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# PARTITIONING THE $5 \times 5$ ARRAY INTO RESTRICTIONS OF CIRCLES

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ABSTRACT. We show that there is a unique way to partition a $5\times5$ array of lattice points into restrictions of five circles. This result is extended to the $6\times5$ array, and used to show the optimality of a six-circle solution for the $6 \times 6$ array.

## 1. INTRODUCTION

In the notes of the late Ross Honsberger, the following problem was found, labeled as "Eddie's problem" [3]:

*Given a $5 \times 5$ array of lattice points, draw a set of circles that collectively pass through each of the lattice points exactly once.*

(Here, and throughout, by "circle" we will mean proper circles, not straight lines.)

This problem appeared in Crux Mathematicorum as problem CC226. One reader [2] gave a solution in the form of a set of concentric circles (Fig. 1$d$) including circles with as many radii as necessary to cover the array. Clearly, such a solution exists for any set of points; and for an $n \times n$ square grid array (which we will refer to as $G_{n,n}$) it requires at most

$$\binom{\lceil n/2 \rceil}{2} \tag{1.1}$$

circles. In fact, for even $n \geq 8$ and odd $n \geq 11$ there are nontrivial coincidences in radii, and the number of circles required is correspondingly fewer.

As stated, this problem is somewhat obvious; and it seems unlikely that it was what Honsberger actually had in mind. The note may well have been intended merely as an *aide-mémoire* for a more challenging problem. The following possible reconstruction was suggested in an editorial comment [5]:

*(1) Given a $5 \times 5$ array of lattice points, show that you can draw a set of 5 circles that collectively pass through each of the lattice points exactly once.*

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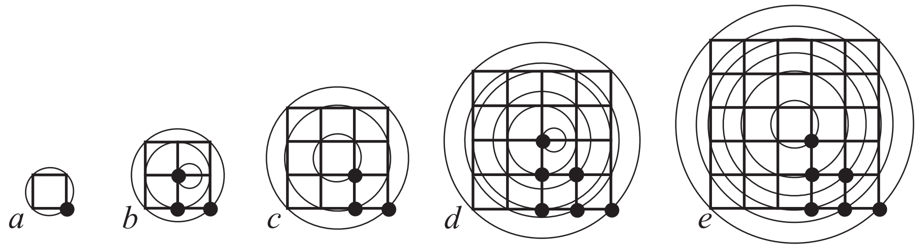



FIGURE 1. A simple family of solutions, nonoptimal for the $5 \times 5$ grid

*(2) Show that this cannot be done with four circles, even if two circles are allowed to pass through the same point.*

A solution to part (1) is shown in Fig. 2. On the left, the circles are shown as curves; on the right, the same circles are shown as discrete sets, partitioning $G_{5,5}$.

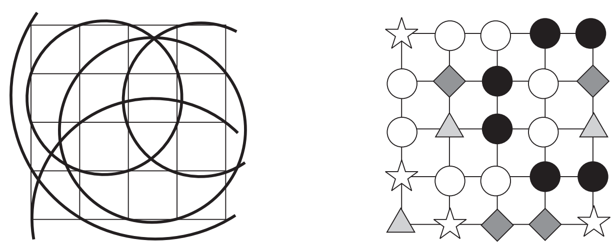


FIGURE 2. The unique partitioning of $G_{5,5}$ into restrictions of circles

**Theorem 1.1.** *The partitioning of $G_{5,5}$ shown in Fig. 2 is unique up to symmetry.*

(The proof is given in section 2.)

While the configuration of Fig. 2 is asymmetric, inspection shows that it can be extended to a symmetric partition of a $6 \times 5$ array, as shown in Fig. 3.

**Corollary 1.2.** *The partitioning of $G_{6,5}$ shown in Fig. 3 is unique up to symmetry.*

Fig. 1*e* shows that a $6 \times 6$ grid can be partitioned into restrictions of six circles. However, it is easily seen that the circles of Fig. 3 do not cover a further rank of six points above or below the grid shown. We thus have

**Corollary 1.3.** *The array $G_{6,6}$ can be partitioned into restrictions of six circles but not with five.*

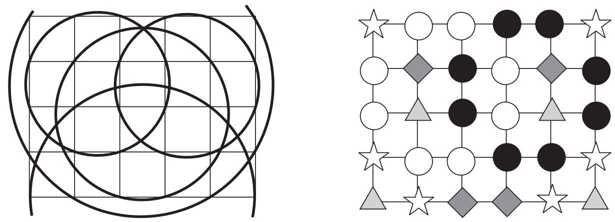


FIGURE 3. The unique partitioning of $G_{6,5}$ into restrictions of circles

If we consider sets of circles as shown in Figs. 1–3 to be "partitionings", then a set of circles that contains a specified set in its union, and may cover a point more than once, is a "covering." It follows from Theorem 1.1 that there is no partitioning of $G_{5,5}$ into restrictions of four circles; but in fact more is true.

**Theorem 1.4.** *The array $G_{5,5}$ cannot be covered with four circles.*

(The proof is given in section 3.)

We may ask how many circles are needed to partition larger grids. For odd $n > 5$, we can replace the inner $5\times 5$ grid of the concentric-circle solution of Fig. 1 with the configuration of Fig. 2, reducing the number of circles by one. (Thus, for instance, $G_{7,7}$ can be partitioned into restrictions of nine circles, rather than ten.)

An upper bound on the number of circles required to cover all points of $G_{n,n}$ is $S(2n^2)$, where $S(k)$ is the number of positive integers not exceeding $k$ that can be expressed as the sum of two squares. Landau [4] gave an estimate for this:

**Theorem 1.5** (Landau)**.**

$$\lim_{k\to\infty} S(k)\frac{\sqrt{\ln k}}{k} = \lambda \; ,$$

*where* $\lambda = 0.76422\ldots$ *is the* Landau–Ramanujan constant*.*

Thus, for large $n$, the number of circles needed to cover all points of $G_{n,n}$ is approximately $\sqrt{2}\lambda n^2/\sqrt{\ln n}$, a significant improvement on (1.1).

## 2. PROOF OF THEOREM 1.1

**Proposition 2.1.** *The two configurations $A_5$ and $B_5$ shown in Fig. 4 are, up to symmetry, the only ways in which a circle can intersect $G_{5,5}$ in exactly five points.*

*Proof.* We note that each of $A_5$ and $B_5$ has a single mirror symmetry, and hence can be embedded in $G_{5,5}$ in exactly four ways. To show that there are no other five-point circles, we use MAPLE to search noncollinear triples for

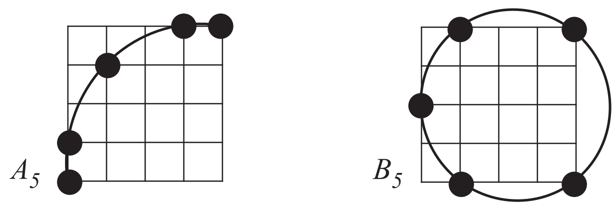



FIGURE 4. The two five-point circles

those that can be extended in exactly two ways to yield concyclic quadruples. We use the `geometry` and `combinat` packages, and create a list `Allpoints` that contains the 25 points of the grid $[P_{00},P_{01},P_{02},...,P_{44}]$ where $P_{ij}$ represents the point $(i, j)$.

```
Triplets := select(j → not AreCollinear(j[1], j[2], j[3]),
choose(Allpoints, 3)):
numelems(select(i → evalb(numelems(select(j →
AreConcyclic(i[1], i[2], i[3], j),Allpoints)) = 5),Triplets));
```

The first command creates a list of all noncollinear triplets on the grid. In the second command we count the triplets that can be extended *using a fourth point of the grid* to a concyclic quadruple in exactly five ways. We check for five values of $j$, not two, because whenever $j$ duplicates one of the elements of the triplet the resulting degenerate quadruple will be trivially concyclic. It returns the value 80 ($\binom{5}{3} \times 4 \times 2$) and we conclude that the two types of circle described above are the only five-point circles on a $G_{5,5}$ grid. $\square$

**Proposition 2.2.** *Up to symmetry, the two configurations $C_8$ and $D_8$ shown in Fig. 5 are the only ways in which a circle can intersect $G_{5,5}$ in exactly eight points; and the two configurations $C_6$ and $D_6$ are the only ways in which a circle can intersect $G_{5,5}$ in exactly six points. No circle can intersect $G_{5,5}$ in exactly seven points.*

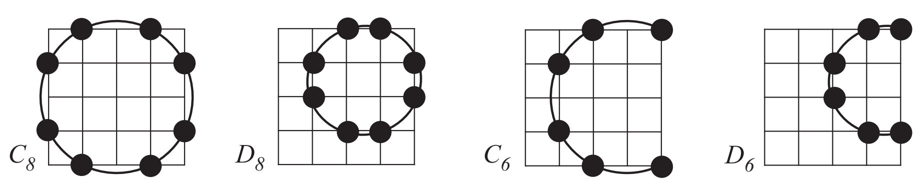



FIGURE 5. The two eight-point circles and their six-point restrictions

*Proof.* A circle of six or more points on $G_{5,5}$ must have two points on one file and two on one rank. These define the center of the circle as the intersection of perpendicular bisectors $x = a$ and $y = b$ where $2a$ and $2b$ are integers in the range $[1, \ldots, 7]$. Without loss of generality let

$$0 < a \leq b \leq 2 \; . \tag{2.1}$$

Suppose that $(a - p/2, b + q/2)$ and $(a + p/2, b + q/2)$ generate $x = a$ as a perpendicular bisector; then (2.1) implies that $p + q \leq 8$, so

$$p^2 + q^2 \leq 64 \; . \tag{2.2}$$

Suppose that the point $(a + r/2, b + s/2)$ is on a different orbit of the symmetry group of the circle restricted to $\mathbb{Z} \times \mathbb{Z}$ (that is, $\{|p|, |q|\} \neq \{|r|, |s|\}$). Then we have a nontrivial solution to

$$r^2 + s^2 = p^2 + q^2 \; . \tag{2.3}$$

The only solutions to (2.2) and (2.3) are $\{(7, 1), (5, 5)\}$ and $\{(5, 0), (3, 4)\}$; but these give the five-point circles above. We conclude that any circle or partial circle with six or more points within the $G_{5,5}$ grid must have a single orbit. Six of the eight points $(a \pm p/2, b \pm q/2)$, $(a \pm q/2, b \pm p/2)$ must lie in $G_{5,5}$ and be distinct; thus (without loss of generality) $0 < p < q \leq 4$, and $p$ and $q$ must have the same parity. The only solutions are $p = 2, q = 4$ and $p = 1$, $q = 3$.

The former gives a circle $C_8$ (see Fig. 5) with a gridpoint center and radius $\sqrt{5}$, while the latter gives a circle $D_8$ with radius $\sqrt{10}/2$, whose center has half-integer coordinates. Either of these can be placed entirely on $G_{5,5}$ or with two points missing to yield six-point circles $C_6$ and $D_6$. $C_8$ is unique, the other three are unique up to symmetry. □

Inspection shows that only the following pairs of "large circles" can coexist on a $5 \times 5$ grid: $A_5$ with itself, $D_8$ or $D_6$; $B_5$ with itself, $C_6$, or $D_8$; $D_6$ with $D_8$; and $C_6$ with $C_8$. All of these are unique up to symmetry except for $\{A_5, D_8\}$ which has two forms; and none allow a third large circle.

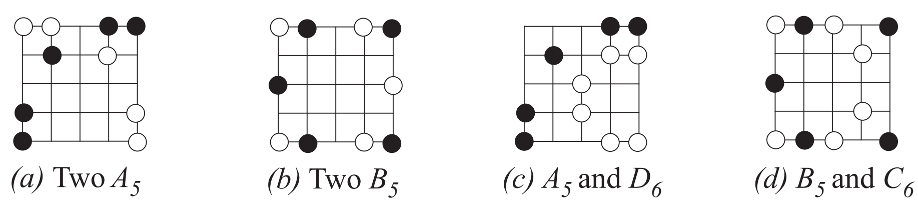
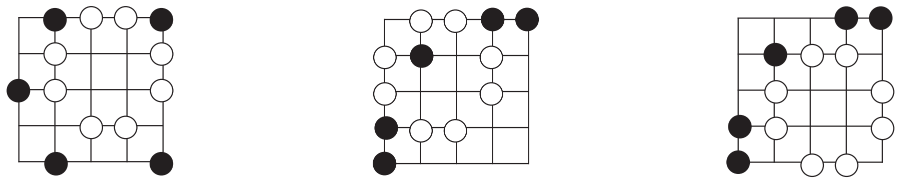



Figure 6. Disjoint pairs of circles with five or six points

**Proposition 2.3.** *No five-circle partition of $G_{5,5}$ uses $A_5$ or $B_5$.*

*Proof.* Placing a five-point circle on $G_{5,5}$ leaves 20 points to cover with four circles, only one of which can have more than four points — hence an eight-point circle and three four-point circles. Using Maple, we create a list `Used` of points already used. For instance, in Fig. 7*a*, we get



(a) $B_5$ and $D_8$ (b) $A_5$ and $D_8$ (I) (c) $A_5$ and $D_8$ (II)

FIGURE 7. Disjoint pairs of circles with five and eight points

(a) $C_6$ and $C_8$ (b) $D_6$ and $D_8$

FIGURE 8. Disjoint pairs of circles with six and eight points

```
Used := [P02, P10, P12, P13, P14, P21, P24, P31, P34, P40,
P42, P43, P44]
```

The list `Free` is its complement, omitting one point (here $P_{41}$) that we wish to test.

```
Free := [P00, P01, P03, P04, P11, P20, P22, P23, P30, P32,
P33]
```

We create a list `FreeTriplets` of all triplets from `Free`:

```
FreeTriplets := select(j → not AreCollinear(j[1],j[2],j[3]),
choose(Free, 3)):
```

We select from those the subset that are concyclic with the chosen point:

```
Cyc := select(i → AreConcyclic(i[1],i[2],i[3],P41),
FreeTriplets)
```

and we choose from `Cyc` those points (if any) that yield a circle that does not intersect `Used`:

```
CanAddThis := select(i→evalb(numelems(select(j→AreConcyclic
(i[1],i[2],i[3],j), Used)) = 0), Cyc)
```

For `Used` as shown, and $P_{41}$ as the test point (Fig. 9), the search algorithm returns an empty list: that is, no four-point circle passes through $P_{41}$ and avoids $B_5$ and $D_8$. We conclude that the configuration of Fig. 7$a$ cannot be part of a five-circle partition of $G_{5,5}$.

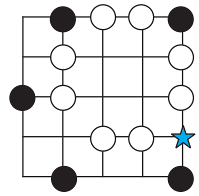


FIGURE 9. Trying to extend the configuration of Fig. 7$a$

Now let `Used` be the union of the $A_5$ and the $D_8$ shown in Fig. 7$b$, with test point $P_{31}$ (Fig. 10$a$). The algorithm returns a single four-point circle (Fig. 10$b$). We move those points from `Free` to `Used`, and use $P_{04}$ as our test point; no four-point circle passes through that point and avoids the already-assigned points. Thus, the configuration of Fig. 7$b$ cannot be part of a five-circle partition of $G_{5,5}$.

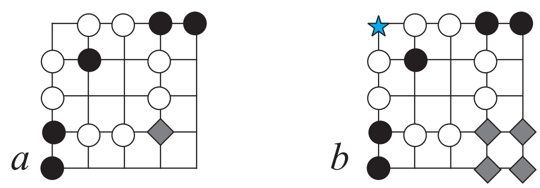


FIGURE 10. Extending the configuration of Fig. 7$b$

Finally, we try the arrangement of Fig. 7$c$. With $P_{40}$ as test point (Fig. 11$a$), we get the unique four-point circle shown in Fig. 11$b$. As before, we move those points to `Used` and use $P_{10}$ as test point; this time we get no four-point circle, and we conclude that there are no partitions of $G_{5,5}$ into restrictions of five circles, one of which is $A_5$ or $B_5$. □

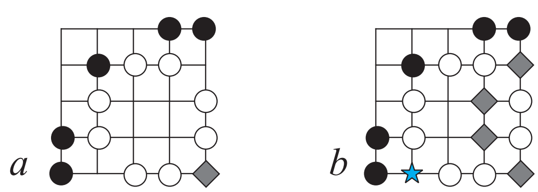


FIGURE 11. Extending the configuration of Fig. 7$c$

Any five-circle partition, then, extends one of the configurations of Fig. 8; and, as well as the eight-point circle and six-point circle, must contain two four-point circles and a three-point circle.

**Proposition 2.4.** *There is no five-circle partition of $G_{5,5}$ using $C_8$ and $C_6$.*

*Proof.* As above, we search for four-point circles through a chosen point that do not overlap existing circles. We find that for the $\{C_8, C_6\}$ configuration, oriented as shown in Fig. 8*a*, no four-point circle can be placed through $P_{02}$ (see Fig. 12*a*). This does not rule out the original configuration, but shows that $P_{02}$ must be on the three-point circle.

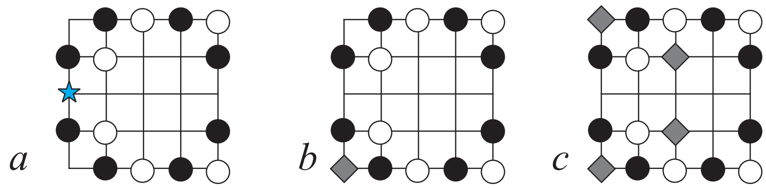


FIGURE 12. Extending the configuration of Fig. 8*a*

We now test $P_{00}$ (Fig. 12*b*), and find that if it is on a four-point circle, the other points are $P_{04}$, $P_{21}$, and $P_{23}$, as shown in Fig. 12*c*. This, however, leaves the third rank to be covered by two circles; and this is impossible, as a circle can only meet a line twice. It follows that that $P_{00}$, too, is on a three-point circle; and the same argument holds, by symmetry, for $P_{04}$.

But $P_{00}$, $P_{02}$, and $P_{04}$ are collinear, and cannot all lie on one circle; we conclude that there is no five-circle tiling of $G_{5,5}$ using $C_8$ and $C_6$. □

It follows that a five-circle partition of $G_{5,5}$ must include a $D_8$ and a $D_6$, positioned (without loss of generality) as shown in Fig. 8*b*; we are now ready to complete the proof of Theorem 1. We find that there are no four-point circles disjoint from $D_8$ and $D_6$ passing through $P_{00}$; $P_{00}$ must therefore be on the three-point circle.

Every four-point circle through $P_{30}$ that does not intersect $D_6$ or $D_8$ also contains $P_{20}$. Thus, if $P_{20}$ were on the three-point circle, it would have to be with $P_{00}$ and $P_{30}$; but these are collinear, a contradiction. We conclude that $P_{20}$ must be on a four-point circle; there are only two possibilities.

*Case 1*: $P_{20}$, $P_{10}$, $P_{42}$ and $P_{43}$ are on a circle (Fig. 13*a*).

There are only two four-point circles among the remaining six points (Fig. 13*b*,*c*). Either one leaves two points collinear with $P_{00}$, and these do not lie on a three-point circle.

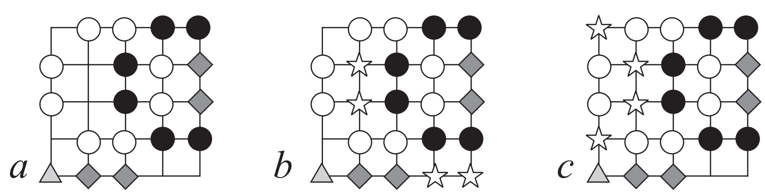


FIGURE 13. Configurations with $D_8$ and $D_6$

*Case 2*: $P_{20}$,$P_{13}$,$P_{30}$, and $P_{43}$ are on a circle (Fig. 14*a*).

There are only two four-point circles among the remaining six points; both include $P_{10}$ and $P_{40}$. If the other two points on the circle are $P_{12}$ and $P_{42}$, (Fig. 14*b*,) then the remaining two points ($P_{01}$ and $P_{04}$) are collinear with $P_{00}$, so there is no three-point circle. We are left, as our only remaining option, with the configuration of Fig. 14*c*—which is that of Fig. 2.

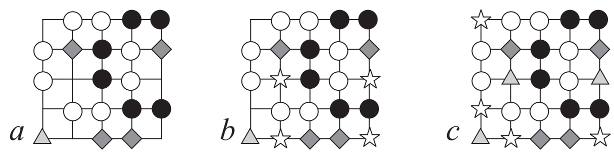



FIGURE 14. More configurations with $D_8$ and $D_6$

## 3. Proof of Theorem 2

Four circles, each covering fewer than eight points, cannot cover more than 24 points; thus, to cover $G_{5,5}$, an eight-point circle would be required. There can be no three- or four-point circle in the covering, because that and the eight-point circle would leave at least 13 points to be covered by two more circles; and any further eight-point circle covers only six points in the complement of the first.

But no eight-point circle covers $P_{00}$, $P_{04}$, $P_{22}$, $P_{40}$, or $P_{44}$, and no five- or six-point circle can cover more than two of these five points. We must therefore have three five- or six-point circles. Any two six-point circles have at least one point in common, so a set of circles of cardinalities 8, 6, 6, and 5 has a union of less than 25 points. We thus rule out any five-point circle.

Moreover, $C_6$ is the only large circle that covers $P_{22}$; so one circle of this type is required. It must intersect the eight-point circle in two points (Fig. 15 *a*, *b*, *c*, *d*) or not at all (Fig. 15 *e*). If the $C_6$ intersects the eight-point circle, they only cover twelve points between them, and the remaining thirteen points cannot be covered by two six-point circles. If they do not intersect, then between them they leave a set of five collinear points uncovered, which cannot be covered by the two remaining circles.

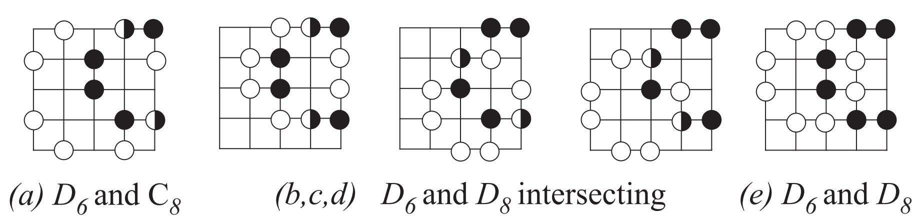



FIGURE 15. Ways in which $D_6$ can meet an eight-point circle

## 4. Conclusions and open questions

We have shown that $G_{5,5}$ can be partitioned uniquely into restrictions of five circles, and that this is optimal even for coverings. That partition extends to a five-circle partition of $G_{6,5}$, also necessarily unique.

**Question.** *The obvious six-circle partition of $G_{6,6}$ is optimal. Is it unique?*

**Question.** *Theorem 1.5 (Landau) shows that, for large $n$, $G_{n,n}$ can be partitioned into restrictions of approximately $\sqrt{2}\lambda n^2/\ln n$ concentric circles. Can this be improved?*

## References

1. D. M. Burton, Elementary Number Theory, 1976, Allyn and Bacon
2. S. Chow, *Solution to Problem CC266*, Crux Mathematicorum **44(3)** (2018), 137
3. R. Honsberger, *Problem CC266*, Crux Mathematicorum **43(4)** (2017), 124–125
4. E. Landau, *Über die Einteilung der positiven ganzen Zahlen in vier Klassen nach der Mindeszahl der zu ihrer additiven Zusammensetzung erforderlichen Quadrate*, *Arch. Math. Phys.* **13** (1908), 305–312.
5. J. G. MacLauchlin, comment on [2], Crux Mathematicorum **44(3)** (2018), 137

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