



3-UNIFORM HYPERGRAPHS: DECOMPOSITION AND REALIZATION

A. BOUSSAÏRI, B. CHERGUI, P. ILLE, AND M. ZAIDI

ABSTRACT. Let H be a 3-uniform hypergraph. A tournament T defined on $V(T) = V(H)$ is a realization of H if the edges of H are exactly the 3-element subsets of $V(T)$ that induce 3-cycles. We characterize the 3-uniform hypergraphs that admit realizations by using a suitable modular decomposition.

1. INTRODUCTION

Let H be a 3-uniform hypergraph. A tournament T , with the same vertex set as H , is a realization of H if the edges of H are exactly the 3-element subsets of the vertex set of T that induce 3-cycles. The aim of the paper is to characterize the 3-uniform hypergraphs that admit realizations (see [2, Problem 1]). This characterization is in the vein of that of comparability graphs, that is, the graphs admitting a transitive orientation (see [9]).

In Section 2, we recall some of the classic results on modular decomposition of tournaments.

In the section below, we introduce a new notion of module for hypergraphs. We introduce also the notion of a modular covering, which generalizes the notion of a partitive family. In Appendices A and B, we show that the set of modules of a hypergraph is a modular covering.

In Section 3, we consider the notion of a strong module, which is the usual strengthening of the notion of a module (for instance, see Subsection 2.1 for tournaments). We establish the analogue of Gallai's modular decomposition theorem for hypergraphs.

Let H be a realizable and 3-uniform hypergraph. Clearly, the modules of the realizations of H are modules of H as well, but the converse is false. Consider a realization T of H . In Section 4, we characterize the modules of H that are not modules of T . We deduce that a realizable and 3-uniform hypergraph and its realizations share the same strong modules. Using Gallai's modular decomposition theorem, we prove that a realizable and 3-uniform hypergraph is prime (i.e. all its modules are trivial) if and only if each of its realizations is prime too. We have parallel results when we consider a

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comparability graph and its transitive orientations (for instance, see [11, Theorem 3] and [11, Corollary 1]).

In Section 5, by using the modular decomposition tree, we demonstrate that a 3-uniform hypergraph is realizable if and only if all its prime, 3-uniform and induced subhypergraphs are realizable. We pursue an inductive characterization of the prime and 3-uniform hypergraphs that are realizable. Hence [2, Problem 1] is solved. From this characterization, we deduce a new proof of [2, Corollary 1], the main result of [2] for tournaments. We conclude by counting the realizations of a realizable and 3-uniform hypergraph by using the modular decomposition tree. There is an analogue counting when we determine the number of transitive orientations of a comparability graph by using the modular decomposition tree of the comparability graph. The number of transitive orientations of a comparability graph was determined by Filippov and Shevrin [6]. They used the notion of a saturated module, which is close to that of a strong module.

Next, we formalize our presentation. We consider only finite structures. A hypergraph H is defined by a vertex set $V(H)$ and an edge set $E(H)$, where $E(H) \subseteq 2^{V(H)} \setminus \{\emptyset\}$. Given a hypergraph H , $v(H)$ denotes the cardinality of $V(H)$. In the sequel, we consider only hypergraphs H such that

$$E(H) \subseteq 2^{V(H)} \setminus (\{\emptyset\} \cup \{\{v\} : v \in V(H)\}).$$

Given $k \geq 2$, a hypergraph H is *k-uniform* if

$$E(H) \subseteq \binom{V(H)}{k}.$$

A hypergraph H is *empty* if $E(H) = \emptyset$. Let H be a hypergraph. With each $W \subseteq V(H)$, we associate the *subhypergraph* $H[W]$ of H induced by W , which is defined by $V(H[W]) = W$ and $E(H[W]) = \{e \in E(H) : e \subseteq W\}$.

Definition 1. Let H be a hypergraph. A subset M of $V(H)$ is a *module* of H if for each $e \in E(H)$ such that $e \cap M \neq \emptyset$ and $e \setminus M \neq \emptyset$, there exists $m \in M$ such that $e \cap M = \{m\}$ and for every $n \in M$, we have

$$(e \setminus \{m\}) \cup \{n\} \in E(H).$$

Definition 1 is not the classic definition of a module of a hypergraph. The classic definition is provided in Definition 24. We compare both definitions in Remark 25. We motivate the choice of Definition 1 in Remark 26.

Notation 2. Given a hypergraph H , the set of the modules of H is denoted by $\mathcal{M}(H)$. For instance, if H is an empty hypergraph, then $\mathcal{M}(H) = 2^{V(H)}$.

We study the set of the modules of a hypergraph. Let S be a set. A family \mathcal{F} of subsets of S is a *partitive family* [3, Definition 6] on S if it satisfies the following assertions.

- $\emptyset \in \mathcal{F}$, $S \in \mathcal{F}$, and for every $x \in S$, $\{x\} \in \mathcal{F}$.
- For any $M, N \in \mathcal{F}$, $M \cap N \in \mathcal{F}$.

- For any $M, N \in \mathcal{F}$, if $M \cap N \neq \emptyset$, $M \setminus N \neq \emptyset$ and $N \setminus M \neq \emptyset$, then $M \cup N \in \mathcal{F}$ and $(M \setminus N) \cup (N \setminus M) \in \mathcal{F}$.

Proposition 3. *Given a hypergraph H , $\mathcal{M}(H)$ is a partitive family.*

Proposition 3 is well-known for 2-uniform hypergraphs, that is, graphs. Its proof for graphs is easy whereas it is more difficult in the general case. Since the proof of Proposition 3 is long and technical in the general case, we provide it in Appendix A. We generalize the notion of a partitive family as follows.

Definition 4. Let S be a set. A *modular covering* of S is a function \mathfrak{M} which associates with each $W \subseteq S$ a set $\mathfrak{M}(W)$ of subsets of W , and which satisfies the following assertions.

(A1) For each $W \subseteq S$, $\mathfrak{M}(W)$ is a partitive family on W .

(A2) For any $W, W' \subseteq S$, if $W \subseteq W'$, then

$$\{M' \cap W : M' \in \mathfrak{M}(W')\} \subseteq \mathfrak{M}(W).$$

(A3) For any $W, W' \subseteq S$, if $W \subseteq W'$ and $W \in \mathfrak{M}(W')$, then

$$\{M' \in \mathfrak{M}(W') : M' \subseteq W\} = \mathfrak{M}(W).$$

(A4) Let $W, W' \subseteq S$ such that $W \subseteq W'$. For any $M \in \mathfrak{M}(W)$ and $M' \in \mathfrak{M}(W')$, if $M \cap M' = \emptyset$ and $M' \cap W \neq \emptyset$, then $M \in \mathfrak{M}(W \cup M')$.

(A5) Let $W, W' \subseteq S$ such that $W \subseteq W'$. For any $M \in \mathfrak{M}(W)$ and $M' \in \mathfrak{M}(W')$, if $M \cap M' \neq \emptyset$, then $M \cup M' \in \mathfrak{M}(W \cup M')$.

We obtain the following result.

Proposition 5. *Given a hypergraph H , the function defined on $2^{V(H)}$, which maps each $W \subseteq V(H)$ to $\mathcal{M}(H[W])$, is a modular covering of $V(H)$.*

As for Proposition 3, we provide the proof of Proposition 5 in Appendix B.

Let H be a hypergraph. By Proposition 3, \emptyset , $V(H)$, and $\{v\}$, where $v \in V(H)$, are modules of H , called *trivial*. A hypergraph H is *indecomposable* if all its modules are trivial, otherwise it is *decomposable*. An hypergraph H is *prime* if it is indecomposable with $v(H) \geq 3$.

To state Gallai's modular decomposition theorem for hypergraphs, we need to define the quotient of a hypergraph by a modular partition (see Subsection 2.1 for tournaments).

Definition 6. Let H be a hypergraph. A partition P of $V(H)$ is a *modular partition* of H if $P \subseteq \mathcal{M}(H)$. Given a modular partition P of H , the *quotient* H/P of H by P is defined on $V(H/P) = P$ as follows. For $\mathcal{E} \subseteq P$, $\mathcal{E} \in E(H/P)$ if $|\mathcal{E}| \geq 2$, and there exists $e \in E(H)$ such that $\mathcal{E} = \{X \in P : X \cap e \neq \emptyset\}$.

As for tournaments, we introduce the following strengthening of the notion of a module. Let H be a hypergraph. A module M of H is *strong* if for every module N of H , we have

$$\text{if } M \cap N \neq \emptyset, \text{ then } M \subseteq N \text{ or } N \subseteq M.$$

Notation 7. Let H be a hypergraph. Recall that a subset W of $V(H)$ is a *proper* subset of $V(H)$ if $W \neq V(H)$. We denote by $\Pi(H)$ the set of proper strong modules of H that are maximal under inclusion. Clearly, $\Pi(H)$ is a modular partition of H when $v(H) \geq 2$.

Gallai's modular decomposition theorem for hypergraphs follows. It is the analogue of Theorem 17.

Theorem 8. *Given a hypergraph H with $v(H) \geq 2$, $H/\Pi(H)$ is an empty hypergraph, a prime hypergraph, or a complete graph (i.e. $E(H/\Pi(H)) = \binom{\Pi(H)}{2}$).*

A realization of a 3-uniform hypergraph is defined as follows. To begin, we associate with each tournament a 3-uniform hypergraph in the following way.

Definition 9. The 3-cycle is the tournament $C_3 = (\{0, 1, 2\}, \{01, 12, 20\})$. Given a tournament T , the C_3 -structure of T is the 3-uniform hypergraph $C_3(T)$ defined on $V(C_3(T)) = V(T)$ by

$$E(C_3(T)) = \{X \subseteq V(T) : T[X] \text{ is isomorphic to } C_3\} \text{ (see [2]).}$$

Definition 10. Given a 3-uniform hypergraph H , a tournament T , with $V(T) = V(H)$, *realizes* H if $H = C_3(T)$. We say also that T is a *realization* of H .

Whereas a realizable and 3-uniform hypergraph and its realizations do not have the same modules, they share the same strong modules.

Theorem 11. *Consider a realizable and 3-uniform hypergraph H . Given a realization T of H , H and T share the same strong modules.*

Theorem 11 is a key result, it necessitates long and technical preliminaries (see Proposition 42). The next result follows from Theorems 8 and 11.

Theorem 12. *Consider a realizable and 3-uniform hypergraph H . For a realization T of H , we have H is prime if and only if T is prime.*

Lastly, we characterize the realizable and 3-uniform hypergraphs. To begin, we establish the following theorem by using the modular decomposition tree.

Theorem 13. *Given a 3-uniform hypergraph H , H is realizable if and only if for every $W \subseteq V(H)$ such that $H[W]$ is prime, $H[W]$ is realizable.*

We conclude by characterizing the prime and 3-uniform hypergraphs that are realizable (see Theorems 49 and 52).

2. BACKGROUND ON TOURNAMENTS

Given a tournament T , $v(T)$ denotes the cardinality of $V(T)$. A tournament is a *linear order* if it does not contain C_3 as a subtournament.

Given $n \geq 2$, the usual linear order on $\{0, \dots, n - 1\}$ is the tournament $L_n = (\{0, \dots, n - 1\}, \{pq : 0 \leq p < q \leq n - 1\})$. With each tournament T , associate its *dual* T^* defined on $V(T^*) = V(T)$ by $A(T^*) = \{vw : wv \in A(T)\}$.

2.1. Modular decomposition of tournaments. To begin, we consider a digraph D . A subset M of $V(D)$ is a *module* [16] of D provided that for any $x, y \in M$ and $v \in V(D) \setminus M$, we have $xv \in A(D)$ if and only if $yv \in A(D)$, and $vx \in A(D)$ if and only if $vy \in A(D)$. (A module is also called a *closed set* [8], an *autonomous set* [13], or a *homogeneous set* [14].)

Let T be a tournament. We obtain that a subset M of $V(T)$ is a *module* of T provided that for any $x, y \in M$ and $v \in V(T)$, if $xv, vy \in A(T)$, then $v \in M$. Note that the notions of a module and of a convex subset [10] coincide for tournaments. Moreover, note that the notions of a module and of an interval coincide for linear orders.

Notation 14. Given a tournament T , the set of the modules of T is denoted by $\mathcal{M}(T)$.

We study the set of the modules of a tournament. We need the following weakening of the notion of a partitive family. Given a set S , a family \mathcal{F} of subsets of S is a *weakly partitive family* [12] on S if it satisfies the following assertions.

- $\emptyset \in \mathcal{F}$, $S \in \mathcal{F}$, and for every $x \in S$, $\{x\} \in \mathcal{F}$.
- For any $M, N \in \mathcal{F}$, $M \cap N \in \mathcal{F}$.
- For any $M, N \in \mathcal{F}$, if $M \cap N \neq \emptyset$, then $M \cup N \in \mathcal{F}$.
- For any $M, N \in \mathcal{F}$, if $M \setminus N \neq \emptyset$, then $N \setminus M \in \mathcal{F}$.

The set of the modules of a tournament is a weakly partitive family (for instance, see [5]). We generalize the notion of a weakly partitive family as follows.

Definition 15. Let S be a set. A *weak modular covering* of S is a function \mathfrak{M} which associates with each $W \subseteq S$ a set $\mathfrak{M}(W)$ of subsets of W , and which satisfies assertions (A2)–(A5) (see Definition 4), and the following assertion. For each $W \subseteq S$, $\mathfrak{M}(W)$ is a weakly partitive family on W .

Since the proof of the next proposition is easy and long, we omit it.

Proposition 16. *Given a tournament T , the function defined on $2^{V(T)}$, which maps each $W \subseteq V(T)$ to $\mathcal{M}(T[W])$, is a weak modular covering of $V(T)$.*

Let T be a tournament. By Proposition 16, \emptyset , $V(T)$ and $\{v\}$, where $v \in V(T)$, are modules of T , called *trivial*. A tournament is *indecomposable* if all its modules are trivial, otherwise it is *decomposable*. A tournament T is *prime* if it is indecomposable with $v(T) \geq 3$.

We define the quotient of a tournament by considering a partition of its vertex set in modules. Precisely, let T be a tournament. A partition P of $V(T)$ is a *modular partition* of T if $P \subseteq \mathcal{M}(T)$. With each modular partition

P of T , associate the *quotient* T/P of T by P defined on $V(T/P) = P$ as follows. Given $X, Y \in P$ such that $X \neq Y$, $XY \in A(T/P)$ if $xy \in A(T)$, where $x \in X$ and $y \in Y$.

We need the following strengthening of the notion of module to obtain an uniform decomposition theorem. Given a tournament T , a subset X of $V(T)$ is a *strong module* of T provided that X is a module of T and for every module M of T , if $X \cap M \neq \emptyset$, then $X \subseteq M$ or $M \subseteq X$. With each tournament T , with $v(T) \geq 2$, associate the set $\Pi(T)$ of the maximal strong module of T under the inclusion amongst all the proper and strong modules of T . Gallai's modular decomposition theorem follows.

Theorem 17 (Gallai [8, 14]). *Given a tournament T such that $v(T) \geq 2$, $\Pi(T)$ is a modular partition of T , and $T/\Pi(T)$ is a linear order or a prime tournament.*

Theorem 17 is deduced from the following result.

Theorem 18. *Given a tournament T , all the strong modules of T are trivial if and only if T is a linear order or a prime tournament.*

Definition 19. Given a tournament T , the set of the nonempty strong modules of T is denoted by $\mathcal{D}(T)$. Clearly, $\mathcal{D}(T)$ ordered by inclusion is a tree called the *modular decomposition tree* of T .

Let T be a tournament. The next proposition allows us to obtain all the elements of $\mathcal{D}(T)$ by using successively Theorem 17 from $V(T)$ to the singletons.

Proposition 20 (Ehrenfeucht et al. [4]). *Given a tournament T , consider a strong module M of T . For every $N \subseteq M$, the following two assertions are equivalent:*

- (1) N is a strong module of T ;
- (2) N is a strong module of $T[M]$.

We use the analogue of Proposition 20 for hypergraphs (see Proposition 39) to prove Proposition 42.

2.2. Critical tournaments.

Definition 21. Given a prime tournament T , a vertex v of T is *critical* if $T - v$ is decomposable. A prime tournament is *critical* if all its vertices are critical.

Schmerl and Trotter [15] characterized the critical tournaments. They obtained the tournaments T_{2n+1} , U_{2n+1} , and W_{2n+1} defined on $\{0, \dots, 2n\}$, where $n \geq 1$, as follows.

- The tournament T_{2n+1} is obtained from L_{2n+1} by reversing all the arcs between even and odd vertices (see Figure 1).
- The tournament U_{2n+1} is obtained from L_{2n+1} by reversing all the arcs between even vertices (see Figure 2).

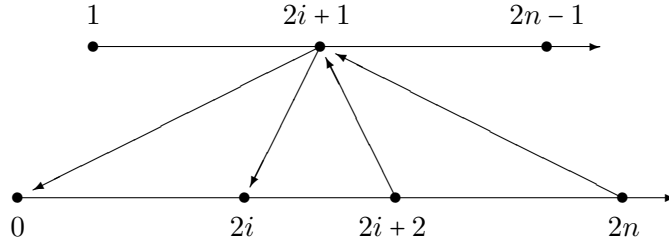


FIGURE 1. The tournament T_{2n+1} .

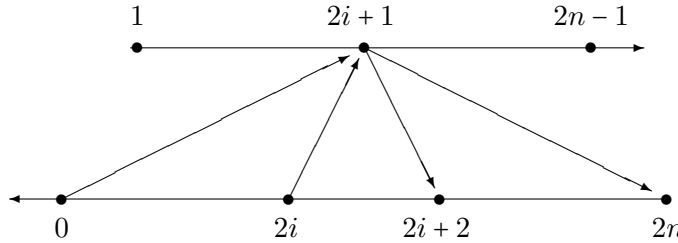


FIGURE 2. The tournament U_{2n+1} .

- The tournament W_{2n+1} is obtained from L_{2n+1} by reversing all the arcs between $2n$ and the even elements of $\{0, \dots, 2n-1\}$ (see Figure 3).

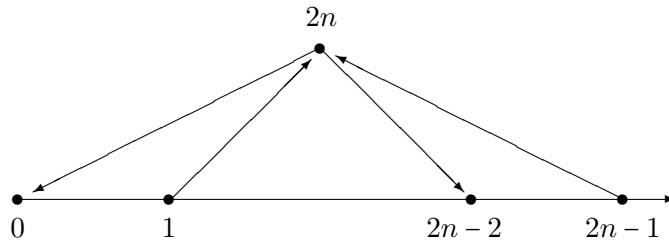


FIGURE 3. The tournament W_{2n+1} .

Theorem 22 (Schmerl and Trotter [15]). *Given a tournament τ , with $v(\tau) \geq 5$, τ is critical if and only if $v(\tau)$ is odd, and τ is isomorphic to $T_{v(\tau)}$, $U_{v(\tau)}$, or $W_{v(\tau)}$.*

2.3. The C_3 -structure of a tournament. The C_3 -structure of a tournament (see Definition 9) is clearly a 3-uniform hypergraph. We recall the main theorem of [2].

Theorem 23 (Boussairi et al. [2]). *Let T be a prime tournament. For every tournament T' , if $C_3(T') = C_3(T)$, then $T' = T$ or T^* .*

We provide a new proof of Theorem 23 at the end of Section 5. It follows easily from the proof of Theorem 52 (see Corollary 53).

3. MODULAR DECOMPOSITION OF HYPERGRAPHS

The classic definition of a module for hypergraphs follows.

Definition 24. Let H be a hypergraph. A subset M of $V(H)$ is a *module* [1, Definition 2.4] of H if for any $e, f \subseteq V(H)$ such that $|e| = |f|$, $e \setminus M = f \setminus M$, and $e \setminus M \neq \emptyset$, we have $e \in E(H)$ if and only if $f \in E(H)$.

In this paper, we use Definition 1 instead of Definition 24.

Remark 25. Definitions 1 and 24 coincide for 2-uniform hypergraphs, that is, for graphs. They do not in the general case. Given a hypergraph H , a module of H in the sense of Definition 1 is a module in the sense of Definition 24. The converse is not true. Given $n \geq 3$, consider the 3-uniform hypergraph H defined by $V(H) = \{0, \dots, n-1\}$ and $E(H) = \{\{0, 1, p\} : 2 \leq p \leq n-1\}$. In the sense of Definition 24, $\{0, 1\}$ is a module of H whereas it is not a module of H in the sense of Definition 1.

Remark 26. Let H be a realizable and 3-uniform hypergraph. Consider a realization T of H . Given $e \in E(H)$, all the modules of $T[e]$ are trivial. In order to have modular decompositions for H and T as close as possible, we try to find a definition of a module of H for which all the modules of $H[e]$ are trivial as well. This is the case with Definition 1, and not with Definition 24. Moreover, note that, with Definition 24, H and T do not share the same strong modules, but they do with Definition 1 (see Theorem 11). Indeed, consider the 3-uniform hypergraph H defined on $\{0, \dots, n-1\}$ in Remark 25. In the sense of Definition 24, $\{0, 1\}$ is a strong module of H . Now, consider the tournament T obtained from L_n by reversing all the arcs between 0 and $p \in \{2, \dots, n-1\}$. Clearly, T realizes H . Since $T[\{0, 1, 2\}]$ is a 3-cycle, $\{0, 1\}$ is not a module of T , so it is not a strong module.

The purpose of this section is to demonstrate Theorem 8. We use the following notation and definition.

Notation 27. Let P be a partition of a set S . For $W \subseteq S$, W/P denotes the subset $\{X \in P : X \cap W \neq \emptyset\}$ of P . For $Q \subseteq P$, set

$$\cup Q = \bigcup_{X \in Q} X.$$

Definition 28. Let P be a partition of a set S . Consider $Q \subseteq P$. A subset W of S is a *transverse* of Q if $W \subseteq \cup Q$ and $|W \cap X| = 1$ for each $X \in Q$.

The next remark makes Definition 6 clearer.

Remark 29. Consider a modular partition P of a hypergraph H . Let $e \in E(H)$ such that $|e/P| \geq 2$. Given $X \in e/P$, we have $e \cap X \neq \emptyset$, and $e \setminus X \neq \emptyset$ because $|e/P| \geq 2$. Since X is a module of H , we obtain $|e \cap X| = 1$. Therefore, e is a transverse of e/P . Moreover, since each element of e/P is a module of H , we obtain that each transverse of e/P is an edge of H .

Given $\mathcal{E} \subseteq P$ such that $|\mathcal{E}| \geq 2$, it follows that $\mathcal{E} \in E(H/P)$ if and only if every transverse of \mathcal{E} is an edge of H .

Lastly, consider a transverse t of P . The function θ_t from t to P , which maps each $x \in t$ to the unique element of P containing x , is an isomorphism from $H[t]$ onto H/P .

In the next proposition, we study the links between the modules of a hypergraph with those of its quotients.

Proposition 30. *Given a modular partition P of a hypergraph H , the following two assertions hold:*

- (1) *if M is a module of H , then M/P is a module of H/P ;*
- (2) *if \mathcal{M} is a module of H/P , then $\cup\mathcal{M}$ is a module of H .*

Proof. For the first assertion, consider a module M of H . Consider a transverse t of P such that

$$(1) \quad \text{for each } X \in M/P, t \cap X \subseteq M.$$

Clearly $M \cap t$ is a module of $H[t]$. Since θ_t is an isomorphism from $H[t]$ onto H/P (see Remark 29),

$$\theta_t(M \cap t), \text{ that is, } M/P$$

is a module of H/P .

For the second assertion, consider a module \mathcal{M} of H/P . Let t be any transverse of P . Since θ_t is an isomorphism from $H[t]$ onto H/P , $(\theta_t)^{-1}(\mathcal{M})$ is a module of $H[t]$. Set

$$\mu = (\theta_t)^{-1}(\mathcal{M}).$$

Denote the elements of \mathcal{M} by X_0, \dots, X_m . We verify by induction on $i \in \{0, \dots, m\}$ that $\mu \cup (X_0 \cup \dots \cup X_i)$ is a module of $H[t \cup (X_0 \cup \dots \cup X_i)]$. It follows from Lemma 63 that $\mu \cup X_0$ is a module of $H[t \cup X_0]$. Given $0 \leq i < m$, suppose that $\mu \cup (X_0 \cup \dots \cup X_i)$ is a module of $H[t \cup (X_0 \cup \dots \cup X_i)]$. Similarly, it follows from Lemma 63 that $\mu \cup (X_0 \cup \dots \cup X_{i+1})$ is a module of $H[t \cup (X_0 \cup \dots \cup X_{i+1})]$. By induction, we obtain that $\mu \cup (X_0 \cup \dots \cup X_m)$ is a module of $H[t \cup (X_0 \cup \dots \cup X_m)]$. Observe that

$$\mu \cup (X_0 \cup \dots \cup X_m) = \cup\mathcal{M}.$$

Lastly, denote the elements of $P \setminus \mathcal{M}$ by Y_0, \dots, Y_n . Using Lemma 62, we show by induction on $0 \leq j \leq n$ that $(\cup\mathcal{M})$ is a module of $H[t \cup (X_0 \cup \dots \cup X_m) \cup (Y_0 \cup \dots \cup Y_j)]$. Consequently, we obtain that $(\cup\mathcal{M})$ is a module of $H[t \cup (X_0 \cup \dots \cup X_m) \cup (Y_0 \cup \dots \cup Y_n)]$, that is, H . \square

The next proposition is similar to Proposition 30, but is devoted to strong modules.

Proposition 31. *Given a modular partition P of a hypergraph H , the following two assertions hold.*

- (1) *If M is a strong module of H , then M/P is a strong module of H/P .*
- (2) *Suppose that all the elements of P are strong modules of H . If \mathcal{M} is a strong module of H/P , then $\cup\mathcal{M}$ is a strong module of H .*

Proof. For the first assertion, consider a strong module M of H . By the first assertion of Proposition 30, M/P is a module of H/P . To show that M/P is strong, consider a module \mathcal{M} of H/P such that $(M/P) \cap \mathcal{M} \neq \emptyset$. By the second assertion of Proposition 30, $\cup\mathcal{M}$ is a module of H . Furthermore, since $(M/P) \cap \mathcal{M} \neq \emptyset$, there exists $X \in (M/P) \cap \mathcal{M}$. We get $X \cap M \neq \emptyset$ and $X \subseteq \cup\mathcal{M}$. Therefore, $M \cap (\cup\mathcal{M}) \neq \emptyset$. Since M is a strong module of H , we obtain $\cup\mathcal{M} \subseteq M$ or $M \subseteq \cup\mathcal{M}$. In the first instance, we get $\mathcal{M} \subseteq M/P$, and, in the second one, we get $M/P \subseteq \mathcal{M}$.

For the second assertion, suppose that all the elements of P are strong modules of H . Consider a strong module \mathcal{M} of H/P . To begin, we make two observations. First, if $\mathcal{M} = \emptyset$, then $\cup\mathcal{M} = \emptyset$, and hence $\cup\mathcal{M}$ is a strong module of H . Second, if $|\mathcal{M}| = 1$, then $\cup\mathcal{M} \in P$, and hence $\cup\mathcal{M}$ is a strong module of H because all the elements of P are. Now, suppose that

$$(2) \quad |\mathcal{M}| \geq 2.$$

By the second assertion of Proposition 30, $\cup\mathcal{M}$ is a module of H . To show that $\cup\mathcal{M}$ is strong, consider a module M of H such that $M \cap (\cup\mathcal{M}) \neq \emptyset$. Let $x \in M \cap (\cup\mathcal{M})$. Denote by X the unique element of P containing x . We get $X \in (M/P) \cap \mathcal{M}$. Since \mathcal{M} is a strong module of H/P , we obtain $M/P \subseteq \mathcal{M}$ or $\mathcal{M} \subseteq M/P$. In the first instance, we obtain $\cup(M/P) \subseteq \cup\mathcal{M}$, so we have $M \subseteq \cup(M/P) \subseteq \cup\mathcal{M}$. Lastly, suppose $\mathcal{M} \subseteq M/P$. It follows from (2) that

$$|M/P| \geq 2.$$

Let $Y \in M/P$. We have $Y \cap M \neq \emptyset$. Let $Z \in (M/P) \setminus Y$. We have $Z \cap M \neq \emptyset$, and hence $M \setminus Y \neq \emptyset$. Since Y is a strong module of H , we obtain $Y \subseteq M$. It follows that $M = \cup(M/P)$. Since $\mathcal{M} \subseteq M/P$, we obtain $\cup\mathcal{M} \subseteq \cup(M/P)$, and hence $\cup\mathcal{M} \subseteq M$. \square

We use the characterization of disconnected hypergraphs in terms of the quotient (see Lemma 35 below) to prove the analogue of Theorem 18 (see Theorem 36 below). Recall the following definition.

Definition 32. A hypergraph H is *connected* if for distinct $v, w \in V(H)$, there exists a sequence (e_0, \dots, e_n) of edges of H , where $n \geq 0$, satisfying $v \in e_0$, $w \in e_n$, and (when $n \geq 1$) $e_i \cap e_{i+1} \neq \emptyset$ for every $0 \leq i \leq n-1$. Given a hypergraph H , a maximal connected subhypergraph of H is called a *component* of H .

Notation 33. Given a hypergraph H , the set of the components of H is denoted by $\mathfrak{C}(H)$.

Remark 34. Let H be a hypergraph. For each component C of H , $V(C)$ is a module of H . Thus, $\{V(C) : C \in \mathfrak{C}(H)\}$ is a modular partition of H . Furthermore, for each component C of H , $V(C)$ is a strong module of H . We conclude the remark with the following result.

Lemma 35. *Given a hypergraph H with $v(H) \geq 2$, the following assertions are equivalent:*

- (1) H is disconnected;
- (2) H admits a modular bipartition P such that H/P is empty;
- (3) $\Pi(H) = \{V(C) : C \in \mathfrak{C}(H)\}$, $|\Pi(H)| \geq 2$, and $H/\Pi(H)$ is empty.

Let H be a hypergraph such that $v(H) \geq 2$. Because of the maximality of the elements of $\Pi(H)$ (see Notation 7), it follows from the second assertion of Proposition 31 that all the strong modules of $H/\Pi(H)$ are trivial. To prove Theorem 8, we establish the following result, which is the analogue of Theorem 18.

Theorem 36. *Given a hypergraph H , all the strong modules of H are trivial if and only if H is an empty hypergraph, a prime hypergraph, or a complete graph.*

Proof. Clearly, if H is an empty hypergraph, a prime hypergraph, or a complete graph, then all the strong modules of H are trivial.

To demonstrate the converse, we prove the following. Given a hypergraph H , if all the strong modules of H are trivial, and H is decomposable, then H is an empty hypergraph or a complete graph.

To begin, we show that H admits a modular bipartition. (This part also appears in the proof of Theorem 18, see [2, Proposition 2].) Since H is decomposable, we can consider a maximal nontrivial module M of H under inclusion. Since M is a nontrivial module, M is not strong. Consequently, there exists a module N of H such that $M \cap N \neq \emptyset$, $M \setminus N \neq \emptyset$, and $N \setminus M \neq \emptyset$. Since $M \cap N \neq \emptyset$, $M \cup N$ is a module of H by Lemma 57. Clearly, $M \not\subseteq M \cup N$ because $N \setminus M \neq \emptyset$. Since M is a maximal nontrivial module of H , $M \cup N$ is a trivial module of H , so $M \cup N = V(H)$. Since $M \setminus N \neq \emptyset$, $N \setminus M$ is a module of H by Lemma 58. But, $N \setminus M = V(H) \setminus M$ because $M \cup N = V(H)$. It follows that $\{M, V(H) \setminus M\}$ is a modular bipartition of H .

We have $H/\{M, V(H) \setminus M\}$ is an empty hypergraph or a complete graph. We distinguish the following two cases.

- (1) Suppose that $H/\{M, V(H) \setminus M\}$ is an empty hypergraph. We prove that H is an empty hypergraph. By Lemma 35, H is disconnected. Let $C \in \mathfrak{C}(H)$. As recalled in Remark 34, $V(C)$ is a strong module of H . By hypothesis, $V(C)$ is trivial. Since H is disconnected, $V(C) \not\subseteq V(H)$. It follows that $v(C) = 1$. Therefore, H is isomorphic to $H/\{V(C) : C \in \mathfrak{C}(H)\}$. It follows from Lemma 35 that H is empty.

(2) Suppose that $H/\{M, V(H) \setminus M\}$ is a complete graph. We prove that H is a complete graph. Consider the graph H^c defined on $V(H)$ by

$$(3) \quad E(H^c) = (E(H) \setminus \binom{V(H)}{2}) \cup (\binom{V(H)}{2} \setminus E(H)).$$

It is easy to verify that H and H^c share the same modules. Therefore, they share the same strong modules. Consequently, all the strong modules of H^c are trivial, H^c is decomposable, and $\{M, V(H) \setminus M\}$ is a modular bipartition of H . Since $H/\{M, V(H) \setminus M\}$ is a complete graph, $H^c/\{M, V(H) \setminus M\}$ is empty. It follows from the first case that H^c is empty. Hence $E(H^c) = \emptyset$, and it follows from (3) that $E(H) = \binom{V(H)}{2}$. \square

Remark 37. Theorem 36 is stated as follows for hypergraphs that are not graphs. Given a hypergraph H such that

$$E(H) \setminus \binom{V(H)}{2} \neq \emptyset,$$

all the strong modules of H are trivial if and only if H is empty or prime.

Proof of Theorem 8. For a contradiction, suppose that $H/\Pi(H)$ admits a nontrivial strong module \mathcal{S} . By the second assertion of Proposition 31, $\cup \mathcal{S}$ is a strong module of H . Given $X \in \mathcal{S}$, we obtain $X \subsetneq \cup \mathcal{S} \subsetneq V(H)$, which contradicts the maximality of X . Consequently, all the strong modules of $H/\Pi(H)$ are trivial. To conclude, it suffices to apply Theorem 36 to $H/\Pi(H)$. \square

Definition 38. Let H be a hypergraph. As for tournaments (see Definition 19), the set of the nonempty strong modules of H is denoted by $\mathcal{D}(H)$. Clearly, $\mathcal{D}(H)$ ordered by inclusion is a tree. It is called the *modular decomposition tree* of H . For convenience, set

$$\mathcal{D}_{\geq 2}(H) = \{X \in \mathcal{D}(H) : |X| \geq 2\}.$$

Moreover, we associate with each $X \in \mathcal{D}_{\geq 2}(H)$, the label $\varepsilon_H(X)$ defined as follows

$$\varepsilon_H(X) = \begin{cases} \triangle & \text{if } H[X]/\Pi(H[X]) \text{ is prime,} \\ \circ & \text{if } H[X]/\Pi(H[X]) \text{ is empty,} \\ \text{or} \\ \bullet & \text{if } H[X]/\Pi(H[X]) \text{ is a complete graph.} \end{cases}$$

To conclude, we prove the analogue of Proposition 20 for hypergraphs.

Proposition 39. *Given a hypergraph H , consider a strong module M of H . For every $N \subseteq M$, the following two assertions are equivalent:*

- (1) N is a strong module of H ;
- (2) N is a strong module of $H[M]$.

Proof. Let N be a subset of M . To begin, suppose that N is a strong module of H . Since N is a module of H , N is a module of $H[M]$ by Lemma 60. To show that N is a strong module of $H[M]$, consider a module X of $H[M]$ such that $N \cap X \neq \emptyset$. Since M is a module of H , X is a module of H by Lemma 61. Since N is a strong module of H , we obtain $N \subseteq X$ or $X \subseteq N$.

Conversely, suppose that N is a strong module of $H[M]$. Since M is a module of H , N is a module of H by Lemma 61. To show that N is a strong module of H , consider a module X of H such that $N \cap X \neq \emptyset$. We have $M \cap X \neq \emptyset$ because $N \subseteq M$. Since M is a strong module of H , we obtain $M \subseteq X$ or $X \subseteq M$. In the first instance, we get $N \subseteq M \subseteq X$. Hence, suppose that $X \subseteq M$. By Lemma 60, X is a module of $H[M]$. Since N is a strong module of $H[M]$ and $N \cap X \neq \emptyset$, we obtain $N \subseteq X$ or $X \subseteq N$. \square

4. REALIZATION AND DECOMPOSABILITY

We need the following notation for the next proposition.

Notation 40. Let H be a 3-uniform hypergraph. For $W \subseteq V(H)$ such that $W \neq \emptyset$, \widetilde{W}^H denotes the intersection of the strong modules of H containing W . Note that \widetilde{W}^H is the smallest strong module of H containing W .

Notation 41. Let T be a tournament. For a subset W of $V(T)$, set

$$\neg_T W = \{v \in V(T) \setminus W : W \text{ is not a module of } T[W \cup \{v\}]\}.$$

Consider a realizable and 3-uniform hypergraph. Let T be a realization of H . A module of T is clearly a module of H , but the converse is false. Nevertheless, we have the following result. Its proof is arduous and somewhat long, but it is central to establish Theorem 11.

Proposition 42. *Let H be a realizable and 3-uniform hypergraph. Consider a realization T of H . Let M be a module of H . If M is not a module of T , then the following four assertions hold:*

- (1) $M \cup (\neg_T M)$ is a module of T ;
- (2) M is not a strong module of H ;
- (3) $M \cup (\neg_T M) \subseteq \widetilde{M}^H$;
- (4) $\varepsilon_H(\widetilde{M}^H) = \circ$ (see Definition 38) and $|\Pi(H[\widetilde{M}^H])| \geq 3$.

Proof. Since M is not a module of T , we have $\neg_T M \neq \emptyset$. Let $v \in \neg_T M$. Since M is not a module of $T[M \cup \{v\}]$, we obtain

$$(4) \quad \begin{cases} N_T^-(v) \cap M \neq \emptyset \\ \text{and} \\ N_T^+(v) \cap M \neq \emptyset. \end{cases}$$

Furthermore, consider $v^- \in N_T^-(v) \cap M$ and $v^+ \in N_T^+(v) \cap M$. Since M is a module of H , $v^- v v^+ \notin E(H)$. Hence $v^- v v^+ \notin E(C_3(T))$. Since $v^- v, v v^+ \in A(T)$, we get $v^- v^+ \in A(T)$. Therefore, for each $v \in \neg_T M$, we have

$$(5) \quad \text{for } v^- \in N_T^-(v) \cap M \text{ and } v^+ \in N_T^+(v) \cap M, v^- v^+ \in A(T).$$

Now, consider $v, w \in \neg_T M$ such that $vw \in A(T)$. Let $v^- \in N_T^-(v) \cap M$. Suppose for a contradiction that $v^- \in N_T^+(w) \cap M$. We get $v^-vw \in E(C_3(T))$, and hence $v^-vw \in E(H)$. Since M is a module of H , we obtain $\mu vw \in E(H)$ for every $\mu \in M$. Thus, since $vw \in A(T)$, $\mu v \in A(T)$ for every $\mu \in M$. Therefore, $M \subseteq N_T^-(v)$, so $N_T^+(v) \cap M = \emptyset$, which contradicts (4). It follows that for $v, w \in \neg_T M$, we have

$$(6) \quad \text{if } vw \in A(T), \text{ then } N_T^-(v) \cap M \subseteq N_T^-(w) \cap M.$$

For the first assertion, set

$$\begin{cases} M^- = \{v \in V(H) \setminus M : vm \in A(T) \text{ for every } m \in M\} \\ \text{and} \\ M^+ = \{v \in V(H) \setminus M : mv \in A(T) \text{ for every } m \in M\}. \end{cases}$$

Note that $\{M^-, M, \neg_T M, M^+\}$ is a partition of $V(H)$. Let $m^- \in M^-$ and $v \in \neg_T M$. By (4), there exist $v^- \in N_T^-(v) \cap M$ and $v^+ \in N_T^+(v) \cap M$. Suppose for a contradiction that $vm^- \in A(T)$. We get $vm^-v^- \in E(C_3(T))$. Hence $vm^-v^- \in E(H)$. Since $m^-v^+, vv^+ \in A(T)$, we have $vm^-v^+ \notin E(C_3(T))$. Thus $vm^-v^+ \notin E(H)$, which contradicts the fact that M is a module of H . It follows that $m^-v \in A(T)$ for any $m^- \in M^-$ and $v \in \neg_T M$. Similarly, $vm^+ \in A(T)$ for any $m^+ \in M^+$ and $v \in \neg_T M$. It follows that $M \cup (\neg_T M)$ is a module of T .

For the second assertion, consider $v \in (\neg_T M)$. Set

$$(7) \quad N_v = (N_T^-(v) \cap M) \cup \{w \in (\neg_T M) : N_T^-(w) \cap M \subseteq N_T^-(v) \cap M\}.$$

We show that N_v is a module of T . If $m^- \in M^-$, then $m^-n \in A(T)$ for every $n \in N_v$ because $M \cup (\neg_T M)$ is a module of T . Similarly, if $m^+ \in M^+$, then $nm^+ \in A(T)$ for every $n \in N_v$. Now, consider $m \in M \setminus N_v$. We get $m \in M \setminus N_T^-(v)$. Therefore, we have $m \in M \setminus N_T^-(w')$ for every $w' \in \{w \in (\neg_T M) : N_T^-(w) \cap M \subseteq N_T^-(v) \cap M\}$. Thus, $m \in N_T^+(w') \cap M$ for every $w' \in \{w \in (\neg_T M) : N_T^-(w) \cap M \subseteq N_T^-(v) \cap M\}$. Since $m \in N_T^+(v) \cap M$, it follows from (5) that $v^-m \in A(T)$ for every $v^- \in N_T^-(v) \cap M$. Furthermore, since $m \in N_T^+(w') \cap M$ for every $w' \in \{w \in (\neg_T M) : N_T^-(w) \cap M \subseteq N_T^-(v) \cap M\}$, we have $w'm \in A(T)$ for every $w' \in \{w \in (\neg_T M) : N_T^-(w) \cap M \subseteq N_T^-(v) \cap M\}$. Therefore, we obtain $nm \in A(T)$ for every $n \in N_v$. Lastly, consider $u \in (\neg_T M) \setminus N_v$. We get $u \in (\neg_T M)$ and $N_T^-(u) \cap M \not\subseteq N_T^-(v) \cap M$. It follows from (6) that $vu \in A(T)$. By (6) again, we have $N_T^-(v) \cap M \not\subseteq N_T^-(u) \cap M$. Thus $v^-u \in A(T)$ for each $v^- \in N_T^-(v) \cap M$. Let $w' \in \{w \in (\neg_T M) : N_T^-(w) \cap M \subseteq N_T^-(v) \cap M\}$. We get $N_T^-(w') \cap M \not\subseteq N_T^-(u) \cap M$. It follows from (6) that $w'u \in A(T)$. Consequently, N_v is a module of T for each $v \in (\neg_T M)$. Hence N_v is a module of H for each $v \in (\neg_T M)$. Let $v \in (\neg_T M)$. Clearly, $v \in N_v \setminus M$. Moreover, it follows from (4) that there exist $v^- \in N_T^-(v) \cap M$ and $v^+ \in N_T^+(v) \cap M$. We get $v^- \in M \cap N_v$ and $v^+ \in M \setminus N_v$. Since N_v is a module of H , M is not a strong module of H .

For the third assertion, consider $v \in (\neg_T M)$. As previously proved, N_v is a module of H . Furthermore, by considering $v^- \in N_T^-(v) \cap M$ and $v^+ \in$

$N_T^+(v) \cap M$, we obtain $M \cap N_v \neq \emptyset$ and $M \setminus N_v \neq \emptyset$. Hence $\widetilde{M}^H \cap N_v \neq \emptyset$ and $\widetilde{M}^H \setminus N_v \neq \emptyset$. Since \widetilde{M}^H is a strong module of H , we get $N_v \subseteq \widetilde{M}^H$. Thus $v \in \widetilde{M}^H$ for every $v \in (\neg_T M)$. Therefore $M \cup (\neg_T M) \subseteq \widetilde{M}^H$.

For the fourth assertion, we prove that for each $v \in (\neg_T M)$,

$$P_v = \{N_T^-(v) \cap M, N_T^+(v) \cap M, \neg_T M\}$$

is a modular partition of $H[M \cup (\neg_T M)]$. Let $v \in (\neg_T M)$. By (5), $N_T^-(v) \cap M$ and $N_T^+(v) \cap M$ are modules of $T[M]$. Thus, $N_T^-(v) \cap M$ and $N_T^+(v) \cap M$ are modules of $H[M]$. Since M is a module of H , it follows from Lemma 61 that $N_T^-(v) \cap M$ and $N_T^+(v) \cap M$ are modules of H . By Lemma 60, $N_T^-(v) \cap M$ and $N_T^+(v) \cap M$ are modules of $H[M \cup (\neg_T M)]$. Now, we prove that $\neg_T M$ is a module of $H[M \cup (\neg_T M)]$. It suffices to prove that there exists no $e \in E(H[M \cup (\neg_T M)])$ such that $e \cap (\neg_T M) \neq \emptyset$ and $e \cap M \neq \emptyset$. Indeed, suppose to the contrary that there exists $e \in E(H[M \cup (\neg_T M)])$ such that $e \cap (\neg_T M) \neq \emptyset$ and $e \cap M \neq \emptyset$. Since M is a module of H , we get $|e \cap M| = 1$ and $|e \cap (\neg_T M)| = 2$. Therefore, there exist $v, w \in e \cap (\neg_T M)$ and $m \in e \cap M$ such that $vwm \in E(H)$. By replacing v by w if necessary, we can suppose that $vw \in A(T)$. Since $H = C_3(T)$, we obtain $vw, wm, mv \in A(T)$, which contradicts (6). Therefore, $\neg_T M$ is a module of $H[M \cup (\neg_T M)]$. Consequently, $P_v = \{N_T^-(v) \cap M, N_T^+(v) \cap M, \neg_T M\}$ is a modular partition of $H[M \cup (\neg_T M)]$. Furthermore, given $v \in (\neg_T M)$, consider $v^- \in N_T^-(v) \cap M$ and $v^+ \in N_T^+(v) \cap M$. It follows from (5) that $v^-v^+v \notin E(C_3(T))$, and hence $v^-v^+v \notin E(H)$. Consequently,

$$(8) \quad H[M \cup (\neg_T M)]/P_v \text{ is empty.}$$

Since $M \cup (\neg_T M)$ is a module of T by the first assertion above, $M \cup (\neg_T M)$ is a module of H . By Lemma 60, $M \cup (\neg_T M)$ is a module of $H[\widetilde{M}^H]$. Given $v \in (\neg_T M)$, it follows from Lemma 61 that each element of P_v is a module of $H[\widetilde{M}^H]$.

Let $v \in (\neg_T M)$. For a contradiction, suppose that there exist $Y \in P_v$ and $X \in \Pi(H[\widetilde{M}^H])$ such that $Y \not\subseteq X$. We get $X \cap (M \cup (\neg_T M)) \neq \emptyset$. Since $M \cup (\neg_T M)$ is a module of $H[\widetilde{M}^H]$ and X is a strong module of $H[\widetilde{M}^H]$, we have $M \cup (\neg_T M) \subseteq X$ or $X \not\subseteq M \cup (\neg_T M)$. Furthermore, since X is a strong module of $H[\widetilde{M}^H]$ and \widetilde{M}^H is a strong module of H , it follows from Proposition 39 that X is a strong module of H . Since $X \not\subseteq \widetilde{M}^H$, it follows from the minimality of \widetilde{M}^H that we do not have $M \cup (\neg_T M) \subseteq X$. Therefore, $X \not\subseteq M \cup (\neg_T M)$. Let $x \in X \setminus Y$. We have $x \in (M \cup (\neg_T M)) \setminus Y$. Denote by Y' the unique element of $P_v \setminus \{Y\}$ such that $x \in Y'$. Also, denote by Z the unique element of $P_v \setminus \{Y, Y'\}$. We get $X \cap Y' \neq \emptyset$ and $Y \subseteq X \setminus Y'$. Since X is a strong module of $H[\widetilde{M}^H]$, we get $Y' \subseteq X$. Since $X \not\subseteq M \cup (\neg_T M)$, we obtain $X \cap Z = \emptyset$. Thus $X = Y \cup Y'$. Since $H[M \cup (\neg_T M)]/P_v$ is empty, $\{Y, Z\}$ is a module of $H[M \cup (\neg_T M)]/P_v$. By the second assertion of Proposition 30, $Y \cup Z$ is a module of $H[M \cup (\neg_T M)]$. As previously seen, $M \cup (\neg_T M)$ is a module of $H[\widetilde{M}^H]$. By Lemma 61, $Y \cup Z$ is a module of $H[\widetilde{M}^H]$, which contradicts the fact that X is a strong module of $H[\widetilde{M}^H]$.

Consequently,

(9) for any $Y \in P_v$ and $X \in \Pi(H[\widetilde{M}^H])$, we do not have $Y \not\subseteq X$.

Let $Y \in P_v$. Set

$$Q_Y = \{X \in \Pi(H[\widetilde{M}^H]) : X \cap Y \neq \emptyset\}.$$

For every $X \in Q_Y$, we have $Y \not\subseteq X$ or $X \subseteq Y$ because X is a strong module of $H[\widetilde{M}^H]$. By (9), we have $X \subseteq Y$. It follows that

(10) for each $Y \in P_v$, we have $Y = \cup Q_Y$.

Therefore, $|\Pi(H[\widetilde{M}^H])| \geq |P_v|$, that is,

$$|\Pi(H[\widetilde{M}^H])| \geq 3.$$

Finally, we prove that $H[\widetilde{M}^H]/\Pi(H[\widetilde{M}^H])$ is empty. Suppose that $M \cup (\neg_T M) \not\subseteq \widetilde{M}^H$, and set

$$Q_{M \cup (\neg_T M)} = \{X \in \Pi(H[\widetilde{M}^H]) : X \cap (M \cup (\neg_T M)) \neq \emptyset\}.$$

Since $M \cup (\neg_T M)$ is a module of $H[\widetilde{M}^H]$, it follows from the first assertion of Proposition 30 that $Q_{M \cup (\neg_T M)}$ is a module of $H[\widetilde{M}^H]/\Pi(H[\widetilde{M}^H])$. Moreover, it follows from (10) that $|Q_{M \cup (\neg_T M)}| \geq 3$. Since each element of $\Pi(H[\widetilde{M}^H])$ is a strong element of \widetilde{M}^H , we get $M \cup (\neg_T M) = \cup Q_{M \cup (\neg_T M)}$. Since $M \cup (\neg_T M) \not\subseteq \widetilde{M}^H$, we obtain that $Q_{M \cup (\neg_T M)}$ is a nontrivial module of $H[\widetilde{M}^H]/\Pi(H[\widetilde{M}^H])$. Hence $H[\widetilde{M}^H]/\Pi(H[\widetilde{M}^H])$ is decomposable. It follows from Theorem 8 that $H[\widetilde{M}^H]/\Pi(H[\widetilde{M}^H])$ is empty. Lastly, suppose that $M \cup (\neg_T M) = \widetilde{M}^H$. Suppose also that there exists $Y \in P_v$ such that $|Q_Y| \geq 2$. As previously, we obtain that Q_Y is a nontrivial module of $H[\widetilde{M}^H]/\Pi(H[\widetilde{M}^H])$, and hence $H[\widetilde{M}^H]/\Pi(H[\widetilde{M}^H])$ is empty. Therefore, suppose that $|Q_Y| = 1$ for every $Y \in P_v$. By (10), $\Pi(H[\widetilde{M}^H]) = P_v$. Hence $H[\widetilde{M}^H]/\Pi(H[\widetilde{M}^H])$ is empty by (8). \square

Remark 43. Let T be a tournament. Consider a subset M of $V(T)$ such that M is not a module of T . In general, $M \cup (\neg_T M)$ is not a module of T . Nevertheless, if M is a module of $C_3(T)$, then it follows from the first assertion of Proposition 42 that $M \cup (\neg_T M)$ is a module of T . In this case, $M \cup (\neg_T M)$ is the convex envelope of M in T .

The next result is an easy consequence of Proposition 42.

Corollary 44. *Consider a realizable 3-uniform hypergraph H , and a realization T of H . The following two assertions are equivalent:*

- H and T share the same modules;
- for each strong module X of H such that $|X| \geq 2$, we have

$$\text{if } \varepsilon_H(X) = \circ, \text{ then } |\Pi(H[X])| = 2.$$

Proof. To begin, suppose that H and T do not share the same modules. There exists a module M of H , which is not a module of T . By the last assertion of Proposition 42, we obtain $\varepsilon_H(\widetilde{M}^H) = \circ$ and $|\Pi(H[\widetilde{M}^H])| \geq 3$.

Conversely, suppose that there exists a strong module X of H , with $|X| \geq 2$, such that $\varepsilon_H(X) = \circ$ and $|\Pi(H[X])| \geq 3$. It follows from the second assertion of Proposition 42 that X is a module of T . Observe that $T[X]$ realizes $H[X]$. Let $Y \in \Pi(H[X])$. Since Y is a strong module of $H[X]$, it follows from the second assertion of Proposition 42 applied to $H[X]$ and $T[X]$ that Y is a module of $T[X]$. Thus, $\Pi(H[X])$ is a modular partition of $T[X]$. Since $H[X]/\Pi(H[X])$ is empty, $T[X]/\Pi(H[X])$ is a linear order. Denote by Y_{\min} the smallest element of $T[X]/\Pi(H[X])$. Similarly, denote by Y_{\max} the largest element of $T[X]/\Pi(H[X])$. Since $H[X]/\Pi(H[X])$ is empty, $\{Y_{\min}, Y_{\max}\}$ is a module of $H[X]/\Pi(H[X])$. By the second assertion of Proposition 30, $Y_{\min} \cup Y_{\max}$ is a module of $H[X]$. Since X is a module of H , it follows from Lemma 61 that $Y_{\min} \cup Y_{\max}$ is a module of H . Lastly, since $|\Pi(H[X])| \geq 3$, there exists $Y \in \Pi(H[X]) \setminus \{Y_{\min}, Y_{\max}\}$. Since Y_{\min} is the smallest element of $T[X]/\Pi(H[X])$ and Y_{\max} is the largest one, we obtain $Y_{\min}Y, YY_{\max} \in A(T[X]/\Pi(H[X]))$. Therefore, for $y_{\min} \in Y_{\min}$, $y \in Y$ and $y_{\max} \in Y_{\max}$, we have $y_{\min}y, yy_{\max} \in A(T[X])$, and hence $y_{\min}y, yy_{\max} \in A(T)$. Consequently, $Y_{\min} \cup Y_{\max}$ is not a module of T . \square

Now, we prove Theorem 11 by using Proposition 42.

Proof of Theorem 11. To begin, consider a strong module M of H . By the second assertion of Proposition 42, M is a module of T . Let N be a module of T such that $M \cap N \neq \emptyset$. Since N is a module of T , N is a module of H . Furthermore, since M is a strong module of H , we obtain $M \subseteq N$ or $N \subseteq M$. Therefore, M is a strong module of T .

Conversely, consider a strong module M of T . Since M is a module of T , M is a module of H . Let N be a module of H such that $M \cap N \neq \emptyset$. If N is a module of T , then $M \subseteq N$ or $N \subseteq M$ because M is a strong module of T . Hence suppose that N is not a module of T . By the last assertion of Proposition 42, $\varepsilon_H(\widetilde{N}^H) = \circ$ and $|\Pi(H[\widetilde{N}^H])| \geq 3$. Since $M \cap N \neq \emptyset$, $M \cap \widetilde{N}^H \neq \emptyset$. Since \widetilde{N}^H is a strong module of H , we get $\widetilde{N}^H \subseteq M$ or $M \not\subseteq \widetilde{N}^H$. Clearly, if $\widetilde{N}^H \subseteq M$, then $N \subseteq M$. Thus, suppose that

$$M \not\subseteq \widetilde{N}^H.$$

We prove that $M \subseteq N$. As previously proved, \widetilde{N}^H is a strong module of T because it is a strong module of H . Since M is a strong module of T , it follows from Proposition 20 that M is a strong module of $T[\widetilde{N}^H]$. For each $X \in \Pi(H[\widetilde{N}^H])$, X is a strong module of $T[\widetilde{N}^H]$ because it is a strong module of $H[\widetilde{N}^H]$. Therefore, $\Pi(H[\widetilde{N}^H])$ is a modular partition of $T[\widetilde{N}^H]$. Set

$$Q_M = \{X \in \Pi(H[\widetilde{N}^H]) : M \cap X \neq \emptyset\}.$$

By the first assertion of Proposition 31, Q_M is a strong module of $T[\tilde{N}^H]/\Pi(H[\tilde{N}^H])$. Since $H[\tilde{N}^H]/\Pi(H[\tilde{N}^H])$ is empty, $T[\tilde{N}^H]/\Pi(H[\tilde{N}^H])$ is a linear order. Therefore, Q_M is a trivial module of $T[\tilde{N}^H]/\Pi(H[\tilde{N}^H])$. For a contradiction, suppose that $Q_M = \Pi(H[\tilde{N}^H])$. Since M is a strong module of $T[\tilde{N}^H]$, we get $M = \tilde{N}^H$, which contradicts $M \not\subseteq \tilde{N}^H$. It follows that $|Q_M| = 1$. Hence there exists $X_M \in \Pi(H[\tilde{N}^H])$ such that

$$M \subseteq X_M.$$

Since N is not a module of T , it follows from the second assertion of Proposition 42 that N is not a strong module of H . Thus $N \not\subseteq \tilde{N}^H$. Set

$$Q_N = \{X \in \Pi(H[\tilde{N}^H]) : N \cap X \neq \emptyset\}.$$

Since \tilde{N}^H is a strong module of H , it follows from Proposition 39 that each element of $\Pi(H[\tilde{N}^H])$ is a strong module of H . It follows from the minimality of \tilde{N}^H that $|Q_N| \geq 2$. Since each element of $\Pi(H[\tilde{N}^H])$ is a strong module of H , we obtain

$$N = \cup Q_N.$$

Since $M \cap N \neq \emptyset$, we get $X_M \in Q_N$. We obtain $M \subseteq X_M \subseteq N$. \square

Lastly, we establish Theorem 12 by using Theorems 8 and 11.

Proof of Theorem 12. Suppose that H is prime. Since all the modules of T are modules of H , T is prime.

Conversely, suppose that T is prime. Hence, all the strong modules of T are trivial. By Theorem 11, all the strong modules of H are trivial. We obtain

$$\Pi(H) = \{\{v\} : v \in V(H)\}.$$

Thus, H is isomorphic to $H/\Pi(H)$. It follows from Theorem 8 that H is an empty hypergraph, a prime hypergraph, or a complete graph. Since T is prime, we have $E(C_3(T)) \neq \emptyset$. Since $E(C_3(T)) = E(H)$, there exists $e \in E(H)$ such that $|e| = 3$. Therefore, H is not an empty hypergraph and H is not a graph. It follows that H is prime. \square

5. REALIZABILITY OF 3-UNIFORM HYPERGRAPHS

The next proposition is useful to construct realizations from the modular decomposition tree. We need the following notation and remark.

Notation 45. Let H be a 3-uniform hypergraph. We denote by $\mathcal{R}(H)$ the set of the realizations of H .

Remark 46. Let H be a 3-uniform hypergraph. Consider $T \in \mathcal{R}(H)$. It follows from Theorem 11 that

$$\mathcal{D}(H) = \mathcal{D}(T).$$

By the same theorem, for each $X \in \mathcal{D}_{22}(H)$, we have

$$\Pi(H[X]) = \Pi(T[X]).$$

Therefore, for each $X \in \mathcal{D}_{\geq 2}(H)$, $T[X]/\Pi(T[X])$ realizes $H[X]/\Pi(H[X])$, that is,

$$T[X]/\Pi(T[X]) \in \mathcal{R}(H[X]/\Pi(H[X])).$$

Set

$$\mathcal{R}_{\mathcal{D}}(H) = \bigcup_{X \in \mathcal{D}_{\geq 2}(H)} \mathcal{R}(H[X]/\Pi(H[X])).$$

We denote by $\theta_H(T)$ the function

$$\begin{aligned} \mathcal{D}_{\geq 2}(H) &\longrightarrow \mathcal{R}_{\mathcal{D}}(H) \\ Y &\longmapsto T[Y]/\Pi(T[Y]). \end{aligned}$$

Lastly, we denote by $\mathcal{Q}(H)$ the set of the functions f from $\mathcal{D}_{\geq 2}(H)$ to $\mathcal{R}_{\mathcal{D}}(H)$ satisfying $f(Y) \in \mathcal{R}(H[Y]/\Pi(H[Y]))$ for each $Y \in \mathcal{D}_{\geq 2}(H)$. Under this notation, we obtain the function

$$\begin{aligned} \theta_H: \mathcal{R}(H) &\longrightarrow \mathcal{Q}(H) \\ T &\longmapsto \theta_H(T). \end{aligned}$$

Proposition 47. *For a 3-uniform hypergraph, θ_H is a bijection.*

Proof. To begin, we show that θ_H is injective. Let T and T' be distinct realizations of H . There exist distinct $v, w \in V(H)$ such that $vw \in A(T)$ and $wv \in A(T')$. Consider $Z_v, Z_w \in \Pi(H[\overline{\{v, w\}}^H])$ (see Notation 40) such that $v \in Z_v$ and $w \in Z_w$. Since $\overline{\{v, w\}}^H$ is the smallest strong module of H containing $\{v, w\}$, we obtain $Z_v \neq Z_w$. It follows from Theorem 11 that $\Pi(H[\overline{\{v, w\}}^H]) = \Pi(T[\overline{\{v, w\}}^H])$ and $\Pi(H[\overline{\{v, w\}}^H]) = \Pi(T'[\overline{\{v, w\}}^H])$. Since $vw \in A(T)$ and $wv \in A(T')$, we obtain

$$\begin{cases} Z_v Z_w \in A(T[\overline{\{v, w\}}^H])/\Pi(T[\overline{\{v, w\}}^H]) \\ \text{and} \\ Z_w Z_v \in A(T'[\overline{\{v, w\}}^H])/\Pi(T'[\overline{\{v, w\}}^H]). \end{cases}$$

Consequently, $\theta_H(T)(\overline{\{v, w\}}^H) \neq \theta_H(T')(\overline{\{v, w\}}^H)$. Thus, $\theta_H(T) \neq \theta_H(T')$.

Now, we prove that θ_H is surjective. Consider $f \in \mathcal{Q}(H)$, that is, f is a function from $\mathcal{D}_{\geq 2}(H)$ to $\mathcal{R}_{\mathcal{D}}(H)$ satisfying $f(Y) \in \mathcal{R}(H[Y]/\Pi(H[Y]))$ for each $Y \in \mathcal{D}_{\geq 2}(H)$. We construct $T \in \mathcal{R}(H)$ such that $\theta_H(T) = f$ in the following manner. Consider distinct vertices v and w of H . Clearly, $\overline{\{v, w\}}^H$ is a strong module of H such that $|\overline{\{v, w\}}^H| \geq 2$. There exist $Z_v, Z_w \in \Pi(H[\overline{\{v, w\}}^H])$ such that $v \in Z_v$ and $w \in Z_w$. Since $\overline{\{v, w\}}^H$ is the smallest strong module of H containing v and w , we obtain $Z_v \neq Z_w$. Set

$$(11) \quad \begin{cases} vw \in A(T) \text{ if } Z_v Z_w \in A(f(\overline{\{v, w\}}^H)), \\ \text{and} \\ wv \in A(T) \text{ if } Z_w Z_v \in A(f(\overline{\{v, w\}}^H)). \end{cases}$$

We obtain a tournament T defined on $V(H)$.

Lastly, we verify that T realizes H . First, consider distinct vertices u, v, w of H such that $uvw \in E(H)$. There exist $Z_u, Z_v, Z_w \in \Pi(H[\{\widetilde{u, v, w}^H\}])$ such that $u \in Z_u$, $v \in Z_v$, and $w \in Z_w$. For a contradiction, suppose that $Z_u = Z_v$. Since Z_u is a module of H and $uvw \in E(H)$, we get $w \in Z_u$. Thus, $Z_u = Z_v = Z_w$, which contradicts the fact that $\{\widetilde{u, v, w}^H\}$ is the smallest strong module of H containing u, v and w . It follows that $Z_u \neq Z_v$. Similarly, we have $Z_u \neq Z_w$ and $Z_v \neq Z_w$. It follows that $Z_u Z_v Z_w \in E(H[\{\widetilde{u, v, w}^H\}]/\Pi(H[\{\widetilde{u, v, w}^H\}]))$. Since $f(\{\widetilde{u, v, w}^H\})$ realizes $H[\{\widetilde{u, v, w}^H\}]/\Pi(H[\{\widetilde{u, v, w}^H\}])$, we obtain $Z_u Z_v, Z_v Z_w, Z_w Z_u \in A(f(\{\widetilde{u, v, w}^H\}))$ or $Z_u Z_w, Z_w Z_v, Z_v Z_u \in A(f(\{\widetilde{u, v, w}^H\}))$. By exchanging u and v if necessary, assume that

$$Z_u Z_v, Z_v Z_w, Z_w Z_u \in A(f(\{\widetilde{u, v, w}^H\})).$$

Since $Z_u \neq Z_v$, we obtain $\{\widetilde{u, v}^H\} = \{\widetilde{u, v, w}^H\}$. Similarly, we have $\{\widetilde{u, w}^H\} = \{\widetilde{u, v, w}^H\}$ and $\{\widetilde{v, w}^H\} = \{\widetilde{u, v, w}^H\}$. It follows from (11) that $uv, vw, wu \in A(T)$. Hence, $T[\{u, v, w\}]$ is a 3-cycle.

Conversely, consider distinct vertices u, v, w of T such that $T[\{u, v, w\}]$ is a 3-cycle. There exist $Z_u, Z_v, Z_w \in \Pi(H[\{\widetilde{u, v, w}^H\}])$ such that $u \in Z_u$, $v \in Z_v$, and $w \in Z_w$. For a contradiction, suppose that $Z_u = Z_v$. Since $\{\widetilde{u, v, w}^H\}$ is the smallest strong module of H containing u, v , and w , we obtain $Z_u \neq Z_w$. Therefore, we have $\{\widetilde{u, w}^H\} = \{\widetilde{u, v, w}^H\}$ and $\{\widetilde{v, w}^H\} = \{\widetilde{u, v, w}^H\}$. For instance, assume that $Z_u Z_w \in A(f(\{\widetilde{u, v, w}^H\}))$. It follows from (11) that $uw, vw \in A(T)$, which contradicts the fact that $T[\{u, v, w\}]$ is a 3-cycle. Consequently, $Z_u \neq Z_v$. It follows that $\{\widetilde{u, v}^H\} = \{\widetilde{u, v, w}^H\}$. Similarly, we have $\{\widetilde{u, w}^H\} = \{\widetilde{u, v, w}^H\}$ and $\{\widetilde{v, w}^H\} = \{\widetilde{u, v, w}^H\}$. For instance, assume that $uv, vw, wu \in A(T)$. It follows from (11) that $Z_u Z_v, Z_v Z_w, Z_w Z_u \in A(f(\{\widetilde{u, v, w}^H\}))$. Since $f(\{\widetilde{u, v, w}^H\})$ realizes

$$H[\{\widetilde{u, v, w}^H\}]/\Pi(H[\{\widetilde{u, v, w}^H\}]),$$

we obtain $Z_u Z_v Z_w \in E(H[\{\widetilde{u, v, w}^H\}]/\Pi(H[\{\widetilde{u, v, w}^H\}]))$. It follows that $uvw \in E(H[\{\widetilde{u, v, w}^H\}])$, and hence $uvw \in E(H)$.

Consequently, $T \in \mathcal{R}(H)$. Let $X \in \mathcal{D}_{\geq 2}(H)$. As seen at the beginning of Remark 46, we have $\Pi(H[X]) = \Pi(T[X])$, and

$$T[X]/\Pi(T[X]) \in \mathcal{R}(H[X]/\Pi(H[X])).$$

Consider distinct elements Y and Z of $\Pi(H[X])$. For instance, suppose that $YZ \in A(T[X]/\Pi(T[X]))$. Let $v \in Y$ and $w \in Z$. We obtain $vw \in A(T)$. Moreover, we have $\{\widetilde{v, w}^H\} = X$ because $Y, Z \in \Pi(H[X])$ and $Y \neq Z$. It

follows from (11) that $YZ \in A(f(X))$. Therefore,

$$(12) \quad T[X]/\Pi(T[X]) = f(X).$$

Since (12) holds for every $X \in \mathcal{D}_{\geq 2}(H)$, we have $\theta_H(T) = f$. □

Theorem 13 is an easy consequence of Proposition 47.

Proof of Theorem 13. Clearly, if H is realizable, then $H[W]$ is also for every $W \subseteq V(H)$. Conversely, suppose that $H[W]$ is realizable for every $W \subseteq V(H)$ such that $H[W]$ is prime. We define an element f of $\mathcal{Q}(H)$ as follows. Consider $Y \in \mathcal{D}_{\geq 2}(H)$. By Theorem 8, $H[Y]/\Pi(H[Y])$ is empty or prime.

First, suppose that $H[Y]/\Pi(H[Y])$ is empty. We choose for $f(Y)$ any linear order defined on $\Pi(H[Y])$. Clearly, $f(Y) \in \mathcal{R}(H[Y]/\Pi(H[Y]))$.

Second, suppose that $H[Y]/\Pi(H[Y])$ is prime. Consider a transverse W of $\Pi(H[Y])$ (see Definition 28). The function

$$\begin{aligned} \varphi_W \quad W : & \longrightarrow \Pi(H[Y]) \\ w & \longmapsto Z, \text{ where } w \in Z, \end{aligned}$$

is an isomorphism from $H[W]$ onto $H[Y]/\Pi(H[Y])$. Thus, $H[W]$ is prime. By hypothesis, $H[W]$ admits a realization T_W . We choose for $f(Y)$ the unique tournament defined on $\Pi(H[Y])$ such that φ_W is an isomorphism from T_W onto $f(Y)$. Clearly, $f(Y) \in \mathcal{R}(H[Y]/\Pi(H[Y]))$.

By Proposition 47, $(\theta_H)^{-1}(f)$ is a realization of H . □

Theorem 13 leads us to study the realization of prime and 3-uniform hypergraphs. We need to introduce the analogue of Definition 21 for 3-uniform hypergraphs.

Definition 48. Given a prime and 3-uniform hypergraph H , a vertex v of H is *critical* if $H - v$ is decomposable. A prime and 3-uniform hypergraph is *critical* if all its vertices are critical.

For critical and 3-uniform hypergraphs, we obtain the following characterization, which is an immediate consequence of Theorems 12 and 22.

Theorem 49. *Given a critical and 3-uniform hypergraph H , H is realizable if and only if $v(H)$ is odd and H is isomorphic to $C_3(T_{v(H)})$, $C_3(U_{v(H)})$, or $C_3(W_{v(H)})$.*

Now, we characterize the noncritical, prime, and 3-uniform hypergraphs that are realizable. We need the following notation.

Notation 50. Let H be a 3-uniform hypergraph. Consider a vertex w of H . Set

$$V_w = V(H) \setminus \{w\}.$$

We denote by G_w the graph defined on V_w as follows. Given distinct elements v and v' of V_w ,

$$vv' \in E(G_w) \text{ if } wvv' \in E(H) \text{ (note that the graph } G_w \text{ is used in [7]) .}$$

Also, we denote by I_w the set of the isolated vertices of G_w .

Notation 51. Let T be a tournament. Consider $W, W' \subseteq V(T)$ such that $W \cap W' = \emptyset$. We denote by $(W \rightarrow W')_T$ the subset of $w' \in W'$ such that there exists a sequence w_0, \dots, w_n satisfying

- $w_0 \in W$ and $w_n = w'$;
- $w_1, \dots, w_n \in W'$;
- for $i = 0, \dots, n-1$, $w_i w_{i+1} \in A(T)$.

Theorem 52. *Let H be a noncritical, prime, and 3-uniform hypergraph. Consider a vertex w of H such that $H - w$ is prime. The 3-uniform hypergraph H is realizable if and only if $H - w$ admits a realization, say T_w , satisfying the following two assertions.*

- (M1) *There exists a bipartition $\{X, Y\}$ of $V_w \setminus I_w$ (see Notation 50) satisfying*
- *for each component C of G_w , with $v(C) \geq 2$, C is bipartite with bipartition $\{X \cap V(C), Y \cap V(C)\}$;*
 - *for $x \in X$ and $y \in Y$, we have*

$$(13) \quad xy \in E(G_w) \text{ if and only if } xy \in A(T_w).$$

- (M2) *We have $(X \rightarrow I_w)_{T_w} \cap (Y \rightarrow I_w)_{(T_w)^*} = \emptyset$ (see Notation 51) and $(X \rightarrow I_w)_{T_w} \cup (Y \rightarrow I_w)_{(T_w)^*} = I_w$. Furthermore, for $x \in X$, $y \in Y$, $x^+ \in (X \rightarrow I_w)_{T_w}$, and $y^- \in (Y \rightarrow I_w)_{(T_w)^*}$, we have $y^-x, yx^+, y^-x^+ \in A(T_w)$.*

Proof. To begin, suppose that H admits a realization T . Clearly, $T - w$ is a realization of $H - w$. Set

$$T_w = T - w.$$

For assertion (M1), consider a component C of G_w such that $v(C) \geq 2$. Consider distinct vertices c_0, c_1, c_2 of C such that $c_0c_1, c_1c_2 \in E(G_w)$. We show that

$$(14) \quad \begin{cases} c_0c_1, c_2c_1 \in A(T) \\ \text{or} \\ c_1c_0, c_1c_2 \in A(T). \end{cases}$$

Suppose that $c_0c_1 \in A(T)$. Since $c_0c_1 \in E(G_w)$, $T[\{w, c_0, c_1\}]$ is a 3-cycle. Hence, $wc_0, c_1w \in A(T)$ because $c_0c_1 \in A(T)$. Since $c_1c_2 \in E(G_w)$, $T[\{w, c_1, c_2\}]$ is a 3-cycle. Since $c_1w \in A(T)$, we obtain $c_2c_1 \in A(T)$. Dually, if $c_1c_0 \in A(T)$, then $c_1c_2 \in A(T)$. It follows that (14) holds. We denote by $V(C)^-$ the set of the vertices c^- of C such that there exists $c^+ \in V(C)$ satisfying $c^-c^+ \in E(G_x)$ and $c^-c^+ \in A(T)$. Dually, we denote by $V(C)^+$ the set of the vertices c^+ of C such that there exists $c^- \in V(C)$ satisfying $c^-c^+ \in E(G_x)$ and $c^-c^+ \in A(T)$. Since C is a component of G_x , we have $V(C) = V(C)^- \cup V(C)^+$. Moreover, it follows from (14) that

$$V(C)^- \cap V(C)^+ = \emptyset.$$

Therefore, it follows from the definition of $V(C)^-$ and $V(C)^+$ that $V(C)^-$ and $V(C)^+$ are stable subsets of C . Therefore, C is bipartite with bipartition $\{V(C)^-, V(C)^+\}$. Set

$$X = \bigcup_{C \in \mathcal{C}(G_w)} V(C)^- \text{ and } Y = \bigcup_{C \in \mathcal{C}(G_w)} V(C)^+ \text{ (see Notation 33).}$$

Clearly, $\{X, Y\}$ is a bipartition of $V_w \setminus I_w$. Consider again a component C of G_w such that $v(C) \geq 2$. Since $V(C)^- = X \cap V(C)$ and $V(C)^+ = Y \cap V(C)$, C is bipartite with bipartition $\{X \cap V(C), Y \cap V(C)\}$. To prove that (13) holds, consider $x \in X$ and $y \in Y$. Denote by C the component of G_w containing x . Since $x \in X$, $x \notin I_w$, so $v(C) \geq 2$. We obtain $x \in V(C)^-$. First, suppose that $xy \in E(G_w)$. Hence $y \in V(C)$. We obtain $y \in V(C)^+$. By definition of $V(C)^-$, there exists $x^+ \in V(C)$ such that $xx^+ \in E(G_w)$ and $xx^+ \in A(T)$. Since $xy \in E(G_w)$, it follows from (14) that $xy \in A(T)$. Second, suppose that $xy \in A(T)$. Since $x \in V(C)^-$, there exists $x^+ \in V(C)$ such that $xx^+ \in E(G_w)$ and $xx^+ \in A(T)$. It follows that $wx, x^+w \in A(T)$. Denote by D the component of G_w containing y . Since $y \in Y$, we obtain $y \in V(D)^+$. Similarly, there exists $y^- \in V(D)$ such that $wy^-, yw \in A(T)$. We obtain $wx, xy, yw \in A(T)$. Thus, $T[\{w, x, y\}]$ is a 3-cycle, so $xy \in E(G_w)$.

For assertion (M2), consider $x \in X$. Denote by C the component of G_w such that $x \in V(C)$. We have $x \in V(C)^-$. Therefore, there exists $x^+ \in V(C)$ satisfying $xx^+ \in E(G_x)$ and $xx^+ \in A(T)$. Since $T[\{w, x, x^+\}]$ is a 3-cycle and $xx^+ \in A(T)$, we get $wx \in A(T)$. Hence,

$$(15) \quad wx \in A(T) \text{ for every } x \in X.$$

Dually, we have

$$(16) \quad yw \in A(T) \text{ for every } y \in Y.$$

Set

$$X^+ = (X \rightarrow I_w)_{(T-w)} \text{ and } Y^- = (Y \rightarrow I_w)_{(T-w)^*}.$$

Now, consider $x^+ \in X^+$. There exists a sequence x_0, \dots, x_n satisfying $x_0 \in X$, $x_n = x^+$, $x_1, \dots, x_n \in I_w$, and $x_i x_{i+1} \in A(T)$ for $i = 0, \dots, n-1$. We show that $wx_i \in A(T)$ by induction on $i = 0, \dots, n$. By (15), this is the case when $i = 0$. Consider $i \in \{0, \dots, n-1\}$, and suppose that $wx_i \in A(T)$. Since $x_{i+1} \in I_x$, $x_i x_{i+1} \notin E(G_x)$. Thus $T[\{w, x_i, x_{i+1}\}]$ is a linear order. Since $wx_i, x_i x_{i+1} \in A(T)$, we obtain $wx_{i+1} \in A(T)$. It follows that

$$(17) \quad wx^+ \in A(T)$$

for every $x^+ \in X^+$. Dually, we have

$$(18) \quad y^- w \in A(T)$$

for every $y^- \in Y^-$. It follows from (17) and (18) that $X^+ \cap Y^- = \emptyset$. By definition of X^+ and Y^- , $X^+ \subseteq I_x$ and $Y^- \subseteq I_x$. Set

$$Z = I_x \setminus (X^+ \cup Y^-).$$

Let $z \in Z$. Since $z \notin X^+$, we have $zx \in A(T)$ for every $x \in X \cup X^+$. Therefore,

$$(19) \quad zx \in A(T)$$

for $z \in Z$ and $x \in X \cup X^+$. Dually,

$$(20) \quad yz \in A(T)$$

for $z \in Z$ and $y \in Y \cup Y^-$. It follows from (15), (16), (17), (18), (19), and (20) that $\{w\} \cup Z$ is a module of T . Since H is prime, it follows from Theorem 12 that T is prime as well. Therefore, $Z = \emptyset$, so

$$X^+ \cup Y^- = I_x.$$

To conclude, consider $x \in X$, $y \in Y$, $x^+ \in X^+$, and $y^- \in Y^-$. Since $x^+ \notin Y^-$, $yx^+ \in A(T)$. Dually, we have $y^-x \in A(T)$. It follows from (17) and (18) that $y^-w, wx^+ \in A(T)$. Since $x^+, y^- \in I_x$, $x^+y^- \notin E(G_x)$. Thus, $T[\{w, x^+, y^-\}]$ is a linear order. Consequently, we have $y^-x^+ \in A(T)$.

Conversely, suppose that $H-x$ admits a realization T_x such that assertions (M1) and (M2) hold. As previously, set

$$X^+ = (X \rightarrow I_w)_{(T-w)} \text{ and } Y^- = (Y \rightarrow I_w)_{(T-w)^*}.$$

Let T be the tournament defined on $V(H)$ by

$$(21) \quad \begin{cases} T-w = T_w, \\ \text{for every } x \in X \cup X^+, wx \in A(T), \\ \text{and} \\ \text{for every } y \in Y \cup Y^-, yw \in A(T). \end{cases}$$

We verify that T is a realization of H . Since T_w realizes $H-w$, it suffices to verify that for distinct $u, v \in V_w$, $uv \in E(G_w)$ if and only if $T[\{u, v, w\}]$ is a 3-cycle. Hence, consider distinct $v, w \in V_x$.

First, suppose that $uv \in E(G_w)$. Denote by C the component of G_w containing u and v . Since assertion (M1) holds, C is bipartite with bipartition $\{X \cap V(C), Y \cap V(C)\}$. By exchanging u and v if necessary, we can assume that $u \in X \cap V(C)$ and $v \in Y \cap V(C)$. It follows from (13) that $uv \in A(T_w)$. Furthermore, it follows from (21) that $uv \in A(T)$, $wu \in A(T)$, and $vw \in A(T)$. Therefore, $T[\{u, v, w\}]$ is a 3-cycle.

Second, suppose that $T[\{u, v, w\}]$ is a 3-cycle. By exchanging v and w if necessary, we can assume that $uv, vw, wu \in A(T)$. It follows from (21) that $uv \in A(T_w)$, $u \in X \cup X^+$, and $v \in Y \cup Y^-$. Moreover, since assertion (M2) holds and $uv \in A(T_w)$, we obtain $u \in X$ and $v \in Y$. It follows from (13) that $uv \in E(G_w)$. \square

The next result is an easy consequence of Theorem 52.

Corollary 53. *Let H be a noncritical, prime, and 3-uniform hypergraph. Consider a vertex w of H such that $H-w$ is prime. Suppose that $H-w$ is realizable, and consider a realization T_w of $H-w$. If H is realizable, then there exists a unique realization T of H such that $T-w = T_w$.*

Proof. Consider a realization T of H such that $T - w = T_w$. It follows from the proof of Theorem 52 that T satisfies (15), (16), (17), and (18), that is, $wx, yw, wx^+, y^-w \in A(T)$ for $x \in X, y \in Y, x^+ \in X^+,$ and $y^- \in Y^-$. It follows that T is uniquely determined. \square

A new proof of Theorem 23. Let T be a prime tournament. Consider a tournament T' such that $C_3(T') = C_3(T)$. We prove by induction on $v(T)$ that $T' = T$ or T^* . Since T is prime, we have $v(T) = 3$ or $v(T) \geq 5$. The result is clear when $v(T) = 3$ because C_3 is the only prime tournament defined on 3 vertices. Hence, suppose that $v(T) \geq 5$.

First, suppose that T is critical. By Theorem 22, $v(T)$ is odd, and T is isomorphic to $T_{v(\tau)}, U_{v(\tau)},$ or $W_{v(\tau)}$. For instance, suppose that $T = T_{2n+1}$, where $v(T) = 2n + 1$. Hence, T' is also defined on $\{0, \dots, 2n\}$, and $C_3(T') = C_3(T_{2n+1})$. Suppose that $(2n)(2n - 1) \in A(T')$. We have to show that $T' = T_{2n+1}$ because $(2n)(2n - 1) \in A(T_{2n+1})$. Since $C_3(T') = C_3(T_{2n+1})$, we obtain

$$(22) \quad (2i - 1)(2n - 1), (2n)(2i - 1), (2n - 1)(2i), (2i)(2n) \in A(T')$$

for $i = 0, \dots, n - 1$. Moreover, we obtain

$$(23) \quad (2n - 2)(2n - 3) \in A(T').$$

We have $C_3(T' - \{2n - 1, 2n\}) = C_3(T_{2n+1} - \{2n - 1, 2n\})$ because $C_3(T') = C_3(T_{2n+1})$. Therefore $C_3(T' - \{2n - 1, 2n\}) = C_3(T_{2n-1})$. Furthermore, it follows from (23) that $(2n - 2)(2n - 3) \in A(T' - \{2n - 1, 2n\}) \cap A(T_{2n-1})$. By induction hypothesis, we have $T' - \{2n - 1, 2n\} = T_{2n-1}$. It follows from (22) that $T' = T_{2n+1}$. We proceed in a similar way when $T = U_{2n+1}$ or W_{2n+1} .

Second, suppose that T is not critical. There exists a vertex w of T such that $T - w$ is prime. We have $C_3(T' - w) = C_3(T - w)$ because $C_3(T') = C_3(T)$. By induction hypothesis, we have $T' - w = T - w$ or $(T - w)^*$. By exchanging T' and $(T')^*$, we can assume that $T' - w = T - w$. It follows from Corollary 53 that $T' = T$. \square

We conclude by counting the number of realizations of a realizable and 3-uniform hypergraph. This counting is an immediate consequence of Proposition 47. We need the following notation.

Notation 54. Let H be a 3-uniform hypergraph. Set

$$\begin{cases} \mathcal{D}_\Delta(H) = \{X \in \mathcal{D}_{\geq 2}(H) : \varepsilon_H(X) = \Delta\} \text{ (see Definition 38)} \\ \text{and} \\ \mathcal{D}_\circ(H) = \{X \in \mathcal{D}_{\geq 2}(H) : \varepsilon_H(X) = \circ\}. \end{cases}$$

Corollary 55. For a realizable and 3-uniform hypergraph, we have

$$|\mathcal{R}(H)| = 2^{|\mathcal{D}_\Delta(H)|} \times \prod_{X \in \mathcal{D}_\circ(H)} |\Pi(H[X])|! .$$

APPENDICES

APPENDIX A. PROOF OF PROPOSITION 3

Given a hypergraph H , \emptyset , $V(H)$, and $\{v\}$ (for $v \in V(H)$) are clearly modules of H . Therefore, Proposition 3 is a direct consequence of the next four lemmas.

Lemma 56. *Let H be a hypergraph. For any $M, N \in \mathcal{M}(H)$, we have $M \cap N \in \mathcal{M}(H)$.*

Proof. Consider $M, N \in \mathcal{M}(H)$. To show that $M \cap N \in \mathcal{M}(H)$, consider $e \in E(H)$ such that $e \cap (M \cap N) \neq \emptyset$ and $e \setminus (M \cap N) \neq \emptyset$. Since $e \setminus (M \cap N) \neq \emptyset$, assume for instance that $e \setminus M \neq \emptyset$. Since M is a module of H and $e \cap M \neq \emptyset$, there exists $m \in M$ such that $e \cap M = \{m\}$. Since $e \cap (M \cap N) \neq \emptyset$, we obtain

$$e \cap (M \cap N) = \{m\}.$$

Let $n \in M \cap N$. Since M is a module of H , $(e \setminus \{m\}) \cup \{n\} \in E(H)$. \square

Lemma 57. *Let H be a hypergraph. For any $M, N \in \mathcal{M}(H)$, if $M \cap N \neq \emptyset$, then $M \cup N \in \mathcal{M}(H)$.*

Proof. Consider $M, N \in \mathcal{M}(H)$ such that $M \cap N \neq \emptyset$. To show that $M \cup N \in \mathcal{M}(H)$, consider $e \in E(H)$ such that $e \cap (M \cup N) \neq \emptyset$ and $e \setminus (M \cup N) \neq \emptyset$. Since $e \cap (M \cup N) \neq \emptyset$, assume for instance that $e \cap M \neq \emptyset$. Clearly $e \setminus M \neq \emptyset$ because $e \setminus (M \cup N) \neq \emptyset$. Since M is a module of H , there exists $m \in M$ such that $e \cap M = \{m\}$, and

$$(24) \quad (e \setminus \{m\}) \cup \{n\} \in E(H) \text{ for every } n \in M.$$

Consider $n \in M \cap N$. By (24), $(e \setminus \{m\}) \cup \{n\} \in E(H)$. Set

$$f = (e \setminus \{m\}) \cup \{n\}.$$

Clearly $n \in f \cap N$. Furthermore, consider $p \in e \setminus (M \cup N)$. Since $m \in M$, we have $p \neq m$, and hence $p \in f \setminus N$. Since N is a module of H , we obtain $f \cap N = \{n\}$ and

$$(25) \quad (f \setminus \{n\}) \cup \{n'\} \in E(H) \text{ for every } n' \in N.$$

Since $(f \setminus \{n\}) \cup \{n'\} = (e \setminus \{m\}) \cup \{n'\}$ for every $n' \in N$, it follows from (25) that

$$(26) \quad (e \setminus \{m\}) \cup \{n'\} \in E(H) \text{ for every } n' \in N.$$

Therefore, it follows from (24) and (26) that

$$(e \setminus \{m\}) \cup \{n\} \in E(H) \text{ for every } n \in M \cup N.$$

Moreover, since $f \cap N = \{n\}$, we have

$$\begin{aligned} e \cap N &= (\{m\} \cup (e \setminus \{m\})) \cap N \\ &= (\{m\} \cup (f \setminus \{n\})) \cap N \\ &= \{m\} \cap N, \end{aligned}$$

and hence $e \cap N \subseteq \{m\}$. Since $e \cap M = \{m\}$, we obtain $e \cap (M \cup N) = \{m\}$. Consequently, $M \cup N$ is a module of H . \square

Lemma 58. *Let H be a hypergraph. For any $M, N \in \mathcal{M}(H)$, if $M \setminus N \neq \emptyset$, then $N \setminus M \in \mathcal{M}(H)$.*

Proof. Consider $M, N \in \mathcal{M}(H)$ such that $M \setminus N \neq \emptyset$. To show that $N \setminus M \in \mathcal{M}(H)$, consider $e \in E(H)$ such that $e \cap (N \setminus M) \neq \emptyset$ and $e \setminus (N \setminus M) \neq \emptyset$. We distinguish the following two cases.

- (1) Suppose that $e \setminus N \neq \emptyset$. Since N is a module of H and $e \cap N \neq \emptyset$, there exists $n \in N$ such that $e \cap N = \{n\}$, and $(e \setminus \{n\}) \cup \{n'\} \in E(H)$ for every $n' \in N$. Since $e \cap N = \{n\}$ and $e \cap (N \setminus M) \neq \emptyset$, we obtain $e \cap (N \setminus M) = \{n\}$. Therefore,

$$(27) \quad \begin{cases} e \cap (N \setminus M) = \{n\} \\ \text{and} \\ (e \setminus \{n\}) \cup \{n'\} \in E(H) \text{ for every } n' \in N \setminus M. \end{cases}$$

- (2) Suppose that $e \subseteq N$. Since $e \setminus (N \setminus M) \neq \emptyset$, we have $e \cap (M \cap N) \neq \emptyset$. Thus $e \cap M \neq \emptyset$, and $e \setminus M \neq \emptyset$ because $e \cap (N \setminus M) \neq \emptyset$. Since M is a module of H , there exists $m \in M$ such that $e \cap M = \{m\}$, and

$$(28) \quad (e \setminus \{m\}) \cup \{m'\} \in E(H) \text{ for every } m' \in M.$$

Since $e \cap (M \cap N) \neq \emptyset$, $m \in M \cap N$. Consider $p \in M \setminus N$ and $q \in e \cap (N \setminus M)$. Set

$$f = (e \setminus \{m\}) \cup \{p\}.$$

By (28), $f \in E(H)$. Clearly, $p \in f \setminus N$ and $q \in f \cap N$. Since N is a module of H , we have $f \cap N = \{q\}$, and

$$(29) \quad (f \setminus \{q\}) \cup \{r\} \in E(H) \text{ for every } r \in N \setminus M.$$

Since $f \cap N = \{q\}$, we obtain $e = mq$, and hence

$$(30) \quad e \cap (N \setminus M) = \{q\}.$$

Since $e = mq$, we get $f = pq$. Moreover, for each $r \in N \setminus M$, set

$$g_r = (f \setminus \{q\}) \cup \{r\}.$$

Since $f = pq$, we have $g_r = pr$. By (29), $g_r \in E(H)$. Clearly, $p \in g_r \cap M$ and $r \in g_r \setminus M$. Since M is a module of H , we obtain $(g_r \setminus \{p\}) \cup \{m\} \in E(H)$. Since $g_r = pr$, we have

$$(g_r \setminus \{p\}) \cup \{m\} = mr = (e \setminus \{q\}) \cup \{r\}$$

because $e = mq$. Thus, for each $r \in N \setminus M$, $(e \setminus \{q\}) \cup \{r\} \in E(H)$. It follows from (30) that

$$(31) \quad \begin{cases} e \cap (N \setminus M) = \{q\} \\ \text{and} \\ \text{for each } r \in N \setminus M, (e \setminus \{q\}) \cup \{r\} \in E(H). \end{cases}$$

Consequently, it follows from (27) and (31) that, in both cases, there exists $n \in N \setminus M$ such that $e \cap (N \setminus M) = \{n\}$, and $(e \setminus \{n\}) \cup \{n'\} \in E(H)$ for each $n' \in N \setminus M$. Thus $N \setminus M$ is a module of H . \square

Lemma 59. *Let H be a hypergraph. For any $M, N \in \mathcal{M}(H)$, if $M \setminus N \neq \emptyset$, $N \setminus M \neq \emptyset$, and $M \cap N \neq \emptyset$, then $(M \setminus N) \cup (N \setminus M) \in \mathcal{M}(H)$.*

Proof. Consider $M, N \in \mathcal{M}(H)$ such that $M \setminus N \neq \emptyset$, $N \setminus M \neq \emptyset$, and $M \cap N \neq \emptyset$. We show that $(M \setminus N) \cup (N \setminus M) \in \mathcal{M}(H)$. Hence consider $e \in E(H)$ such that $e \cap ((M \setminus N) \cup (N \setminus M)) \neq \emptyset$ and $e \setminus ((M \setminus N) \cup (N \setminus M)) \neq \emptyset$. Since $e \cap ((M \setminus N) \cup (N \setminus M)) \neq \emptyset$, assume for instance that $e \cap (M \setminus N) \neq \emptyset$. Clearly $e \setminus (M \setminus N) \neq \emptyset$ because $e \setminus ((M \setminus N) \cup (N \setminus M)) \neq \emptyset$. Since $N \setminus M \neq \emptyset$, it follows from Lemma 58 that $M \setminus N$ is a module of H . Thus, there exists $m \in M \setminus N$ such that $e \cap (M \setminus N) = \{m\}$. We distinguish the following two cases.

- (1) Suppose that $e \subseteq M$. Since $e \setminus ((M \setminus N) \cup (N \setminus M)) \neq \emptyset$, $e \cap (M \cap N) \neq \emptyset$. Therefore $e \cap N \neq \emptyset$. Furthermore, since $e \cap (M \setminus N) \neq \emptyset$, we have $e \setminus N \neq \emptyset$. Since N is a module of H , there exists $n \in N$ such that $e \cap N = \{n\}$. Since $e \cap (M \cap N) \neq \emptyset$, we get $e \cap (M \cap N) = \{n\}$. Since $e \subseteq M$ and $e \cap (M \setminus N) = \{m\}$, we obtain $e = mn$. It follows that

$$(32) \quad e \cap ((M \setminus N) \cup (N \setminus M)) = \{m\}.$$

Let $p \in (M \setminus N) \cup (N \setminus M)$. We have to show that

$$(33) \quad (e \setminus \{m\}) \cup \{p\} = np \in E(H).$$

Recall that $M \setminus N$ is a module of H . Consequently (33) holds whenever $p \in M \setminus N$. Suppose that $p \in N \setminus M$. Since N is a module of H and $mn \in E(H)$, we get $mp \in E(H)$. Now, since M is a module of H and $mp \in E(H)$, we obtain $np \in E(H)$. It follows that (33) holds for each $p \in (M \setminus N) \cup (N \setminus M)$. Lastly, it follows from (32) that there exists $m \in M \setminus N$ such that

$$(34) \quad \begin{cases} e \cap ((M \setminus N) \cup (N \setminus M)) = \{m\} \\ \text{and} \\ \text{for each } p \in (M \setminus N) \cup (N \setminus M), (e \setminus \{m\}) \cup \{p\} \in E(H). \end{cases}$$

- (2) Suppose that $e \setminus M \neq \emptyset$. Since $e \cap (M \setminus N) = \{m\}$, $m \in e \cap M$. Since M is a module of H , there exists $m' \in M$ such that $e \cap M = \{m'\}$. Since $e \cap (M \setminus N) = \{m\}$, we have $m = m'$, and hence

$$e \cap (M \setminus N) = e \cap M = \{m\}.$$

It follows that $e \cap (M \cap N) = \emptyset$. Since $e \setminus ((M \setminus N) \cup (N \setminus M)) \neq \emptyset$, we obtain

$$e \setminus (M \cup N) \neq \emptyset.$$

Since $M \cap N \neq \emptyset$, it follows from Lemma 57 that $M \cup N$ is a module of H . Therefore, there exists $p \in M \cup N$ such that $e \cap (M \cup N) = \{p\}$,

and for every $q \in M \cup N$, $(e \setminus \{p\}) \cup \{q\} \in E(H)$. Since $e \cap M = \{m\}$, we get $p = m$. Thus, $e \cap (M \cup N) = \{m\}$, and hence

$$(35) \quad e \cap ((M \setminus N) \cup (N \setminus M)) = \{m\}.$$

Since $p = m$, we have $(e \setminus \{m\}) \cup \{q\} \in E(H)$ for every $q \in M \cup N$. It follows that

$$(36) \quad (e \setminus \{m\}) \cup \{q\} \in E(H) \text{ for every } q \in (M \setminus N) \cup (N \setminus M).$$

Combining (35) and (36), we obtain that there exists $m \in M \setminus N$ such that

$$(37) \quad \begin{cases} e \cap ((M \setminus N) \cup (N \setminus M)) = \{m\} \\ \text{and} \\ \text{for every } q \in (M \setminus N) \cup (N \setminus M), (e \setminus \{m\}) \cup \{q\} \in E(H). \end{cases}$$

Consequently, it follows from (34) and (37) that, in both cases, there exists $m \in M \setminus N$ such that $e \cap ((M \setminus N) \cup (N \setminus M)) = \{m\}$, and for every $r \in (M \setminus N) \cup (N \setminus M)$, we have $(e \setminus \{m\}) \cup \{r\} \in E(H)$. \square

APPENDIX B. PROOF OF PROPOSITION 5

Proposition 5 follows from Proposition 3, and from the next four lemmas.

Lemma 60. *Given a hypergraph H , consider subsets W and W' of $V(H)$. If $W \subseteq W'$, then $\{M' \cap W : M' \in \mathcal{M}(H[W'])\} \subseteq \mathcal{M}(H[W])$ (see Definition 4, assertion (A2)).*

Proof. Let M' be a module of $H[W']$. To show that $M' \cap W$ is a module of $H[W]$, consider $e \in E(H[W])$ such that $e \cap (M' \cap W) \neq \emptyset$ and $e \setminus (M' \cap W) \neq \emptyset$. We obtain $e \in E(H[W'])$ and $e \cap M' \neq \emptyset$. Since $e \setminus (M' \cap W) \neq \emptyset$ and $e \subseteq W$, we get $e \setminus M' \neq \emptyset$. Since M' is a module of $H[W']$, there exists $m' \in M'$ such that $e \cap M' = \{m'\}$, and $(e \setminus \{m'\}) \cup \{n'\} \in E(H[W'])$ for each $n' \in M'$. Let $n' \in M' \cap W$. Since $e \subseteq W$, $(e \setminus \{m'\}) \cup \{n'\} \subseteq W$. Hence $(e \setminus \{m'\}) \cup \{n'\} \in E(H[W])$ because $(e \setminus \{m'\}) \cup \{n'\} \in E(H[W'])$. Moreover, since $e \cap (M' \cap W) \neq \emptyset$ and $e \cap M' = \{m'\}$, we obtain $e \cap (M' \cap W) = \{m'\}$. \square

Lemma 61. *Given a hypergraph H , consider subsets W and W' of $V(H)$ such that $W \subseteq W'$. If $W \in \mathcal{M}(H[W'])$, then $\{M' \in \mathcal{M}(H[W']) : M' \subseteq W\} = \mathcal{M}(H[W])$ (see Definition 4, assertion (A3)).*

Proof. By Lemma 60, $\{M' \in \mathcal{M}(H[W']) : M' \subseteq W\} \subseteq \mathcal{M}(H[W])$. Conversely, consider a module M of $H[W]$. To prove that M is a module of $H[W']$, consider $e \in E(H[W'])$ such that $e \cap M \neq \emptyset$ and $e \setminus M \neq \emptyset$. We distinguish the following two cases.

- (1) Suppose that $e \subseteq W$. We obtain $e \in E(H[W])$. Since M is a module of $H[W]$, there exists $m \in M$ such that $e \cap M = \{m\}$, and for each $n \in M$, we have $(e \setminus \{m\}) \cup \{n\} \in E(H[W])$. Hence $(e \setminus \{m\}) \cup \{n\} \in E(H[W'])$.

- (2) Suppose that $e \setminus W \neq \emptyset$. Clearly, $e \cap W \neq \emptyset$ because $e \cap M \neq \emptyset$. Since W is a module of $H[W']$, there exists $w \in W$ such that $e \cap W = \{w\}$. Furthermore,

$$(38) \quad \text{for each } w' \in W, (e \setminus \{w\}) \cup \{w'\} \in E(H[W']).$$

Since $e \cap M \neq \emptyset$, we get $e \cap M = \{w\}$. Clearly, it follows from (38) that $(e \setminus \{w\}) \cup \{w'\} \in E(H[W'])$ for each $w' \in M$. \square

Lemma 62. *Given a hypergraph H , consider subsets W and W' of $V(H)$ such that $W \subseteq W'$. For any $M \in \mathcal{M}(H[W])$ and $M' \in \mathcal{M}(H[W'])$, if $M \cap M' = \emptyset$ and $M' \cap W \neq \emptyset$, then $M \in \mathcal{M}(H[W \cup M'])$ (see Definition 4, assertion (A4)).*

Proof. Consider a module M of $H[W]$ and a module M' of $H[W']$ such that $M \cap M' = \emptyset$ and $M' \cap W \neq \emptyset$. We have to show that M is a module of $H[W \cup M']$. Hence consider $e \in E(H[W \cup M'])$ such that $e \cap M \neq \emptyset$ and $e \setminus M \neq \emptyset$. We distinguish the following two cases.

- (1) Suppose that $e \subseteq W$. We obtain $e \in E(H[W])$. Since M is a module of $H[W]$, there exists $m \in M$ such that $e \cap M = \{m\}$, and for each $n \in M$, we have $(e \setminus \{m\}) \cup \{n\} \in E(H[W])$. Hence $(e \setminus \{m\}) \cup \{n\} \in E(H[W \cup M'])$.
- (2) Suppose that $e \setminus W \neq \emptyset$. We obtain $e \cap (M' \setminus W) \neq \emptyset$. Since $e \cap M \neq \emptyset$, we have $e \setminus M' \neq \emptyset$. Since M' is a module of $H[W']$, there exists $m' \in M'$ such that $e \cap M' = \{m'\}$, and

$$(39) \quad \text{for each } n' \in M', (e \setminus \{m'\}) \cup \{n'\} \in E(H[W']).$$

Since $e \cap (M' \setminus W) \neq \emptyset$ and $e \cap M' = \{m'\}$, we get $e \cap (M' \setminus W) = \{m'\}$. Let $w' \in W \cap M'$. Set

$$f = (e \setminus \{m'\}) \cup \{w'\}.$$

By (39), $f \in E(H[W'])$. Furthermore, since $e \cap (M' \setminus W) = \{m'\}$, we obtain $f \subseteq W$, and hence $f \in E(H[W])$. Since $e \cap M \neq \emptyset$, we have $f \cap M \neq \emptyset$. Moreover, $w' \in f \setminus M$ because $w' \in W \cap M'$ and $M \cap M' = \emptyset$. Since M is a module of $H[W]$, there exists $m \in M$ such that $f \cap M = \{m\}$. Since $f = (e \setminus \{m'\}) \cup \{w'\}$, with $m', w' \notin M$, we get $e \cap M = f \cap M$, so

$$(40) \quad e \cap M = \{m\}.$$

Lastly, consider $n \in M$. We have to verify that $(e \setminus \{m\}) \cup \{n\} \in E(H[W'])$. Set

$$g_n = (f \setminus \{m\}) \cup \{n\}.$$

Since M is a module of $H[W]$ such that $f \cap M = \{m\}$ and $w' \in f \setminus M$, $g_n \in E(H[W])$. Hence $g_n \in E(H[W'])$. Since $n \in g_n \cap M$ and $M \cap M' = \emptyset$, $n \in g_n \setminus M'$. Clearly, $w' \in M'$ because $w' \in W \cap M'$. Furthermore, $w' \in f$ because $f = (e \setminus \{m'\}) \cup \{w'\}$. Since $g_n = (f \setminus \{m\}) \cup \{n\}$, $m \in M$, and $M \cap M' = \emptyset$, we have $w' \in g_n$. It

follows that $w' \in g_n \cap M'$. Since M' is a module of $H[W']$, we have $g_n \cap M' = \{w'\}$ and $(g_n \setminus \{w'\}) \cup \{m'\} \in E(H[W'])$. We have

$$\begin{aligned} (g_n \setminus \{w'\}) \cup \{m'\} &= (((f \setminus \{m\}) \cup \{n\}) \setminus \{w'\}) \cup \{m'\} \\ &= (f \setminus \{m, w'\}) \cup \{m', n\} \\ &= (((e \setminus \{m'\}) \cup \{w'\}) \setminus \{m, w'\}) \cup \{m', n\} \\ &= (e \setminus \{m, m', w'\}) \cup \{m', n, w'\} \\ &= (e \setminus \{m\}) \cup \{n\}. \end{aligned}$$

Therefore, $(e \setminus \{m\}) \cup \{n\} \in E(H[W'])$, and hence $(e \setminus \{m\}) \cup \{n\} \in E(H[W \cup M'])$. It follows from (40) that there exists $m \in M$ satisfying $e \cap M = \{m\}$, and for every $n \in M$, $(e \setminus \{m\}) \cup \{n\} \in E(H[W \cup M'])$. \square

Lemma 63. *Given a hypergraph H , consider subsets W and W' of $V(H)$ such that $W \subseteq W'$. For any $M \in \mathcal{M}(H[W])$ and $M' \in \mathcal{M}(H[W'])$, if $M \cap M' \neq \emptyset$, then $M \cup M' \in \mathcal{M}(H[W \cup M'])$ (see Definition 4, assertion (A5)).*

Proof. Consider a module M of $H[W]$ and a module M' of $H[W']$ such that $M \cap M' \neq \emptyset$. We have to prove that $M \cup M'$ is a module of $H[W \cup M']$. Hence consider $e \in E(H[W \cup M'])$ such that $e \cap (M \cup M') \neq \emptyset$ and $e \setminus (M \cup M') \neq \emptyset$. Let $m \in M \cap M'$. We distinguish the following two cases.

- (1) Suppose that $e \cap M' \neq \emptyset$. Clearly $e \in E(H[W'])$. Moreover, $e \setminus M' \neq \emptyset$ because $e \setminus (M \cup M') \neq \emptyset$. Since M' is a module of $H[W']$, there exists $m' \in M'$ such that $e \cap M' = \{m'\}$, and $(e \setminus \{m'\}) \cup \{n'\} \in E(H[W'])$ for every $n' \in M'$. Hence, for every $n' \in M'$, we have

$$(41) \quad (e \setminus \{m'\}) \cup \{n'\} \in E(H[W \cup M']).$$

In particular, $(e \setminus \{m'\}) \cup \{m\} \in E(H[W \cup M'])$. Set

$$f = (e \setminus \{m'\}) \cup \{m\}.$$

Since $e \cap M' = \{m'\}$, we obtain $f \cap M' = \{m\}$. Hence $m \in f \cap M$. It follows that $f \in E(H[W])$ because $e \in E(H[W \cup M'])$. Clearly $e \setminus M \neq \emptyset$ because $e \setminus (M \cup M') \neq \emptyset$. Since M is a module of $H[W]$, there exists $n \in M$ such that $f \cap M = \{n\}$, and $(f \setminus \{n\}) \cup \{p\} \in E(H[W])$ for every $p \in M$. Since $m \in f \cap M$, we get $m = n$. Therefore, $f \cap M = f \cap M' = \{m\}$. It follows that $f \cap (M \cup M') = \{m\}$, so

$$e \cap (M \cup M') = \{m'\}.$$

By (41), it remains to show that $(e \setminus \{m'\}) \cup \{n\} \in E(H[W \cup M'])$ for each $n \in M$. Let $n \in M$. Recall that $f \cap (M \cup M') = \{m\}$ and $e \cap (M \cup M') = \{m'\}$. Thus $e \setminus (M \cup M') = f \setminus (M \cup M')$. Hence $f \setminus (M \cup M') \neq \emptyset$ because $e \setminus (M \cup M') \neq \emptyset$. It follows that $f \setminus M \neq \emptyset$. Recall that $f \in E(H[W])$. Since M is a module of $H[W]$, we obtain

$(f \setminus \{m\}) \cup \{n\} \in E(H[W])$. We have

$$\begin{aligned} (f \setminus \{m\}) \cup \{n\} &= ((e \setminus \{m'\}) \cup \{m\}) \setminus \{m\} \cup \{n\} \\ &= (e \setminus \{m'\}) \cup \{n\}. \end{aligned}$$

Therefore $(e \setminus \{m'\}) \cup \{n\} \in E(H[W])$, so $(e \setminus \{m'\}) \cup \{n\} \in E(H[W \cup M'])$.

- (2) Suppose that $e \cap M' = \emptyset$. We get $e \in E(H[W])$. Clearly $e \setminus M \neq \emptyset$ because $e \setminus (M \cup M') \neq \emptyset$. Furthermore, since $e \cap (M \cup M') \neq \emptyset$ and $e \cap M' = \emptyset$, we obtain $e \cap (M \setminus M') \neq \emptyset$. Since M is a module of $H[W]$, there exists $q \in M$ such that

$$(42) \quad e \cap M = \{q\}$$

and

$$(43) \quad \text{for every } r \in M, (e \setminus \{q\}) \cup \{r\} \in E(H[W]).$$

Since $e \cap M' = \emptyset$, it follows from (42) that $q \in M \setminus M'$ and

$$(44) \quad e \cap (M \cup M') = \{q\}.$$

By (43), $(e \setminus \{q\}) \cup \{m\} \in E(H[W])$. Set

$$e' = (e \setminus \{q\}) \cup \{m\}.$$

Clearly, $m \in e' \cap M'$. Moreover, since $e \cap (M \cup M') = \{q\}$, we obtain

$$\begin{cases} e' \cap (M \cup M') = \{m\} \\ \text{and} \\ e \setminus (M \cup M') = e' \setminus (M \cup M'). \end{cases}$$

Therefore $e' \cap (M \cup M') \neq \emptyset$, and $e' \setminus (M \cup M') \neq \emptyset$ because $e \setminus (M \cup M') \neq \emptyset$. It follows from the first case above applied with e' that

$$(45) \quad \text{for every } s \in M \cup M', (e' \setminus \{m\}) \cup \{s\} \in E(H[W \cup M']).$$

Recall that $e \cap (M \cup M') = \{q\}$ by (44). Consequently, we have to show that $(e \setminus \{q\}) \cup \{s\} \in E(H[W \cup M'])$ for every $s \in M \cup M'$. Let $s \in M \cup M'$. We have

$$\begin{aligned} (e' \setminus \{m\}) \cup \{s\} &= ((e \setminus \{q\}) \cup \{m\}) \setminus \{m\} \cup \{s\} \\ &= (e \setminus \{q\}) \cup \{s\}. \end{aligned}$$

It follows from (45) that $(e \setminus \{q\}) \cup \{s\} \in E(H[W \cup M'])$. \square

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FACULTÉ DES SCIENCES AÏN CHOCK, DÉPARTEMENT DE MATHÉMATIQUES ET
INFORMATIQUE, KM 8 ROUTE D'EL JADIDA, BP 5366 MAARIF, CASABLANCA, MAROC
E-mail address: aboussairi@hotmail.com

FACULTÉ DES SCIENCES AÏN CHOCK, DÉPARTEMENT DE MATHÉMATIQUES ET
INFORMATIQUE, KM 8 ROUTE D'EL JADIDA, BP 5366 MAARIF, CASABLANCA, MAROC
E-mail address: cherguibrahim@gmail.com

AIX MARSEILLE UNIV, CNRS, CENTRALE MARSEILLE, I2M, MARSEILLE, FRANCE
E-mail address: pierre.ille@univ-amu.fr

FACULTÉ DES SCIENCES AÏN CHOCK, DÉPARTEMENT DE MATHÉMATIQUES ET
INFORMATIQUE, KM 8 ROUTE D'EL JADIDA, BP 5366 MAARIF, CASABLANCA, MAROC
E-mail address: zaidi.fsac@gmail.com