $r > 5$, then let
  $\beta(\lfloor r/2 \rfloor + 1) = x$, $\beta(\lfloor r/2 \rfloor + 3) = y$, $\beta(i) = \beta_r(i)$ if $i \notin \{\lfloor r/2 \rfloor + 1, \lfloor r/2 \rfloor + 3\}$, $\beta(x) = 1$, and $\beta(y) = 5$.

We claim that β is a strong packing of $C_r \cup \{x, y\}$. The proof of the cases $r = 4$ and 5 is similar to the case $r > 5$. Next we suppose that $r > 5$. Let $A \in \boldsymbol{C}(C_r \cup \{x, y\})$ and suppose that $\beta(A) \in \boldsymbol{C}(C_r \cup \{x, y\})$. If $x \in A$ or $y \in A$, then $A = \{x\}$ or $A = \{y\}$ or $A = \{x, y\}$. But then $\beta(A) = \{1, 5\}$ (the cases $A = \{x\}$ and $A = \{y\}$ are not possible since $1 < |A|$) which is not connected in $C_r \cup \{x, y\}$. A contradiction. This proves that $A \cap \{x, y\} = \emptyset$. Similarly, if $x \in \beta(A)$ or $y \in \beta(A)$, then $\beta(A) = \{x, y\}$ and hence $A = \{\lfloor r/2 \rfloor + 1, \lfloor r/2 \rfloor + 3\}$ which is not connected in $C_r \cup \{x, y\}$. A contradiction. This proves that $\beta(A) \cap \{x, y\} = \emptyset$. It follows that $A \subseteq V(C_r) \setminus \{\lfloor r/2 \rfloor + 1, \lfloor r/2 \rfloor + 3\}$. Since A is connected it is a set of consecutive integers. Hence, $A \subseteq \{1, ..., \lfloor r/2 \rfloor\}$ or $A \subseteq \{\lfloor r/2 \rfloor + 4, ..., r\}$. But then $\beta(A)$ is either a set of even numbers or a set of odd numbers which are not connected. A contradiction.

Let $\alpha \in \Pi(\mathrm{H} - \{x, y\})$ and ϕ be the union of the maps α and β. From Lemma 3.2 we deduce that it is enough to prove that if $X \in \boldsymbol{C}(\mathrm{G})$ and $\phi(X) \in \boldsymbol{C}(\mathrm{G})$, then $X \neq \{x, y\}$ or $\phi(X) \neq \{x, y\}$. This is obvious from the construction of β. The proof of the corollary is now complete. □

Corollary 3.4. *The graph* $\mathrm{G} = C_r \cup C_s$ *has a strong packing if and only if* $\max(r, s) \geq 4$.

Proof. If $\max(r, s) \leq 3$, then $\mathrm{G} = C_3 \cup C_3$ which is easily seen not to have a strong packing.

Suppose now that $\max(r,s) \geq 4$, say $r \geq 4$. Let $x, y \in V(C_s)$ be any two distinct vertices, and in the case $s \geq 4$, we impose that $\{x,y\} \notin E(C_s)$. Then $C_s - \{x,y\}$ is either a singleton or the disjoint union of two paths which has a strong packing by Theorem 1.8. We now apply Corollary 3.3 to obtain a strong packing of G. ☐

Let $\mathfrak{F}$ be the set of forbidden graphs: $K_2 \cup C_3$, $K_1 \cup 2\,C_3$, $K_1 \cup C_{n-1}$ for $n \geq 4$, and $K_{1,n-4} \cup C_3$ for $n \geq 8$. Now, we begin by disposing of a certain class of $(n, n-1)$ graphs.

Theorem 3.5. *Let* T *be any tree. If* G *is the union of* T *and* $m \geq 1$ *disjoint cycles, and* $\mathrm{G} \notin \mathfrak{F}$*, then* G *has a strong packing.*

Proof. Suppose that among the m cycles of G, there is a cycle C_r with $r \geq 4$. If $m = 1$, then since $\mathrm{G} \notin \mathfrak{F}$, T is nontrivial. Then either $t = |V(\mathrm{T})| = 2$ or T has two distinct vertices x and y such that at least one of x and y has degree at least two. Hence, $\mathrm{T} - \{x,y\}$ is either empty, a singleton, or is disconnected and therefore has a strong packing (use Theorem 1.8 for the last case). We can now apply Corollary 3.3 to deduce a strong packing of G. If $m > 1$, then let x and y be any two distinct vertices of a cycle other than C_r and consider the graph $\mathrm{G}_1 = \mathrm{G} - V(C_r)$. Then $\mathrm{G} = \mathrm{G}_1 \cup C_r$ and $\mathrm{G}_1 - \{x,y\}$ is either empty, a singleton, or is disconnected. Moreover, G_1 has a strong packing since it is a $(k, k-p)$ graph for some $p > 1$. We can now apply Corollary 3.3 to deduce a strong packing of G.

Having proven the theorem for graphs with a cycle C_r with $r \geq 4$, as one of its components, we assume henceforth that the m cycles of G are 3-cycles. The remainder of the proof proceeds by induction on m.

Now assume that G has a strong packing for any graph G satisfying the hypothesis of the theorem, where $m < k$ and $k \geq 2$. Let H be a $(p, p-1)$ graph satisfying the hypothesis, where H is the union of a tree T and k cycles C_3.

Suppose that $\mathrm{T} = K_1$. Then $k > 2$, otherwise $\mathrm{H} \in \mathfrak{F}$. We now exhibit a strong packing of $k\,C_3$. Let $\{x_i, y_i, z_i\}$ be the vertices of the ith copy of C_3 for $0 \leq i \leq k-1$. Define the permutation ϕ of $\cup_{i=1}^{k}\{x_i, y_i, z_i\}$ as follows:

$$\phi(x_i) = x_i,\ \phi(y_i) = y_{i+1 \mod k} \text{ and } \phi(z_i) = z_{i+2 \mod k} \text{ for all } 1 \leq i \leq k.$$

It can be easily verified that ϕ is a strong packing. Hence, H has a strong packing.

Suppose T is the star $K_{1,n}$ and $k = 2, 3$, or 4. Let $\{x_i, y_i, z_i\}$ be the vertices of the ith copy of C_3 for $0 \leq i \leq k-1$ and let $\{0, 1, ..., n-1\}$ be the vertices of $K_{1,n}$ so that the vertex 0 has maximum degree. Define the permutation

$$\begin{gathered} \phi(x_i) = x_i,\ \phi(y_i) = y_{i+1 \mod k} \text{ for all } 0 \leq i \leq k-1, \text{ and} \\ \phi(0) = z_0,\ \phi(1) = z_1,\ \phi(l) = l \text{ for } 2 \leq l \leq n-1, \text{ and} \\ \phi(z_{k-2}) = 0,\ \phi(z_{k-1}) = 1 \text{ and } \phi(z_i) = z_{i+2 \mod k} \text{ otherwise.} \end{gathered}$$

Then it is easy to see that ϕ is a strong packing.

If $k > 4$, the graph $\mathrm{H}_2 = \mathrm{H} - \{3\,C_3\}$ obeys the induction hypothesis; hence, H_2 has a strong packing. And, since $3\,C_3$ has a strong packing, we conclude that H has a strong packing. This completes the analysis for T a star.

We now turn to the general case in which T is neither K_1 nor a star $K_{1,n}$. For $k > 3$, $\mathrm{H}_3 = \mathrm{H} - \{3\,C_3\}$ satisfies the induction hypothesis; hence H_3 has a strong packing and $3\,C_3$ has a strong packing completes the proof that H has a strong packing. We now deal with the cases $k = 2$ and 3. Let $x, y \in V(\mathrm{T})$ be two distinct vertices such that at least one of x and y has degree two and $\{x, y\} \notin E(\mathrm{T})$. Form the graph G_1 induced by $k\,C_3 \cup \{x, y\}$ and $\mathrm{G}_2 = \mathrm{T} - \{x, y\}$. Then G_1 and G_2 are $(p, p-q)$ graphs (for some suitable values of p and $q \geq 2$). From Theorem 1.8 we deduce that both have a strong packing. In fact one can easily see that G_1 has a strong packing β such that $\beta(\{x, y\}) \subseteq V(k\,C_3)$. From Lemma 3.2 we deduce that the union of these packings yields a strong packing of G. □

We are now able to present the complete classification of those disconnected $(n, n-1)$ graphs which have a strong packing.

Clearly, if $\mathrm{G} \in \mathfrak{F}$, then G does not have a strong packing. We now suppose that $\mathrm{G} \notin \mathfrak{F}$ and we prove that G has a strong packing. If v is an isolated vertex of G, then $\mathrm{G} - \{v\}$ is a $(n-1, n-1)$ graph. Hence, $\mathrm{G} - \{v\}$ is either a union of cycles or else it contains a vertex u of degree at least 3. The former case is covered by Theorem 3.5. In the latter case, $\mathrm{G} - \{u, v\}$ is a $(n-2, n-k)$ graph with $k \geq 4$. Thus, by Remark 2.1, we know that there is a strong packing ϕ of $\mathrm{G} - \{u, v\}$. Defining $\phi(v) = u$ and $\phi(u) = v$ provides a strong packing of G.

If G possesses no isolated vertices, then it must have a tree T of order $t \geq 2$ as one of its components (for every cyclic component with n vertices has at least n edges). Then $\mathrm{G} - V(\mathrm{T})$ is a $(n-t, n-t)$ graph. Either $\mathrm{G} - V(\mathrm{T})$ is a disjoint union of cycles or $\mathrm{G} - V(\mathrm{T})$ contains a vertex w whose degree is at least 3. The former alternative, in which G is the union of a tree and cycles, is covered by Theorem 3.5. In the second alternative, $\mathrm{G} - V(\mathrm{T} - \{w\})$ is a $(n-t-1, n-t-s)$ graph with $s \geq 3$; hence, there is a strong packing α of $\mathrm{G} - V(\mathrm{T} - \{w\})$. Also, if z is a vertex of maximal degree in T, then there is a strong packing β of $\mathrm{T} - \{z\}$, because $\mathrm{T} - \{z\}$ is either a $(t-1, t-m)$ graph with $m \geq 3$ or it is K_1. By defining

$$\begin{gathered}\phi(w) = z, \ \ \phi(z) = w,\\ \phi(x) = \alpha(x) \text{ for all } x \in V(\mathrm{G} - V(\mathrm{T} - \{w\})),\\ \phi(y) = \beta(y) \text{ for } y \in V(\mathrm{T} - \{z\}),\end{gathered}$$

we obtain a strong packing of G. This completes the proof of statement (ii) of Theorem 1.9.

4. Proof of Theorem 1.9: connected graphs

4.1. **Basic notions.** An *endpoint* of a tree T is a vertex of valence less than or equal to one. A vertex of T which is not an endpoint is an *inner point* of T. That is the tree P_1 which consists of exactly one vertex has one endpoint and no inner points, the tree P_2 with two vertices has two endpoints the path P_3 with three points has two endpoints and one inner point and the path P_4 with four vertices has two endpoints and two inner points. Let $V(\mathrm{T})$ denote the set of vertices of T and $\boldsymbol{e}(\mathrm{T})$ the set of endpoints of T and $\boldsymbol{i}(\mathrm{T})$ the set of inner points of T.

Then $\Pi(\mathrm{P}_1) \neq \emptyset \neq \Pi(\mathrm{P}_2)$ and $\Pi(\mathrm{P}_4) \neq \emptyset$ while $\Pi(\mathrm{P}_3) = \emptyset$. Note that if $\sigma \in \Pi(\mathrm{T})$ then $\sigma^{-1} \in \Pi(\mathrm{T})$.

Lemma 4.1. *For $n \geq 4$ and P_n a path of length n, with $V(\mathrm{P}_n) = \{1, 2, 3, \ldots, n\}$ and two consecutive numbers adjacent in P_n, there exists a permutation $\pi_n \in \Pi(\mathrm{P}_n)$.*

Proof. For example, let:

$$\pi_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}, \quad \pi_5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 5 & 2 & 4 \end{pmatrix}$$

and

$$\pi_6 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 5 & 2 & 6 & 4 \end{pmatrix}.$$

In general, permutations π_n may be obtained by mapping the first $\lfloor n/2 \rfloor$ numbers to odd numbers in such a way that the only consecutive numbers mapped to numbers of different parity are mapped to the largest number and the smallest number of the other parity. $\square$

It will be notationally convenient for the remainder of this article to demand that a tree has at least two vertices.

Lemma 4.2. *If $\pi \in \Pi(\mathrm{T})$, then $\pi(x) \in \boldsymbol{i}(\mathrm{T})$ for all $x \in \boldsymbol{e}(\mathrm{T})$. If $\Pi(\mathrm{T}) \neq \emptyset$ then the number of endpoints of T is less than or equal to the number of inner points of T, that is $|\boldsymbol{e}(\mathrm{T})| \leq |\boldsymbol{i}(\mathrm{T})|$.*

Proof. For a contradiction let $x_0, x_1 \in \boldsymbol{e}(\mathrm{T})$ with $\pi(x_0) = x_1$. If $x_0 = x_1$ then π maps $V(\mathrm{T} - x_0) \in \boldsymbol{C}(\mathrm{T})$ onto $V(\mathrm{T} - x_0) \in \boldsymbol{C}(\mathrm{T})$. Otherwise π maps $V(\mathrm{T} - x_0) \in \boldsymbol{C}(\mathrm{T})$ onto $V(\mathrm{T} - x_1) \in \boldsymbol{C}(\mathrm{T})$. $\square$

It follows that $\Pi(\mathrm{T}) = \emptyset$ if $|\boldsymbol{e}(\mathrm{T})| > |\boldsymbol{i}(\mathrm{T})|$ and that if $|\boldsymbol{e}(\mathrm{T})| = |\boldsymbol{i}(\mathrm{T})|$ and $\pi \in \Pi(\mathrm{T})$, then π maps every endpoint of T to an interior point of T and every interior point of T to an endpoint of T.

We aim to prove the following Theorem:

Theorem 4.3. *Let $\mathrm{T} = (V(\mathrm{T}), E(\mathrm{T}))$ be a tree. Then $\Pi(\mathrm{T}) \neq \emptyset$ if and only if $|\boldsymbol{i}(\mathrm{T})| \geq |\boldsymbol{e}(\mathrm{T})|$.*

Let $\boldsymbol{T}$ be the set of trees T for which $|\boldsymbol{i}(\mathrm{T})| \geq |\boldsymbol{e}(\mathrm{T})|$ and $|V(\mathrm{T})| \geq 4$. The set of trees T with $|\boldsymbol{i}(\mathrm{T})| \geq |\boldsymbol{e}(\mathrm{T})|$ and $|V(\mathrm{T})| \leq 5$ consists of the path on four vertices and the path on five vertices. It follows from Lemma 4.1 that Theorem 4.3 holds for all trees $\mathrm{T} \in \boldsymbol{T}$ with $|V(\mathrm{T})| \leq 5$. From Lemma 4.2, it follows that it suffices to show that $\Pi(\mathrm{T}) \neq \emptyset$ for all $\mathrm{T} \in \boldsymbol{T}$ with $|V(\mathrm{T})| \geq 6$.

The proof will be by induction on the number of vertices of T. Let $\mathfrak{M} \subseteq \boldsymbol{T}$ consist of all trees T having at least six vertices and $\Pi(\mathrm{T}) = \emptyset$ so that if $\mathrm{T}' \in \boldsymbol{T}$ with $|V(\mathrm{T}')| < |V(\mathrm{T})|$ then $\Pi(\mathrm{T}') \neq \emptyset$. We will prove Theorem 4.3 by showing that $\mathfrak{M} = \emptyset$.

4.2. **Hanging endpoints.** An endpoint of a tree T is a *hanging endpoint* of T if it is adjacent to a vertex of valence two. We will show in this section that a tree in $\mathfrak{M}$ does not have a hanging endpoint. But first the following:

Lemma 4.4. *Let* $\mathrm{T} \in \boldsymbol{T}$ *with* $|\boldsymbol{i}(\mathrm{T})| = |\boldsymbol{e}(\mathrm{T})|$ *and* $p \in V(\mathrm{T})$ *of degree two so that both neighbours* b *and* c *of* p *are vertices in* $\boldsymbol{i}(\mathrm{T})$*. Then* $\mathrm{T} \notin \mathfrak{M}$*.*

Proof. Assume $\mathrm{T} \in \mathfrak{M}$. Then in particular $|V(\mathrm{T})| \geq 6$. Because p is not adjacent to an endpoint there exists a vertex v adjacent to two different endpoints. Note that v has valence at least three. Let x be one of the endpoints adjacent to v. Let the tree T' be obtained from T by removing the vertices x and p from T and adding the edge $\{b, c\}$. Then v, b, and c are inner points of T', $|\boldsymbol{i}(\mathrm{T}')| = |\boldsymbol{e}(\mathrm{T}')|$, and $\mathrm{T}' \in \boldsymbol{T}$. It follows from the minimality of $\mathfrak{M}$ that there is a permutation $\sigma \in \Pi(\mathrm{T}')$. Then it follows from Lemma 4.2 that σ maps the vertices b, c, and v to endpoints of T', which are not adjacent to p in T because p is only adjacent to inner points. Let π be the permutation of $V(\mathrm{T})$ which agrees with σ on $V(\mathrm{T}')$ and for which $\pi(p) = x$ and $\pi(x) = p$. We claim that $\pi \in \Pi(\mathrm{T})$ and assume for a contradiction that there is a set $A \in \boldsymbol{C}(\mathrm{T})$ with $\pi(A) \in \boldsymbol{C}(\mathrm{T})$.

If $\{p, x\} \cap A = \emptyset$ then $A \in \boldsymbol{C}(\mathrm{T}')$ and $\sigma(A) \in \boldsymbol{C}(\mathrm{T}')$. If $p \in A$ and $x \notin A$ then $A \setminus \{p\}$ is connected in T' and so is $\pi(A) \setminus \{x\}$ because x is an endpoint. It follows then from $\pi(A \setminus \{p\}) = \sigma(A \setminus \{p\})$ that either $A \setminus \{p\} = V(\mathrm{T}')$, in which case $\pi(A) = \mathrm{T} \setminus \{p\}$ is not connected, or that $|A \setminus \{p\}| = 1$. Then $A = \{p, b\}$ or $A = \{p, c\}$. But then $\pi(A) = \{x, \pi(b)\}$ or $\pi(A) = \{x, \pi(c)\}$ are sets of two endpoints and hence not connected. If $x \in A$ and $p \notin A$ then $A \setminus \{x\}$ is connected in T' and so is $\pi(A) \setminus \{p\}$. It follows then from $\pi(A \setminus \{x\}) = \sigma(A \setminus \{x\})$ that either $A \setminus \{x\} = V(\mathrm{T}')$, in which case $A = V(\mathrm{T}) \setminus \{p\}$ is not connected, or that $|A \setminus \{x\}| = 1$. Then $A = \{v, x\}$ and then $\pi(A) = \{p, \pi(v)\}$ is not connected because $v \in \boldsymbol{i}(\mathrm{T}')$ and hence $\pi(v) = \sigma(v)$ is an endpoint and therefore not adjacent to p.

If $\{p, x\} \subseteq A$ then $B = A \setminus \{p, x\}$ is connected in T' and so is $\pi(B) = \pi(A) \setminus \{p, x\}$. Hence $B = V(\mathrm{T}')$ or $|B| \leq 1$. If $B = V(\mathrm{T}')$ then $A = V(\mathrm{T})$. Because $\{p, x\}$ is not connected we have $|B| = 1$, say $B = y$. Then $A = \{p, x, y\}$ implying $y \in \boldsymbol{i}(\mathrm{T})$ because p and x are not adjacent and hence y is adjacent to both p and x. But then $\pi(A) = \{p, x, \sigma(y)\}$ is not connected because x and $\sigma(y)$ are two endpoints not adjacent to p. $\square$

Lemma 4.5. *Let* $\mathrm{T} \in \boldsymbol{T}$ *and* $e \in \boldsymbol{e}(\mathrm{T})$ *be a hanging endpoint of* T. *Then* $\mathrm{T} \notin \mathfrak{M}$.

Proof. Assume $\mathrm{T} \in \mathfrak{M}$. Then in particular $|V(\mathrm{T})| \geq 6$. Let e be adjacent to the vertex i which in turn is adjacent to the vertex $c \neq e$. Let T' be the tree obtained from T by removing the vertices e and i. It follows from Lemma 4.4 and $|V(T)| \geq 6$ that if $|\boldsymbol{i}(\mathrm{T})| = |\boldsymbol{e}(\mathrm{T})|$ then the valence of c is larger than two. Hence $\mathrm{T}' \in \boldsymbol{T}$ and there exists a permutation $\sigma \in \Pi(\mathrm{T}')$. Let $w = \sigma^{-1}(c)$, that is $\sigma(w) = c$ and $w \neq c$ because σ is fixed point free.

Let $D = V(\mathrm{T})\setminus\{w, i, e\}$ and π the permutation of $V(\mathrm{T})$ which agrees with σ on D and for which $\pi(w) = i$, $\pi(i) = e$, and $\pi(e) = c$. Then π is fixed point free and we will show that $\pi \in \Pi(\mathrm{T})$. Assume for a contradiction that there exists a set $A \in \boldsymbol{C}(\mathrm{T})$ for which $\pi(A)$ is connected. Let $B = A - \{w, i, e\}$.

If $B = \emptyset$ then $A = \{i, e\}$ and $\pi(A) = \{e, c\}$ which is not connected. Hence $B \neq \emptyset$. Because π restricted to B is equal to σ restricted to B it follows that $A \neq B$ and hence $\{w, i, e\} \cap A \neq \emptyset$.

Suppose next that $i \in A$. Then $w \in A$ for otherwise e is an isolated point of $\pi(A)$. Hence all points on the path from w to i and in particular c, are in A implying $|A| \geq 3$ and in turn that $|\pi(A)| \geq 3$. It follows then from $e \in \pi(A)$ that $c \in \pi(A)$ implying that $e \in A$. We conclude that $\{w, i, e\} \subset A$ and $\{i, e, c\} \subset \pi(A)$. Note that $A\setminus\{i, e\} = B\cup\{w\}$ and $\pi(A)\setminus\{i, e\} = \pi(B)\cup\{c\}$ are connected because e is an endpoint and i is an endpoint after removing e.

But then $\sigma(B \cup \{w\}) = \sigma(B) \cup \{c\} = \pi(B) \cup \{c\}$ provides a contradiction because $|B \cup \{w\}| \geq 2$ and if $B \cup \{w\} = V(\mathrm{T}')$ then $A = V(\mathrm{T}) \notin \boldsymbol{C}(\mathrm{T})$.

Hence we are left with the case that $i \notin A$. Then $e \notin A$ implying $A = B\cup\{w\}$. Because $e \notin A$ the point c is not in $\pi(A)$ leading to the contradiction that i is an isolated point of the connected set $\pi(A)$. □

Lemma 4.6. *If* $\mathrm{T} \in \mathfrak{M}$ *then* $|V(\mathrm{T})| \geq 8$.

Proof. We are left with the cases $|V(\mathrm{T})|$ is equal to six or seven. We will show that if $7 \leq |V(\mathrm{T})| < 8$ then T has a hanging endpoint and hence according to Lemma 4.5 is not in $\mathfrak{M}$.

Note that T can not have more than three endpoints. The set X of inner points of T is connected and hence a tree with at least two different endpoints, say x and y. If an endpoint e adjacent to x is not hanging then there is another endpoint of T also adjacent to x and similarly for y. This is not possible because T has at most three endpoints. □

4.3. **No hanging endpoints.** Let $\mathrm{P} = (P, \leq)$ be a partial order. For $x, y \in P$ the point x *covers* the point y if $y < x$ and $y \leq z \leq x$ implies $y = z$ or $z = x$. The *cover graph* of P is the graph with P as the set of vertices and x adjacent to y if x covers y or y covers x.

Let $\mathrm{T} = (V(\mathrm{T}), E(\mathrm{T}))$ be a tree and $r \in \boldsymbol{i}(\mathrm{T})$. We define a binary relation $\leq$ on $V(\mathrm{T})$ as follows: if $u, v \in V(\mathrm{T})$, then $u \leq v$ if and only if the unique path in T joining r to v contains u. It turns out that $\leq$ is an order relation on $V(\mathrm{T})$. We denote by $\mathrm{T}_{\langle r\rangle} = (V(\mathrm{T}), \leq)$ the corresponding ordered set. Then r is the smallest element of $\mathrm{T}_{\langle r\rangle}$, the set $\boldsymbol{e}(\mathrm{T})$ is the set of maximal elements of $\mathrm{T}_{\langle r\rangle}$, and the elements below every element of $V(\mathrm{T})$ are totally ordered.

Removing any element from $\mathrm{T}_{\langle r\rangle} \setminus \{r\}$ leaves a connected ordered set whose covering graph is a tree. Hence if $X \subseteq \mathrm{T}_{\langle r\rangle} \setminus \{r\}$ then the cover graph T' of $\mathrm{T}_{\langle r\rangle} \setminus X$ is connected and indeed a tree.

Remark 4.7. Let $r \notin B \subseteq V(\mathrm{T})$ and T' the cover graph of $\mathrm{T}_{\langle r\rangle} \setminus B$. Then if u, v are two vertices in a connected component of $\mathrm{T} - B$ the vertices u and v are adjacent in T if and only if the vertices u and v are adjacent in T'. Hence if A is a connected subset of $\mathrm{T} - B$ then it is in one of the connected components of $\mathrm{T} - B$ and hence it is a connected subset of T'.

Lemma 4.8. *Let $A \subseteq V(\mathrm{T})$ be connected in T and $B \subseteq A$ and $r \in (A \setminus B) \cap \boldsymbol{i}(\mathrm{T})$. Then $A \setminus B$ is connected in the covering graph T' of $\mathrm{T}_{\langle r\rangle} \setminus B$.*

Proof. If $A \setminus B$ is not connected in T' then there are two vertices $a, b \in A \setminus B$ having minimal distance in T and so that if $a = x_0, x_1, \ldots, x_n = b$ is the path from a to b at least one of the vertices in $X := \{x_1, x_2, \ldots, x_{n-2}, x_{n-1}\}$ is an element of B. It follows from the minimality of n, the distance from a to b, that $X \subseteq B$. If $a < b$ or $b < a$ in $\mathrm{T}_{\langle r\rangle}$ then a is adjacent to b in $\mathrm{T}_{\langle r\rangle} \setminus B$. Otherwise $r < a$ and $r < b$ with $r \in A \setminus B$, implying that there is a vertex $c \in A \setminus B$ so that $c < a$, $c < b$, and c is adjacent both to a and to b in $\mathrm{T}_{\langle r\rangle} \setminus B$. □

Lemma 4.9. *Let $\mathrm{T} \in \boldsymbol{T}$ and let $e_1, e_2 \in \boldsymbol{e}(\mathrm{T})$ being adjacent to the vertices i_1 and i_2 respectively having distance between i_1 and i_2 at least two. Then $\mathrm{T} \notin \mathfrak{M}$.*

Proof. Let $\mathrm{T} \in \mathfrak{M}$ and r a vertex on the path from i_1 to i_2 so that $i_1 \neq r \neq i_2$ and $c_j \neq e_j$ the points adjacent to i_j. Let T' be the covering graph of $\mathrm{T}_r \setminus \{e_1, e_2, i_1, i_2\}$. Lemma 4.6 implies $|V(\mathrm{T}')| \geq 4$. Because $\mathrm{T} \in \mathfrak{M}$ the tree T has no hanging endpoints according to Lemma 4.5, implying that the vertices i_1 and i_2 have valence at least three. This fact ensures that $|\boldsymbol{i}(\mathrm{T}')| \geq |\boldsymbol{e}(\mathrm{T}')|$ and therefore $\mathrm{T}' \in \boldsymbol{T}$. Because of the minimality of $\mathfrak{M}$ there exists a permutation $\sigma \in \Pi(\mathrm{T}')$.

We define a permutation π of $V(\mathrm{T})$ to $V(\mathrm{T})$ as follows:

$$\pi(e_1) = i_1,\ \pi(i_1) = e_2,\ \pi(e_2) = i_2,\ \pi(i_2) = e_1, \text{ and}$$
$$\pi(v) = \sigma(v) \text{ for all } v \in V(\mathrm{T}) \setminus \{e_1, e_2, i_1, i_2\}.$$

Clearly π has no fixed points. We now prove that $\pi \in \Pi(\mathrm{T})$. Let $A \in \boldsymbol{C}(\mathrm{T})$ and suppose for a contradiction that $\pi(A) \in \boldsymbol{C}(\mathrm{T})$. Then $A \cap \{e_1, e_2, i_1, i_2\} \neq \emptyset$, because otherwise A and $\pi(A) = \sigma(A)$ are connected subsets of T' according to Remark 4.7 and hence $A = V(\mathrm{T}')$. But $V(\mathrm{T}') = V(\mathrm{T}) \setminus \{e_1, e_2, i_1, i_2\}$ is not connected in T because i_1 has valence at least three. But then at least one of i_1 or i_2 is in A because the endpoints e_1 and e_2 would be isolated without i_1 or i_2 in A. Assume without loss of generality that $i_1 \in A$.

Then $e_2 \in \pi(A)$ and hence $i_2 \in \pi(A)$, otherwise e_2 is isolated, implying $e_2 \in A$ which in turn implies that $i_2 \in A$. That is in all cases $\{e_1, i_1, e_2, i_2\} \subseteq A \cap \pi(A)$. This implies that the path from i_1 to i_2 and hence r is in $A \setminus B$ and in $\pi(A) \setminus B$. Then Lemma 4.8 implies that $A \setminus B$ and $\pi(A) \setminus B$ are both connected in T'.

It follows that $A \setminus B$ is either empty, a singleton, or $V(\mathrm{T}')$. This latter case is not possible since by assumption we have $A \neq V(\mathrm{T})$. The case $A = B$ is also not possible because B is not a connected subset of T. So we are left with the case $A \setminus B$ is a singleton, say $A = B \cup \{x\}$. Then both x and $\pi(x) = \sigma(x)$ are neighbours of i_1 and i_2 in T and therefore $\pi(x) = \sigma(x) = x$ contradicting our assumption that σ has no fixed points. $\square$

4.4. **Proof of Theorem 4.3.**

Lemma 4.10. *Every tree* $\mathrm{T} \in \boldsymbol{T}$ *has at least two endpoints* e_1 *and* e_2*, adjacent to vertices* i_1 *and* i_2 *respectively, whose distance is at least two.*

Proof. Let X be the set of vertices in T adjacent to endpoints of T. If there are no two vertices in X of distance two then any two vertices in X are adjacent, implying $|X| = 2$. Let $X = \{i_1, i_2\}$. If T is not the path on four vertices then at least one of i_1 or i_2 is adjacent to an inner point, say a. It follows that the tree T has to have an endpoint not adjacent to i_1 or i_2 in contradiction to the choice of X. $\square$

We now turn to the proof of Theorem 4.3

Proof. It follows from Lemma 4.2 that if $|\boldsymbol{i}(\mathrm{T})| < |\boldsymbol{e}(\mathrm{T})|$ then $\Pi(\mathrm{T}) = \emptyset$.

If there is a tree S with $|\boldsymbol{i}(\mathrm{S})| \geq |\boldsymbol{e}(\mathrm{S})|$ and $\Pi(\mathrm{S}) = \emptyset$ then there is a tree $\mathrm{T} \in \mathfrak{M}$. It follows from Lemma 4.10 that T contains two endpoints adjacent to vertices of distance at least two. But then we arrive at a contradiction to Lemma 4.9. $\square$

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