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SHELLABILITY, VERTEX DECOMPOSABILITY, AND LEXICOGRAPHICAL PRODUCTS OF GRAPHS

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Abstract. In this note we describe when the independence complex of $G[H]$, the lexicographical product of two graphs G and H, is either vertex decomposable or shellable. As an application, we show that there exists an infinite family of graphs whose independence complexes are shellable but not vertex decomposable.

1. Introduction

Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two finite simple graphs. There are a number of constructions in the literature that enable one to make a "product" of two graphs, that is, a new graph on the vertex set $V_G \times V_H$. In this paper we are interested in the lexicographical product. The *lexicographical product* of G and H, denoted $G[H]$, is the graph with the vertex set $V_G \times V_H$, such that (w, x) and (y, z) are adjacent if $\{w, y\} \in E_G$ or if $w = y$ and $\{x, z\} \in E_H$.

Given some property that both G and H possess, it is then natural to ask if $G[H]$ also possesses this property. The property of being well-covered is an example of such an inherited property. Recall that a subset $W \subseteq V_G$ of a graph G is a *vertex cover* if $e \cap W \neq \emptyset$ for all $e \in E_G$. A graph is *well-covered* if every minimal (ordered with respect to inclusion) vertex cover has the same cardinality. Topp and Volkmann [9] showed that G and H are well-covered if and only if $G[H]$ is well-covered.

In this note we focus on the independence complex of $G[H]$. Recall that a subset $W \subseteq V_G$ is an *independent set* if for all $e \in E_G$, $e \not\subseteq W$. Equivalently, $W \subseteq V_G$ is an independent set if and only if $V_G \setminus W$ is a vertex cover of G. The independence complex of a graph G, denoted $\mathrm{Ind}(G)$, is the simplicial complex

$$\mathrm{Ind}(G) = \{W \subseteq V_G \mid W \text{ is a independent set}\}.$$

Because of the duality between vertex covers and independent sets, $\mathrm{Ind}(G)$ is pure (see the next section) if and only if G is well-covered. Topp and

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Volkmann's result can be restated as saying $\mathrm{Ind}(G[H])$ is pure if and only if $\mathrm{Ind}(G)$ and $\mathrm{Ind}(H)$ are both pure.

If a simplicial complex is pure, it may indicate that the complex has a richer combinatorial or topological structure. Two examples relevant to this paper are vertex decomposability or shellability. Inspired by Topp and Volkmann's result, we can ask if $\mathrm{Ind}(G)$ and $\mathrm{Ind}(H)$ are both shellable, respectively, vertex decomposable, does $\mathrm{Ind}(G[H])$ also inherit this property? The purpose of this short note is to prove that this natural guess is too naive. In fact, $\mathrm{Ind}(G[H])$ is rarely shellable or vertex decomposable, as we demonstrate in Theorem 2.6.

We conclude this paper with some applications to circulant graphs. In particular, starting from a circulant graph found in [4], we construct an infinite family of graphs whose independence complexes are shellable but not vertex decomposable. To the best of our knowledge, this is the first known infinite family with this property.

2. Shellability and Vertex Decomposability

A simplicial complex Δ on a vertex set $V = \{x_1, \ldots, x_n\}$ is a subset of 2^V such that (i) if $G \subseteq F \in \Delta$, then $G \in \Delta$, and (ii) $\{x_i\} \in \Delta$ for all $x_i \in V$. Elements of Δ are called *faces*. The maximal faces of Δ with respect to inclusion are called the *facets* of Δ. A simplicial complex is called *pure* if all its facets have the same cardinality. If $F_1, \ldots, F_s$ is a complete list of the facets of Δ, then we sometimes write $\Delta = \langle F_1, \ldots, F_s \rangle$.

Given any simplicial complex Δ and face $F \in \Delta$, we can create two new simplicial complexes. The *deletion* of F from Δ is $\mathrm{del}_\Delta(F) = \{G \in \Delta \mid F \not\subseteq G\}$. The *link* of F in Δ is $\mathrm{link}_\Delta(F) = \{G \in \Delta \mid F \cap G = \emptyset \text{ and } F \cup G \in \Delta\}$. When $F = \{x\}$ for a vertex $x \in V$, we shall abuse notation and simply write $\mathrm{del}_\Delta(x)$ and $\mathrm{link}_\Delta(x)$.

Definition 2.1. *Let Δ be a pure simplicial complex.*

- (i) *Δ is* shellable *if there is an ordering of the facets $F_1, \ldots, F_s$ of Δ such that for all $1 \leq j < i \leq s$, there is some $x \in F_i \setminus F_j$ and some $k \in \{1, \ldots, i-1\}$ such that $\{x\} = F_i \setminus F_k$.*
- (ii) *Δ is* vertex decomposable *if (a) Δ is a simplex (i.e., has a unique facet), or (b) there exists a vertex $x \in V$ such that $\mathrm{del}_\Delta(x)$ and $\mathrm{link}_\Delta(x)$ are vertex decomposable.*

Vertex decomposability and shellability are related as follows:

Lemma 2.2 ([8, Corollary 2.9])**.** *Let Δ be a pure simplicial complex. If Δ is vertex decomposable, then Δ is shellable.*

The independence complex $\mathrm{Ind}(G)$ of graph G is an example of a simplicial complex. We will say G is vertex decomposable, respectively shellable, if $\mathrm{Ind}(G)$ has this property.

The following result, due to Hoshino [6], is the first of two key critical results needed to prove Theorem 2.6. In the proof, $\pi_1 : V_G \times V_H \to V_G$

denotes the projection $\pi_1((x_i, y_j)) = x_i$. In addition, $\alpha(G)$ denotes the cardinality of the largest independent set of G and K_m denotes the complete graph on m vertices. Note that when G is a graph of t isolated vertices, then $G[H]$ is simply t disjoint copies of H.

Theorem 2.3. *Suppose that G is not the graph of isolated vertices. If H is not a complete graph, then $G[H]$ is not shellable.*

Proof. (Based upon [6, Theorem 4.52].) We note that $\mathrm{Ind}(G)$ has at least two facets. Indeed, G must have at least two vertices that are adjacent, say x_1 and x_2. Because $\{x_1\}$ and $\{x_2\}$ are independent sets, there exist facets of $\mathrm{Ind}(G)$, that contain x_1 and x_2. Furthermore, these facets must be distinct because x_1 is adjacent to x_2.

If H is not a complete graph, then there are at least two vertices in V_H that are not adjacent, and thus $\alpha(H) \geq 2$. Furthermore, the construction of $G[H]$ implies that $\alpha(G[H]) = \alpha(G)\alpha(H)$.

Suppose that $\mathrm{Ind}(G[H])$ has a shelling. Let $F_1, \dots, F_s$ be the corresponding shelling. Because $\mathrm{Ind}(G[H])$ is shellable, it is pure, which implies that both $\mathrm{Ind}(G)$ and $\mathrm{Ind}(H)$ are pure (this is a restatement of Topp and Volkmann's [9] result about well-covered graphs). So, every facet of $\mathrm{Ind}(G[H])$ has cardinality $\alpha(G)\alpha(H)$. For each $i \in \{1, \dots, s\}$, it follows that $\pi_1(F_i)$ is a maximal independent set of G, that is, $\pi_1(F_i) \in \mathrm{Ind}(G)$. Because $\mathrm{Ind}(G)$ has at least two facets, there is an index k such that $\pi_1(F_1) = \cdots = \pi_1(F_{k-1}) \neq \pi_1(F_k)$. Then, for each $i = 1, \dots, k-1$, we have

$$|F_i \cap F_k| \leq (\alpha(G) - 1)\alpha(H) < \alpha(G)\alpha(H) - 1 = |F_k| - 1$$

where the strict inequality follows from the fact that $\alpha(H) \geq 2$.

However, because $F_1, \dots, F_s$ is a shelling order, for every $1 \leq j < k$, there exists some $x \in F_k \setminus F_j$ such that $\{x\} = F_k \setminus F_i$ for some $1 \leq i < k$. Because F_k and F_i have the same cardinality, this implies that $|F_i \cap F_k| = |F_k| - 1$, which contradicts the inequality given above. So, $\mathrm{Ind}(G[H])$ cannot be shellable if H is not a complete graph. □

We utilize a result of Moradi and Khosh-Ahang [7] on the expansion of a simplicial complex. Note that this construction appeared earlier in [1] where it was called an inflated simplicial complex. Although the results of [7] apply to any simplicial complex, we only present their results for independence complexes. We first define the expansion of a graph.

Definition 2.4. *Let G be a graph on the vertex set $V = \{x_1, \dots, x_n\}$ and let $(s_1, \dots, s_n) \in \mathbb{N}^n_{>0}$ be an n-tuple of positive integers. The $(s_1, \dots, s_n)$-expansion of G, denoted $G^{(s_1,\dots,s_n)}$, is the graph on the vertex set $V_{G^{(s_1,\dots,s_n)}} = \{x_{1,1}, \dots, x_{1,s_1}, x_{2,1}, \dots, x_{2,s_2}, \dots, x_{n,1}, \dots, x_{n,s_n}\}$ with edge set $E_{G^{(s_1,\dots,s_n)}} = \{\{x_{i,j}, x_{k,l}\} \mid \{x_i, x_k\} \in E_G \text{ or } i = k\}$.*

The next result now follows from more general results of Moradi and Khosh-Ahang.

Theorem 2.5. *Let G be a finite simple graph and $(s_1, \ldots, s_n) \in \mathbb{N}^n_{>0}$.*

- (i) [7, Theorem 2.7] *G is vertex decomposable if and only if $G^{(s_1,\ldots,s_n)}$ is vertex decomposable.*
- (ii) [7, Theorem 2.12] *If G is shellable, then $G^{(s_1,\ldots,s_n)}$ is shellable.*

We now obtain our main theorem.

Theorem 2.6. *Let G and H be finite simple graphs.*

- (a) *Suppose that G is a graph of isolated vertices. Then* $\mathrm{Ind}(H)$ *is vertex decomposable, respectively shellable, if and only if* $\mathrm{Ind}(G[H])$ *is vertex decomposable, respectively shellable.*
- (b) *Suppose that G is not a graph of isolated vertices. Then* $\mathrm{Ind}(G[H])$ *is vertex decomposable if and only if* $\mathrm{Ind}(G)$ *is vertex decomposable and $H = K_m$ for some $m \geq 1$.*
- (c) *Suppose that G is not a graph of isolated vertices. If* $\mathrm{Ind}(G)$ *is shellable and H is a complete graph, then* $\mathrm{Ind}(G[H])$ *is shellable. Furthermore, if* $\mathrm{Ind}(G[H])$ *is shellable, then $H = K_m$ for some $m \geq 1$.*

Proof of Theorem 2.6. Statement (a) follows from [11, Theorem 20].

To prove (b) and (c), observe that if G has n vertices and $(m, \ldots, m) \in \mathbb{N}^n_{>0}$, then $G^{(m,\ldots,m)} = G[K_m]$.

- (b) Suppose that $G[H]$ is vertex decomposable. Because $G[H]$ is also shellable by Lemma 2.2, Theorem 2.3 implies that $H = K_m$ for some $m \geq 1$. So $G[H] = G^{(m,\ldots,m)}$. Because $G^{(m,\ldots,m)}$ is vertex decomposable, G is vertex decomposable by Theorem 2.5. For the converse, because $H = K_m$ and G is vertex decomposable, $G[H] = G^{(m,\ldots,m)}$ is vertex decomposable by Theorem 2.5.
- (c) Suppose $H = K_m$ for some $m \geq 1$. If G is shellable, then $G[H] = G^{(m,\ldots,m)}$ is shellable by Theorem 2.5. As well, if $G[H]$ is shellable, we must have $H = K_m$ for some $m \geq 1$ by Theorem 2.3.

□

3. Applications to circulant graphs

We define a *circulant graph* on $n \geq 1$ vertices as follows. Let $S \subseteq \{1, 2, \ldots, \lfloor n/2 \rfloor\}$. The circulant graph $C_n(S)$ is the graph with vertex set $V = \{x_0, x_1, \ldots, x_{n-1}\}$, such that $\{x_a, x_b\}$ is an edge of $C_n(S)$ if and only if $|a - b| \in S$ or $n - |a - b| \in S$. See [2, 3, 4, 6, 10] for some recent papers on the properties of $C_n(S)$, especially well-covered circulant graphs.

Hoshino proved the following result about the lexicographical products of circulant graphs (in fact, the original result describes how to construct the lexicographical product from the data describing the two initial circulant graphs).

Theorem 3.1. [6, Theorem 2.31] *Let $G = C_n(S_1)$ and $H = C_m(S_2)$ be circulant graphs. Then $G[H]$ is also a circulant graph.*

Because K_m is the circulant graph $K_m = C_n(\{1, 2, \ldots, \lfloor m/2 \rfloor\})$, Theorem 2.6, combined with Theorem 3.1 implies the following result.

Theorem 3.2. *Let G be a circulant graph such that G is vertex decomposable, respectively shellable. Then $G[K_m]$ with $m \geq 1$ is a circulant graph that is also vertex decomposable, respectively, shellable.*

Remark. Many families of vertex decomposable and shellable circulant graphs have been identified [4, 10]. From any such graph, we can now build an infinite family of circulant graphs that is either vertex decomposable or shellable using the above result.

It has long been known that the converse of Lemma 2.2 is false (see [8]). However, it was less clear whether the converse of Lemma 2.2 was still false if we restricted to independence complexes of graphs. To the best of our knowledge, the circulant graph $C_{16}(\{1, 4, 8\})$ found in [4, Theorem 6.1] is the first example of a graph that is shellable but not vertex decomposable. By combining Theorem 2.5 with this example, we have an infinite family of independence complexes which are shellable but not vertex decomposable. In addition, Theorems 2.6 and 3.2 allow us to make an infinite family of circulant graphs with this property.

Theorem 3.3. *If $G = C_{16}(\{1, 4, 8\})$ and $(s_1, \ldots, s_n) \in \mathbb{N}^n_{>0}$, then $G^{(s_1,\ldots,s_n)}$ is shellable but not vertex decomposable. Furthermore, for all $m \geq 1$, $G^{(m,\ldots,m)} = G[K_m]$ is a circulant graph that is shellable but not vertex decomposable.*

Remark. We end with a couple of concluding remarks. The most obvious question to ask is if the converse of Theorem 2.6 (c) holds, or more generally, does the converse of [7, Theorem 2.12] hold. To prove the converse, we would need to determine if $\operatorname{Ind}(G[K_m])$ being shellable implies that G is shellable.

Our strategy to construct an infinite family of shellable but not vertex decomposable graphs is to find an initial graph with this property, and then apply Theorem 2.5. However, finding the initial graph with this property is quite difficult. Besides the graph $G = C_{16}(\{1, 4, 8\})$, we know of only one other graph with this property, namely the circulant graph $C_{20}(\{1, 5, 10\})$, which was verified computationally using *Macaulay2* [5]. We were also able to computationally verify that $C_{24}(\{1, 6, 12\})$ is not vertex decomposable, although we have not verified it is shellable (it is Cohen–Macaulay: see e.g. [10] for a definition). Based upon on this very slim evidence, we suspect that the graphs $G = C_{4s}(\{1, s, 2s\})$ with $s \geq 4$ are shellable but not vertex decomposable. The first three graphs in this family can be seen in Figure 1.

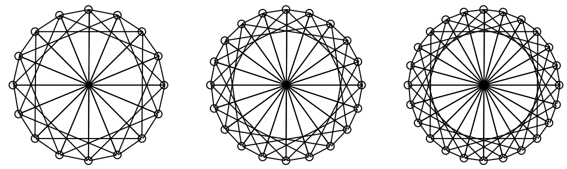

FIGURE 1. The circulant graphs $C_{16}(\{1,4,8\})$, $C_{20}(\{1,5,10\})$, and $C_{24}(\{1,6,12\})$.

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