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MINIMUM SIZE BLOCKING SETS OF CERTAIN LINE SETS WITH RESPECT TO AN ELLIPTIC QUADRIC IN PG$(3, q)$

BART DE BRUYN, PUSPENDU PRADHAN, AND BINOD KUMAR SAHOO

ABSTRACT. For a given nonempty subset $\mathcal{L}$ of the line set of PG$(3, q)$, a set X of points of PG$(3, q)$ is called an $\mathcal{L}$-blocking set if each line in $\mathcal{L}$ contains at least one point of X. Consider an elliptic quadric $Q^-(3, q)$ in PG$(3, q)$. Let $\mathcal{E}$ (respectively, $\mathcal{T}, \mathcal{S}$) denote the set of all lines of PG$(3, q)$ which meet $Q^-(3, q)$ in 0 (respectively, $1, 2$) points. In this paper, we characterize the minimum size $\mathcal{L}$-blocking sets in PG$(3, q)$, where $\mathcal{L}$ is one of the line sets $\mathcal{S}$, $\mathcal{E} \cup \mathcal{S}$, and $\mathcal{T} \cup \mathcal{S}$.

1. Introduction

Throughout the paper, q is a prime power and PG(d, q) is the projective space of dimension d defined over a finite field of order q. For two distinct points x, y of PG(d, q), we denote by xy the unique line of PG(d, q) through x and y.

1.1. **On the elliptic quadric $Q^-(3, q)$ in PG$(3, q)$.** Let $Q^-(3, q)$ be an elliptic quadric in PG$(3, q)$, that is, a nondegenerate quadric in PG$(3, q)$ of Witt index one. We refer to [14] for the basic properties of the points, lines and planes of PG$(3, q)$ with respect to $Q^-(3, q)$. The quadric $Q^-(3, q)$ contains $q^2 + 1$ points and every line of PG$(3, q)$ meets $Q^-(3, q)$ in at most two points. We denote by $\mathcal{E}$, $\mathcal{T}$, and $\mathcal{S}$ the set of lines of PG$(3, q)$ that intersect $Q^-(3, q)$ in respectively 0, 1, and 2 points. The elements of $\mathcal{E}$ are called *external lines*, elements of $\mathcal{T}$ *tangent lines*, and elements of $\mathcal{S}$ *secant lines*. Every point of $Q^-(3, q)$ is contained in $q + 1$ tangent lines,

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Corresponding author: Puspendu Pradhan.

this gives $|\mathcal{T}| = (q+1)(q^2+1)$. We also have $|\mathcal{S}| = q^2(q^2+1)/2$ and $|\mathcal{E}| = (q^2+1)(q^2+q+1) - q^2(q^2+1)/2 - (q+1)(q^2+1) = q^2(q^2+1)/2$.

With the quadric $Q^-(3,q)$, there is a naturally associated polarity τ which is symplectic if q is even, and orthogonal if q is odd. Thus τ is an inclusion reversing bijection of order two on the set of all subspaces of PG$(3,q)$. It fixes the line set of PG$(3,q)$, and interchanges the point set and the set of all planes of PG$(3,q)$. For every point x of $Q^-(3,q)$, the plane x^τ of PG$(3,q)$ intersects $Q^-(3,q)$ at the point x and the $q+1$ tangent lines through x are precisely the lines through x contained in x^τ. In this case, we call x^τ a *tangent plane* (which is *tangent to* $Q^-(3,q)$ *at the point* x). For every point x of PG$(3,q) \setminus Q^-(3,q)$, the plane x^τ of PG$(3,q)$ is called a *secant plane* and it intersects $Q^-(3,q)$ in an irreducible conic $\mathcal{C}_{x^\tau}$ of x^τ. The map $x \mapsto \mathcal{C}_{x^\tau}$ defines a bijection between PG$(3,q) \setminus Q^-(3,q)$ and the set of conics contained in $Q^-(3,q)$.

Suppose that q is even. For every point x of PG$(3,q) \setminus Q^-(3,q)$, the secant plane x^τ contains the point x and the tangent lines contained in x^τ are precisely the $q+1$ lines of x^τ through x. Thus the point x is the nucleus of the conic $\mathcal{C}_{x^\tau}$ of x^τ.

Suppose that q is odd. For every point x of PG$(3,q) \setminus Q^-(3,q)$, the secant plane x^τ does not contain the point x and the tangent lines through x are precisely the $q+1$ lines through x meeting the conic $\mathcal{C}_{x^\tau}$.

There are q^2+1 tangent planes and q^3+q secant planes. Every tangent line is contained in one tangent plane and q secant planes. Every secant line is contained in $q+1$ secant planes. Every external line is contained in two tangent planes and $q-1$ secant planes. Every point of $Q^-(3,q)$ is contained in one tangent plane and q^2+q secant planes. Every point of PG$(3,q) \setminus Q^-(3,q)$ is contained in $q+1$ tangent planes and q^2 secant planes.

1.2. **Blocking sets in** PG$(3,q)$**.** For a given nonempty set $\mathcal{L}$ of lines of PG(d,q), a set X of points of PG(d,q) is called an *$\mathcal{L}$-blocking set* if each line of $\mathcal{L}$ meets X in at least one point. Blocking sets in PG(d,q) are combinatorial objects in finite geometry with several applications and have been the subject of investigation by many authors with respect to varying sets of lines. The first step in the study of blocking sets has been to determine the smallest cardinality of a blocking set and to characterize, if possible, all blocking sets of that cardinality. The following classical result was proved by Bose and Burton in [8, Theorem 1].

Proposition 1.1 ([8])**.** *If $\mathcal{L}$ is the set of all lines of* PG(d,q) *and X is an $\mathcal{L}$-blocking set in* PG(d,q)*, then $|X| \geq (q^d-1)/(q-1)$ and equality holds if and only if X is a hyperplane of* PG(d,q)*.*

Now we consider PG$(3,q)$. When $\mathcal{L} = \mathcal{T} \cup \mathcal{E}$, the minimum size $(\mathcal{T} \cup \mathcal{E})$-blocking sets in PG$(3,q)$ are characterized in [10, Theorem 1.6]. It is proved that if X is a $(\mathcal{T} \cup \mathcal{E})$-blocking set in PG$(3,q)$, then $|X| \geq q^2+q$, and equality holds if and only if $X = x^\tau \setminus \{x\}$ for some point x of $Q^-(3,q)$.

When $\mathcal{L} = \mathcal{E}$, Biondi et al. proved in [6, Theorem 3.5] the following result regarding $\mathcal{E}$-blocking sets for $q \geq 9$. However, their proof also works for $q \in \{2, 4, 8\}$. A different proof is given in [10, Theorem 1.7] for the characterization of minimum size $\mathcal{E}$-blocking sets in $\mathrm{PG}(3, q)$ which works for all q, in particular, for $q = 3, 5, 7$.

Proposition 1.2 ([6, 10])**.** *Let X be an $\mathcal{E}$-blocking set in $\mathrm{PG}(3, q)$. Then $|X| \geq q^2$, and equality holds if and only if $X = \pi \setminus Q^-(3, q)$ for some secant plane π of $\mathrm{PG}(3, q)$.*

Let π be a secant plane of $\mathrm{PG}(3, q)$. The lines of $\mathcal{S}$ and $\mathcal{E}$ that are contained in π will respectively be denoted by $\mathcal{S}_\pi$ and $\mathcal{E}_\pi$. Note that $\mathcal{S}_\pi$ (respectively, $\mathcal{E}_\pi$) is the set of secant (respectively, external) lines of π with respect to the conic $\mathcal{C}_\pi := \pi \cap Q^-(3, q)$. If B is an $\mathcal{S}$-blocking set (respectively, $\mathcal{E}$-blocking set) in $\mathrm{PG}(3, q)$, then the set $B_\pi := \pi \cap B$ is an $\mathcal{S}_\pi$-blocking set (respectively, $\mathcal{E}_\pi$-blocking set) in π. For a given $\mathcal{S}_\pi$-blocking set A in π, we define

$$A(\pi) := A \cup (Q^-(3, q) \setminus \mathcal{C}_\pi),$$

which is a disjoint union. The proof of the following lemma is straightforward.

Lemma 1.3. *Let π be a secant plane of $\mathrm{PG}(3, q)$. If A is an $\mathcal{S}_\pi$-blocking set in π, then $A(\pi)$ is an $\mathcal{S}$-blocking set in $\mathrm{PG}(3, q)$ of size $q^2 - q + |A|$.*

In this paper, we characterize the minimum size $\mathcal{L}$-blocking sets in $\mathrm{PG}(3, q)$, where the line set $\mathcal{L}$ is one of $\mathcal{S}$, $\mathcal{E} \cup \mathcal{S}$, and $\mathcal{T} \cup \mathcal{S}$. The following are the main results of this paper.

Theorem 1.4. *Let B be an $\mathcal{S}$-blocking set in $\mathrm{PG}(3, q)$. Then $|B| \geq q^2$ and equality holds if and only if $B = A(\pi)$ for some secant plane π of $\mathrm{PG}(3, q)$ and for some $\mathcal{S}_\pi$-blocking set A in π of size q.*

As a consequence of Theorem 1.4 and Proposition 2.1 (1) in the next section, we have the following.

Corollary 1.5. *Let B be an $\mathcal{S}$-blocking set in $\mathrm{PG}(3, q)$ of minimum size q^2. If q is odd, then $|B \setminus Q^-(3, q)| \in \{0, 1, 3\}$ and one of the following three cases occurs:*

(i) *$B = Q^-(3, q) \setminus \{x\}$, where x is a point of $Q^-(3, q)$.*
(ii) *$B = (Q^-(3, q) \setminus \{x, y\}) \cup \{a\}$, where x, y are two distinct points of $Q^-(3, q)$ and a is a point (different from x, y) on the secant line through x and y.*
(iii) *$A = (Q^-(3, q) \setminus \{w, x, y, z\}) \cup \{a, b, c\}$, where $\{w, x, y, z\}$ is a quadrangle contained in $\mathcal{C}_\pi$ for some secant plane π of $\mathrm{PG}(3, q)$ and a, b, c are the three diagonal points of this quadrangle.*

Theorem 1.6. *Let B be a $(\mathcal{T} \cup \mathcal{S})$-blocking set in $\mathrm{PG}(3, q)$. Then $|B| \geq q^2+1$ and equality holds if and only if $B = Q^-(3, q)$.*

Theorem 1.7. *Let B be an $(\mathcal{E} \cup \mathcal{S})$-blocking set in* PG$(3, q)$. *Then the following hold:*

(i) *If $q = 2$, then $|B| \geq 6$ and equality holds if and only if $B = L \cup L^\tau$ for some line L secant to $Q^-(3, q)$.*
(ii) *If $q \geq 3$, then $|B| \geq q^2 + q + 1$ and equality holds if and only if B is a plane of* PG$(3, q)$.

When $q \geq 4$ is even, Theorem 1.7 (ii) can be seen from [20, Theorem 1.3] which was proved using properties of the symplectic generalized quadrangle $W(q)$ of order q. In this paper, we give a different proof which works for both q odd and even.

We shall prove Theorems 1.4, 1.6, 1.7 in Sections 2, 3, 4, respectively. We note that the minimum size blocking sets in PG$(3, q)$ of similar line sets with respect to a hyperbolic quadric were characterized in the papers [5, 6, 11, 12, 18, 19].

Remark. One can wonder whether the results of the present paper extend to ovoids of PG$(3, q)$. An *ovoid* of PG$(3, q)$ is a set of q^2+1 points intersecting each plane in either a singleton or an oval of that plane (that need not to be an irreducible conic). If q is odd, then every ovoid is also an elliptic quadric [4, 16], but for q even more examples exist. Theorem 1.6 easily generalizes to ovoids using the same arguments. Also Theorem 1.7 is valid for general ovoids. This follows again from [20, Theorem 1.3], taking into account that with every ovoid of PG$(3, q)$, q even, there is associated a symplectic polarity and generalized quadrangle [21]. The proof of Theorem 1.4 cannot be extended to ovoids since it makes use of Proposition 2.1 (2) for which no generalization to ovals is known. However, if all oval intersections O still have the property mentioned in Proposition 2.1 (2), namely that every blocking set A of minimal size q with respect to the secant lines to O has the property that $A \setminus O$ is contained in a tangent line to O, then the conclusion of Theorem 1.4 would still be valid.

2. $\mathcal{S}$-BLOCKING SETS

2.1. **Blocking sets with respect to the secant lines of an irreducible conic of** PG$(2, q)$**.** The minimum size blocking sets of secant lines in PG$(2, q)$ with respect to an irreducible conic were studied by Aguglia et al. in [3, Theorem 1.1] for q even and in [1, Theorem 1.1] for q odd. We recall their results which are needed in this paper.

A *quadrangle* in PG$(2, q)$ is a set of four points, no three of which are collinear. If a, b, c, d are the points of a quadrangle in PG$(2, q)$, define the three points x, y, z to be the intersections of the lines ab and cd, ac and bd, ad and bc, respectively. The points x, y, z are called the *diagonal points* of the quadrangle. Note that the three points x, y, z are contained in a line of PG$(2, q)$ if and only if q is even [15, 9.63, p. 501].

Proposition 2.1 ([1, 3]). *Let $\mathcal{C}$ be an irreducible conic in* $\mathrm{PG}(2,q)$. *If A is a blocking set of the secant lines in* $\mathrm{PG}(2,q)$ *with respect to $\mathcal{C}$, then $|A| \geq q$. Moreover, the following hold:*

(1) *If q is odd, then $|A| = q$ if and only if $|A \setminus \mathcal{C}| \in \{0,1,3\}$ and one of the following three cases occurs:*
  (i) *$A = \mathcal{C} \setminus \{x\}$ for some point $x \in \mathcal{C}$.*
  (ii) *$A = (\mathcal{C} \setminus \{x,y\}) \cup \{a\}$ for some distinct points $x, y \in \mathcal{C}$, and for some point a (different from x and y) on the secant line to $\mathcal{C}$ through x and y.*
  (iii) *$A = (\mathcal{C} \setminus \{w,x,y,z\}) \cup \{a,b,c\}$ for some quadrangle $\{w,x,y,z\} \subseteq \mathcal{C}$ with diagonal points a,b,c.*

(2) *If q is even and $|A| = q$, then the points of $A \setminus \mathcal{C}$ are contained in a line of* $\mathrm{PG}(2,q)$ *which is tangent to $\mathcal{C}$.*

We note that, when q is even, the description of the minimum size blocking sets of the secant lines in $\mathrm{PG}(2,q)$ with respect to $\mathcal{C}$ is quite different. The statement in Proposition 2.1 (2) above was obtained while proving the main result of [3] in Section 2 (see after case (3) on page 654) of that paper.

2.2. **Proof of Theorem 1.4.** If π is a secant plane of $\mathrm{PG}(3,q)$ and A is an $\mathcal{S}_\pi$-blocking set in π of size q, then Lemma 1.3 implies that $A(\pi)$ is an $\mathcal{S}$-blocking set in $\mathrm{PG}(3,q)$ of size q^2. We prove the other parts of Theorem 1.4 in the rest of this section. Suppose that B is an $\mathcal{S}$-blocking set in $\mathrm{PG}(3,q)$ of minimum possible size.

2.2.1. *General properties.* Observe that $Q^-(3,q) \setminus \{x\}$ is an $\mathcal{S}$-blocking set in $\mathrm{PG}(3,q)$ of size q^2 for every point x of $Q^-(3,q)$. Then the minimality of $|B|$ implies that $|B| \leq q^2$ and hence $Q^-(3,q) \setminus B$ is nonempty.

Lemma 2.2. *The following hold:*

(i) *Every secant line through a point of $Q^-(3,q) \setminus B$ meets B in a unique point.*
(ii) $|B| = q^2$.

Proof. Let w be a point of $Q^-(3,q) \setminus B$. Each of the q^2 secant lines through w meets B and two distinct such lines meet B at different points. This gives $|B| \geq q^2$. Since $|B| \leq q^2$, it follows that both (i) and (ii) hold. □

Suppose that B is contained in $Q^-(3,q)$. Then $|B| = q^2$ implies that $B = Q^-(3,q) \setminus \{x\}$ for some point x of $Q^-(3,q)$. Consider a secant plane π containing the point x. Then $x \in \mathcal{C}_\pi$ and $A = \mathcal{C}_\pi \setminus \{x\}$ is an $\mathcal{S}_\pi$-blocking set in π of size q. We also have $B = A \cup (Q^-(3,q) \setminus \mathcal{C}_\pi) = A(\pi)$. This proves Theorem 1.4 in this case.

From now on we assume that B is not contained in $Q^-(3,q)$. Then both $B \setminus Q^-(3,q)$ and $Q^-(3,q) \setminus B$ are nonempty sets and

$$|Q^-(3,q) \setminus B| = |B \setminus Q^-(3,q)| + 1. \tag{2.1}$$

Lemma 2.3. *$B \cap Q^-(3,q)$ is nonempty.*

Proof. Suppose that $B \cap Q^-(3,q)$ is empty. We count the cardinality of the set

$$Z = \{(x,L) : x \in B, L \in \mathcal{S}, x \in L\}$$

in two ways. There are $q(q-1)/2$ secant lines through every point of B. Since $|B| = q^2$, $|\mathcal{S}| = q^2(q^2+1)/2$, and B is an $\mathcal{S}$-blocking set, we must have

$$q^2 \cdot q(q-1)/2 = |Z| \geq |\mathcal{S}| \cdot 1 = q^2(q^2+1)/2,$$

which is not possible. Hence $B \cap Q^-(3,q)$ is nonempty. □

Corollary 2.4. *Every secant line through a point of $B \setminus Q^-(3,q)$ contains two points of either $B \cap Q^-(3,q)$ or $Q^-(3,q) \setminus B$.*

Proof. This follows from Lemma 2.2 (i). □

Lemma 2.5. *The tangency point of every tangent line through a point of $B \setminus Q^-(3,q)$ is contained in $B \cap Q^-(3,q)$.*

Proof. Let x be a point of $B \setminus Q^-(3,q)$ and $y \in Q^-(3,q)$ be the tangency point of some tangent line through x. We show that y is a point of $B \cap Q^-(3,q)$.

Suppose to the contrary that y is a point of $Q^-(3,q) \setminus B$. Then there would be at least q^2+1 lines through y containing a point of B, namely the q^2 secant lines (see Lemma 2.2) and the line yx, in contradiction with $|B| = q^2$. □

Corollary 2.6. *Every line through a point of $B \setminus Q^-(3,q)$ and a point of $Q^-(3,q) \setminus B$ is a secant line which meets $Q^-(3,q) \setminus B$ in a second point.*

Proof. This follows from Lemma 2.5 and Corollary 2.4. □

Corollary 2.7. *$|Q^-(3,q) \setminus B|$ is even and hence $|B \setminus Q^-(3,q)|$ is odd.*

Proof. Since $B \setminus Q^-(3,q)$ is nonempty by our assumption, the first part follows from Corollary 2.6. The second part follows from (2.1) using the first part. □

Lemma 2.8. *Let L be a secant line containing two points of $Q^-(3,q) \setminus B$ and π be a secant plane through L. Then B_π is an $\mathcal{S}_\pi$-blocking set in π of size q.*

Proof. Let $\pi = \pi_0, \pi_1, \ldots, \pi_q$ be the $q+1$ secant planes through L. By Proposition 2.1, $|B_{\pi_i}| \geq q$ for every $i \in \{0,1,\ldots,q\}$. We show that $|B_{\pi_i}| = q$ for each i and then the lemma will follow from this.

Let $|B_{\pi_i}| = q + s_i$, $0 \leq i \leq q$, for some nonnegative integer s_i. We have $|L \cap B| = 1$ by Lemma 2.2 (i) and $\pi_i \cap \pi_j = L$ for distinct $i,j \in \{0,1,\ldots,q\}$. Since $B = \bigcup_{i=0}^{q} B_{\pi_i}$, we get

$$q^2 = |B| = 1 + \sum_{i=0}^{q} (q + s_i - 1).$$

This gives $\sum_{i=0}^{q} s_i = 0$. Since each $s_i \geq 0$, we must have $s_i = 0$ for all i. $\square$

2.2.2. *Proof of Theorem 1.4 for q even.*

Lemma 2.9. *If q is even, then the line through two distinct points of $B \setminus Q^-(3,q)$ is tangent to $Q^-(3,q)$.*

Proof. Let x, y be two distinct points of $B \setminus Q^-(3,q)$. We show that xy is a tangent line. Consider a point $a \in Q^-(3,q) \setminus B$. Lemma 2.5 implies that the lines xa and ya are secant to $Q^-(3,q)$. Since $x, y \in B$ with $x \neq y$, Lemma 2.2 (i) implies that the secant lines xa and ya are distinct. Then, by Corollary 2.4, there exist distinct points b and c of $Q^-(3,q) \setminus B$ such that $xa \cap Q^-(3,q) = \{a, b\}$ and $ya \cap Q^-(3,q) = \{a, c\}$.

Let $\pi = \langle a, b, c \rangle$ be the secant plane generated by the points a, b, c. Then x, y are points of π. Since $a, b \in Q^-(3,q) \setminus B$, applying Lemma 2.8 to the secant line $xa = ab$, we get that B_π is an $\mathcal{S}_\pi$-blocking set in π of minimum size q. So, by Proposition 2.1 (2), all the points of $B_\pi \setminus \mathcal{C}_\pi$ are contained in a common line L which is tangent to $\mathcal{C}_\pi$ and hence to $Q^-(3,q)$. Since $x, y \in B_\pi \setminus \mathcal{C}_\pi$, it follows that the tangent line L contains both x and y. $\square$

As a consequence of Lemma 2.9, we have the following.

Corollary 2.10. *If q is even, then any secant line contains at most one point of $B \setminus Q^-(3,q)$.*

Lemma 2.11. *If q is even, then all the points of $B \setminus Q^-(3,q)$ are contained in a common tangent line.*

Proof. The statement is clear if $|B \setminus Q^-(3,q)| = 1$. Since $|B \setminus Q^-(3,q)|$ is odd by Corollary 2.7, we assume that $|B \setminus Q^-(3,q)| \geq 3$. By Lemma 2.9, it is enough to show that any three distinct points x, y, z of $B \setminus Q^-(3,q)$ are contained in a line.

By Lemma 2.9, xy, xz, and yz are tangent lines. Suppose that the line xy does not contain the point z. Then the plane π generated by the two tangent lines xy and xz is a secant plane. Since q is even, x must be the nucleus of the conic $\mathcal{C}_\pi$ in π and so all tangent lines contained in π meet at x. But the tangent line yz contained in π does not contain x, a contradiction. $\square$

The following proposition proves Theorem 1.4 when q is even.

Proposition 2.12. *If q is even, then $B = A(\pi)$ for some secant plane π of $\mathrm{PG}(3,q)$ and for some $\mathcal{S}_\pi$-blocking set A in π of size q.*

Proof. By Lemma 2.11, there exists a tangent line T containing all the points of $B \setminus Q^-(3,q)$. Consider a point x of T which is in $B \setminus Q^-(3,q)$. Let L be a secant line through x meeting $Q^-(3,q) \setminus B$ at two points. Such a line L exists by Corollary 2.6 as $Q^-(3,q) \setminus B$ is nonempty. The plane π generated by the two intersecting lines T and L is a secant plane of $\mathrm{PG}(3,q)$. Applying Lemma 2.8 to the secant line L, the set B_π is an $\mathcal{S}_\pi$-blocking set in π of size q.

As every point of $B \setminus Q^-(3,q)$ is contained in $A := B_\pi$, we have $B \subseteq A \cup (Q^-(3,q) \setminus \mathcal{C}_\pi) = A(\pi)$. Since $|A(\pi)| = |A| + q^2 - q = q^2 = |B|$, we see that $B = A(\pi)$. $\square$

2.2.3. *Proof of Theorem 1.4 for q odd.*

Lemma 2.13. *If q is odd, then there is no line of* PG$(3,q)$ *containing more than two points of* $B \setminus Q^-(3,q)$.

Proof. Suppose that L_1 is a line of PG$(3,q)$ containing at least three points, say a, b, c, of $B \setminus Q^-(3,q)$. Let L_2 be a secant line through a which contains two points of $Q^-(3,q) \setminus B$. Such a line L_2 exists by Corollary 2.6 as $Q^-(3,q) \setminus B$ is nonempty. By Lemma 2.2 (i), a is the only point of $B \setminus Q^-(3,q)$ contained in L_2. So $L_1 \neq L_2$. The plane π generated by the two intersecting lines L_1 and L_2 is a secant plane of PG$(3,q)$. By Lemma 2.8, the set B_π is an $\mathcal{S}_\pi$-blocking set in π of size q. Then $|B_\pi \setminus \mathcal{C}_\pi| \leq 3$ by Proposition 2.1 (1). Since the points a, b, c of L_1 are contained in $B_\pi \setminus \mathcal{C}_\pi$, we must have $B_\pi \setminus \mathcal{C}_\pi = \{a, b, c\}$.

By Proposition 2.1 (1) (iii), a, b, c must be the three diagonal points of some quadrangle contained in $\mathcal{C}_\pi$. Then a, b, c cannot be contained in any line of π as q is odd, contradicting that the line L_1 of π contains a, b, c. $\square$

We recall the theorem of Desargues in a projective space. Let x, a_1, a_2, a_3, b_1, b_2, b_3 be seven distinct points of PG(d,q) which is a *Desargues configuration*, that is, they satisfy the following two conditions:

(i) the points a_1, a_2, a_3 (respectively, b_1, b_2, b_3) are not contained in any line of PG(d,q),
(ii) the lines a_1b_1, a_2b_2 and a_3b_3 are mutually distinct and intersect at the point x.

Let z_{12}, z_{13}, z_{23} be the intersection points $a_1a_2 \cap b_1b_2$, $a_1a_3 \cap b_1b_3$, $a_2a_3 \cap b_2b_3$, respectively. Then the theorem of Desargues says that the three points z_{12}, z_{13}, z_{23} are contained in a line of PG(d,q).

Lemma 2.14. *If q is odd, then* $|Q^-(3,q) \setminus B| \leq 4$ *and hence* $|B \setminus Q^-(3,q)| \leq 3$.

Proof. Suppose that $|Q^-(3,q) \setminus B| > 4$. Then $|Q^-(3,q) \setminus B| \geq 6$ as $|Q^-(3,q) \setminus B|$ is even by Corollary 2.7. Fix a point x of $B \setminus Q^-(3,q)$. Let L_1, L_2, L_3 be three secant lines through x each of which meets $Q^-(3,q) \setminus B$ at two points (use Corollary 2.6). Set $L_i \cap (Q^-(3,q) \setminus B) = \{a_i, b_i\}$ for $i \in \{1,2,3\}$. Since a_1, a_2, a_3 (respectively, b_1, b_2, b_3) are points of $Q^-(3,q)$, they are not contained in any line of PG$(3,q)$. Thus the seven points $x, a_1, a_2, a_3, b_1, b_2, b_3$ is a Desarguesian configuration. Set $a_1a_2 \cap b_1b_2 = \{z_{12}\}$, $a_1a_3 \cap b_1b_3 = \{z_{13}\}$, and $a_2a_3 \cap b_2b_3 = \{z_{23}\}$. By the theorem of Desargues, the points z_{12}, z_{13}, z_{23} are contained in a line of PG$(3,q)$.

Let π_{ij} be the plane generated by the two intersecting lines L_i and L_j, where $1 \leq i < j \leq 3$. Then π_{ij} is a secant plane and by Lemma 2.8, the set $B_{\pi_{ij}}$ is an $\mathcal{S}_{\pi_{ij}}$-blocking set in π_{ij} of size q. Since a_i, b_i, a_j, b_j are points of

$\mathcal{C}_{\pi_{ij}} \setminus B_{\pi_{ij}}$, Proposition 2.1 (1) implies that $\mathcal{C}_{\pi_{ij}} \setminus B_{\pi_{ij}} = \{a_i, b_i, a_j, b_j\}$ and the set $B_{\pi_{ij}} \setminus \mathcal{C}_{\pi_{ij}}$ consists of the diagonal points of the quadrangle $\{a_i, b_i, a_j, b_j\}$ contained in $\mathcal{C}_{\pi_{ij}}$. In particular, the point z_{ij} is contained in $B_{\pi_{ij}} \setminus \mathcal{C}_{\pi_{ij}}$ and hence in $B \setminus Q^-(3,q)$. It follows that the three points z_{12}, z_{13}, z_{23} of $B \setminus Q^-(3,q)$ are contained in a line of $\mathrm{PG}(3,q)$, contradicting Lemma 2.13.

Hence $|Q^-(3,q) \setminus B| \leq 4$ and then (2.1) implies that $|B \setminus Q^-(3,q)| \leq 3$. □

The following proposition proves Theorem 1.4 when q is odd.

Proposition 2.15. *If q is odd, then $B = A(\pi)$ for some secant plane π of $\mathrm{PG}(3,q)$ and for some $\mathcal{S}_\pi$-blocking set A in π of size q.*

Proof. We have $|B \setminus Q^-(3,q)| \in \{1,3\}$ by Lemma 2.14 and Corollary 2.7. First assume that $|B \setminus Q^-(3,q)| = 1$. Then $|Q^-(3,q) \setminus B| = 2$ by (2.1). Let $B \setminus Q^-(3,q) = \{a\}$ and $Q^-(3,q) \setminus B = \{x, y\}$. The secant line through x and y meets B at the point a. Consider any secant plane π containing the line through x and y. Then $x, y \in \mathcal{C}_\pi$ and $A = (\mathcal{C}_\pi \setminus \{x,y\}) \cup \{a\}$ is an $\mathcal{S}_\pi$-blocking set in π of size q. It can be seen that $B = A \cup (Q^-(3,q) \setminus \mathcal{C}_\pi) = A(\pi)$.

Now assume that $|B \setminus Q^-(3,q)| = 3$. Then $|Q^-(3,q) \setminus B| = 4$ by (2.1). Let $B \setminus Q^-(3,q) = \{a,b,c\}$ and $Q^-(3,q) \setminus B = \{w,x,y,z\}$. Using Corollary 2.6, there are exactly two secant lines through a point of $B \setminus Q^-(3,q)$ each of which meets $Q^-(3,q) \setminus B$ in at most two points. Conversely, any secant line through two points of $Q^-(3,q) \setminus B$ contains a unique point of $B \setminus Q^-(3,q)$ by Lemma 2.2 (i). Thus the four points w, x, y, z generate a plane π of $\mathrm{PG}(3,q)$ which contains the points a, b, c as well. In fact, a, b, c are the three diagonal points of the quadrangle $\{w,x,y,z\}$ contained in the conic $\mathcal{C}_\pi$. Then $A = (\mathcal{C}_\pi \setminus \{w,x,y,z\}) \cup \{a,b,c\}$ is an $\mathcal{S}_\pi$-blocking set in π of size q. We also have $B = A \cup (Q^-(3,q) \setminus \mathcal{C}_\pi) = A(\pi)$ in this case. □

3. $(\mathcal{T} \cup \mathcal{S})$-BLOCKING SETS

If A is a minimum size blocking set in $\mathrm{PG}(2,q)$ of the tangent and secant lines with respect to an irreducible conic $\mathcal{C}$, then $|A| = q+1$. Further, if A is disjoint from $\mathcal{C}$, then A is a line of $\mathrm{PG}(2,q)$ that is external to $\mathcal{C}$. This result was proved by Bruen and Thas in [9] for q even, and by Segre and Korchmáros in [22] for all q. All such blocking sets A that are different from $\mathcal{C}$ and the lines of $\mathrm{PG}(2,q)$ are described by Boros et al. in [7]. Our proof of Theorem 1.6 given below does not need these results from the planar case.

Proof of Theorem 1.6. Let B be a $(\mathcal{T} \cup \mathcal{S})$-blocking set in $\mathrm{PG}(3,q)$ of minimum possible size. Since the quadric $Q^-(3,q)$ contains q^2+1 points and it blocks every tangent and secant line, the minimality of $|B|$ implies that $|B| \leq |Q^-(3,q)| = q^2+1$. We assert that $B = Q^-(3,q)$. It is enough to show that $Q^-(3,q) \subseteq B$.

Suppose that there exists a point x of $Q^-(3,q)$ which is not in B. Each line through x in $\mathrm{PG}(3,q)$ is either tangent or secant to $Q^-(3,q)$. Since $x \notin B$ and B is a $(\mathcal{T} \cup \mathcal{S})$-blocking set, the q^2+q+1 lines of $\mathrm{PG}(3,q)$

through x would meet B at different points. This gives $|B| \geq q^2 + q + 1$, a contradiction to $|B| \leq q^2 + 1$. Thus $Q^-(3, q) \subseteq B$. □

4. $(\mathcal{E} \cup \mathcal{S})$-BLOCKING SETS

4.1. **Some useful results on blocking sets of** PG$(2, q)$**.** We recall two results related to blocking sets in PG$(2, q)$. The first of these results was proved in [6, Proposition 3.1] and [19, Lemma 2.4], while the second one was proved in [17, Theorems 3.2, 3.3].

Proposition 4.1 ([6, 19])**.** *Let α be a point of* PG$(2, q)$ *and $\mathcal{L}$ be the set of all lines of* PG$(2, q)$ *not containing α. If A is an $\mathcal{L}$-blocking set in* PG$(2, q)$*, then $|A| \geq q$ and equality holds if and only if $A = L \setminus \{\alpha\}$ for some line L through α.*

Proposition 4.2 ([17])**.** *Let $\mathcal{C}$ be an irreducible conic in* PG$(2, q)$ *and A be a blocking set of the external and secant lines of* PG$(2, q)$ *with respect to $\mathcal{C}$. Then the following hold:*

- (i) *If q is even, then $|A| \geq q$ and equality holds if and only if $A = L \setminus \{n\}$ for some line L of* PG$(2, q)$ *tangent to $\mathcal{C}$, where n is the nucleus of $\mathcal{C}$.*
- (ii) *If q is odd, then $|A| \geq 3$ if $q = 3$, and $|A| \geq q + 1$ if $q \geq 5$. Further, if $q = 3$, then $|A| = 3$ if and only if A consists of the three interior points of* PG$(2, q)$ *with respect to $\mathcal{C}$.*

We also need the following result related to blocking sets of the external lines of PG$(2, q)$ with respect to a conic in it, see [2, Theorem 1.1] for q odd and [13, Theorem 1.1] for q even.

Proposition 4.3 ([2, 13])**.** *Let $\mathcal{C}$ be an irreducible conic in* PG$(2, q)$ *and A be a blocking set of the external lines of* PG$(2, q)$ *with respect to $\mathcal{C}$. Then $|A| \geq q - 1$. Further, if $q = 3$, then $|A| = 2$ if and only if one of the following two cases occurs:*

- (i) *$A = L \setminus \mathcal{C}$ for some line L of* PG$(2, 3)$ *which is secant to $\mathcal{C}$.*
- (ii) *A consists of any two interior points of* PG$(2, 3)$ *with respect to $\mathcal{C}$.*

When $q = 3$, the possibility stated in Proposition 4.3 (ii) was not included in the statement of [2, Theorem 1.1]. It was observed in [11, Theorem 2.1] while giving a different proof for the characterization of the minimum size blocking sets of the external lines with respect to a hyperbolic quadric in PG$(3, q)$, q odd.

4.2. **Proof of Theorem 1.7.** For every plane π of PG$(3, q)$, we denote by $(\mathcal{E} \cup \mathcal{S})_\pi$ the set of external and secant lines of PG$(3, q)$ which are contained in π. Note that if π is a tangent plane, then there is no secant line in $(\mathcal{E} \cup \mathcal{S})_\pi$. If π is a secant plane, then $(\mathcal{E} \cup \mathcal{S})_\pi$ is precisely the set of external and secant lines with respect to the conic $\mathcal{C}_\pi = \pi \cap Q^-(3, q)$ of π.

Suppose now that B is an $(\mathcal{E} \cup \mathcal{S})$-blocking set in PG$(3, q)$ of minimum possible size. Then $|B| \leq q^2 + q + 1$, as every plane blocks every line of

$\mathrm{PG}(3, q)$. For every plane π, the set $B_\pi = \pi \cap B$ is an $(\mathcal{E} \cup \mathcal{S})_\pi$-blocking set in π.

We prove two results for q even, which are needed to show that any secant plane contains at least $q+1$ points of B for $q \geq 4$. Recall that, if q is even and x is a point of $\mathrm{PG}(3, q) \setminus Q^-(3, q)$, then x is a point of the secant plane x^τ. In fact, x is the nucleus of the conic $\mathcal{C}_{x^\tau}$ in the plane x^τ, in particular, x^τ is precisely the union of the $q+1$ tangent lines through x.

Lemma 4.4. *Suppose that q is even. Let L be a tangent line containing a point y which is not in B. Then for every point x of $L \setminus \{y\}$, the set B_{x^τ} contains at least q points which are different from the points of $L \cap B$.*

Proof. Let M be a line through y in x^τ which is different from L. If x is a point of $Q^-(3, q)$, then x^τ is a tangent plane and so the line M is external to $Q^-(3, q)$. If $x \notin Q^-(3, q)$, then x is the nucleus of the conic $\mathcal{C}_{x^\tau}$ of x^τ, implying that $L = yx$ is the unique tangent line of x^τ through y and that M is either a secant or an external line. In all cases, B must block each such line M. Since $y \notin B$ and there are q possible choices for M, it follows that B_{x^τ} contains at least q points which are different from those of $L \cap B$. □

Lemma 4.5. *Suppose that q is even and let x be a point of $B \cap Q^-(3, q)$. If there exists a tangent line L through x with $|L \cap B| = q$, then every tangent line through x contains at least q points of B. In particular, $|B_{x^\tau}| \geq 1 + (q+1)(q-1) = q^2$.*

Proof. Let M be a tangent line through x which is different from L. Since L has a point not in B, Lemma 4.4 implies that the tangent plane x^τ through L contains at least $2q$ points of B. Note that x^τ is also the tangent plane through M.

Suppose that $|M \cap B| \leq q-1$. Then M has at least two points which are not in B. Applying Lemma 4.4 carefully to the tangent line M, it follows that each of the q secant planes z^τ, $z \in M \setminus \{x\}$, through M contains at least q points of B which are different from those of $M \cap B$. Counting the points of B contained in the $q+1$ planes through M, we get

$$|B| \geq 2q + q^2 > q^2 + q + 1,$$

which is a contradiction to the fact that $|B| \leq q^2+q+1$. So $|M \cap B| \geq q$. □

Lemma 4.6. *Let π be any plane of $\mathrm{PG}(3, q)$. Then the following hold:*

(i) *Suppose that π is a tangent plane. Then $|B_\pi| \geq q$ and equality holds if and only if $B_\pi = L \setminus \{x\}$ for some tangent line L through x, where $\{x\} = \pi \cap Q^-(3, q)$.*

(ii) *Suppose that π is a secant plane. Then $|B_\pi| \geq q$. Further, if $q \geq 4$, then $|B_\pi| \geq q+1$.*

Proof. (i) Since π is a tangent plane, $(\mathcal{E} \cup \mathcal{S})_\pi$ is precisely the set of all lines of π not containing the point x. Then (i) follows from Proposition 4.1.

(ii) Here B_π is an $(\mathcal{E} \cup \mathcal{S})_\pi$-blocking set in π. The first part for all q and the second part for odd $q \geq 5$ follow from Proposition 4.2.

Assume that $q \geq 4$ is even. Let $\pi = x^\tau$ for some point x of PG$(3, q) \setminus Q^-(3, q)$. We have $|B_{x^\tau}| \geq q$ by Proposition 4.2 (i). Suppose that $|B_{x^\tau}| = q$. Note that $(\mathcal{E} \cup \mathcal{S})_{x^\tau}$ is precisely the set of lines in x^τ not containing x. Then, by Proposition 4.2 (i) again, $B_{x^\tau} = L \setminus \{x\}$ for some tangent line L through x. Let $L = \{x_0, x_1, \dots, x_{q-1}, x_q = x\}$ with tangency point $x_0 \in Q^-(3, q)$. By Lemma 4.4, each of the secant planes x_i^τ, $1 \leq i \leq q-1$, through L contains at least q points of B which are different from those of $L \cap B$. Also we have $|B_{x_0^\tau}| \geq q^2$ for the tangent plane x_0^τ by Lemma 4.5. Counting the points of B contained in the q planes x_i^τ through L for $i \in \{0, 1, \dots, q-1\}$ and using our assumption that $q \geq 4$, we get

$$|B| \geq q^2 + (q-1)q > q^2 + q + 1,$$

which is a contradiction to the fact that $|B| \leq q^2 + q + 1$. Hence $|B_{x^\tau}| \geq q + 1$. $\square$

Corollary 4.7. $|B| \geq q^2 + q$.

Proof. If every tangent line meets B, then B would be blocking set with respect to all lines of PG$(3, q)$ and hence we must have $|B| \geq q^2 + q + 1$ by Proposition 1.1. Suppose that there is a tangent line L which is disjoint from B. Count the points of B contained in the $q + 1$ planes through L. Since $L \cap B = \emptyset$, we get $|B| \geq (q + 1)q = q^2 + q$ using Lemma 4.6. $\square$

The following proposition proves Theorem 1.7 when $q \geq 4$.

Proposition 4.8. *If $q \geq 4$, then $|B| = q^2 + q + 1$ and B is a plane of* PG$(3, q)$.

Proof. By Proposition 1.1, it is enough to show that every tangent line meets B. Suppose that there exists a tangent line L which is disjoint from B. Count the points of B contained in the $q+1$ planes through L. There is one tangent plane and q secant planes containing L. Using the assumption that $q \geq 4$, Lemma 4.6 implies that

$$|B| \geq q + q(q + 1) = q^2 + 2q > q^2 + q + 1,$$

which is a contradiction to the fact that $|B| \leq q^2 + q + 1$. Hence every tangent line meets B. $\square$

The following proposition proves Theorem 1.7 for $q = 2$.

Proposition 4.9. *If $q = 2$, then $|B| = 6$ and $B = L \cup L^\tau$ for some line L secant to $Q^-(3, q)$.*

Proof. By Corollary 4.7, we have $|B| \geq 6$. Let L be a secant line of PG$(3, 2)$ and $L \cap Q^-(3, 2) = \{u, v\}$. Then L^τ is an external line which is common to the two tangent planes u^τ and v^τ. If w is the third point of L, then L^τ is the unique external line contained in the secant plane w^τ. If M is a secant line not containing u and v, then M contains two points of $\mathcal{C}_{w^\tau} = Q^-(3, 2) \setminus \{u, v\}$ and so is a line of w^τ. In the plane w^τ, the lines M and L^τ meet. If M is an external line, then M meets the plane w^τ in at least one point of $\{w\} \cup L^\tau$.

It follows that $L \cup L^\tau$ is an $(\mathcal{E} \cup \mathcal{S})$-blocking set in PG$(3,2)$ of size 6. Thus $|B| = 6$ by the minimality of $|B|$.

Conversely, let B be an $(\mathcal{E} \cup \mathcal{S})$-blocking set in PG$(3,2)$ of size 6. Since $B \setminus Q^-(3,2)$ is an $\mathcal{E}$-blocking set in PG$(3,2)$, we have $|B \setminus Q^-(3,2)| \geq 4$ by Proposition 1.2. There are 10 secant lines to $Q^-(3,2)$, and every point of PG$(3,2) \setminus Q^-(3,2)$ is contained in a unique secant line. If $|B \setminus Q^-(3,2)| = 6$ and $B \cap Q^-(3,2) = \emptyset$, then B blocks precisely 6 secant lines. If $|B \setminus Q^-(3,2)| = 5$ and $|B \cap Q^-(3,2)| = 1$, then B blocks at most $5 + 4 = 9$ secant lines. So we must have $|B \setminus Q^-(3,2)| = 4$, $|B \cap Q^-(3,2)| = 2$, and hence $B \setminus Q^-(3,2) = \pi \setminus Q^-(3,2)$ for some secant plane π of PG$(3,2)$ by Proposition 1.2. Let $B \cap Q^-(3,2) = \{x, y\}$. Since $|B| = 6$ and B blocks all secant lines, it can be seen that $Q^-(3,2) \setminus \mathcal{C}_\pi = \{x, y\}$. The secant line $L = xy$ meets the plane π in the nucleus of $\mathcal{C}_\pi$ and L^τ is precisely the unique external line contained in π. It follows that $B = L \cup L^\tau$. □

In the rest of this section, we prove Theorem 1.7 for $q = 3$.

Lemma 4.10. *If $q = 3$, then $|B| = 13$.*

Proof. We have $|B| \leq 13$ and by Corollary 4.7, $|B| \geq 12$. Suppose that $|B| = 12$. Then Proposition 1.1 implies that there exists a tangent line L of PG$(3,3)$ which is disjoint from B. By Lemma 4.6, each of the four planes through L contains at least three points of B. Since $L \cap B = \emptyset$ and $|B| = 12$, it follows that each plane through L contains exactly three points of B. Clearly, the points of B contained in the tangent plane through L are outside $Q^-(3,3)$. Proposition 4.2 (ii) implies that the points of B contained in a secant plane through L are also outside $Q^-(3,3)$. Thus B is disjoint from $Q^-(3,3)$.

There are three secant lines through each point of B so that B blocks at most 36 secant lines. But there are 45 lines which are secant to $Q^-(3,3)$. It follows that B does not block all the secant lines, a contradiction. Hence $|B| = 13$. □

Lemma 4.11. *Suppose that $q = 3$. If $|B \cap Q^-(3,3)| = 1$, then B is a tangent plane.*

Proof. Since $|B| = 13$ by Lemma 4.10 and $|B \cap Q^-(3,3)| = 1$, we have $|B \setminus Q^-(3,3)| = 12$. There are three secant lines through a point of $B \setminus Q^-(3,3)$, giving that the points of $B \setminus Q^-(3,3)$ block at most 36 secant lines. There are nine secant lines through the point of $B \cap Q^-(3,3)$. Since B blocks each of the 45 secant lines, it follows that each secant line contains exactly one point of B. Thus, if $B \cap Q^-(3,3) = \{x\}$, then none of the secant lines through x contains a point of $B \setminus Q^-(3,3)$. This is equivalent to saying that the 12 points of $B \setminus Q^-(3,3)$ are contained in the tangent lines through x. It follows that B coincides with the tangent plane x^τ. □

Lemma 4.12. *Suppose that $q = 3$. If $B \cap Q^-(3,3)$ contains exactly two points, say x_1, x_2, then every tangent line that is disjoint from B meets $x_1x_2 \setminus \{x_1, x_2\}$.*

Proof. Let K be a tangent line which is disjoint from B. Suppose that K does not meet $x_1x_2 \setminus \{x_1, x_2\}$. Then the planes $\pi_1 = \langle K, x_1\rangle$ and $\pi_2 = \langle K, x_2\rangle$ are distinct secant planes through K. Now, for every $i \in \{1, 2\}$, B_{π_i} is an $(\mathcal{E} \cup \mathcal{S})_{\pi_i}$-blocking set of π_i containing the point $x_i \in \mathcal{C}_{\pi_i}$. By Proposition 4.2 (ii), we then know that $|B_{\pi_i}| \geq 4$. Each of the two remaining planes π_3, π_4 through K distinct from π_1, π_2 contains at least three points of B by Lemma 4.6. As $B \cap K = \emptyset$, we have $|B| = \sum_{i=1}^{4} |B_{\pi_i}| \geq 4 + 4 + 3 + 3 = 14$, in contradiction with $|B| = 13$. □

Proposition 4.13. *If $q = 3$, then B is a plane.*

Proof. We have $|B| = 13$ by Lemma 4.10. There are three secant lines through a point of PG$(3,3) \setminus Q^-(3,3)$. If B is disjoint from $Q^-(3,3)$, then $|B| = 13$ implies that B blocks at most 39 secant lines. Since there are 45 lines which are secant to $Q^-(3,3)$, it would follow that B does not block all the secant lines. So, $|B \setminus Q^-(3,3)| \leq 12$.

We show that every tangent line meets B. Then $|B| = 13$ and Proposition 1.1 would imply that B is a plane.

On the contrary, suppose that there exists a tangent line L of PG$(3,3)$ which is disjoint from B. Let $\pi_0, \pi_1, \pi_2, \pi_3$ be the four planes of PG$(3,3)$ through L, where π_0 is the tangent plane and the other three are secant planes. By Lemma 4.6, $|B_{\pi_i}| \geq 3$ for each $i \in \{0, 1, 2, 3\}$. Since $L \cap B = \emptyset$ and $|B| = 13$, it follows that exactly one of planes π_i contains 4 points of B and each of the remaining three planes contains 3 points of B. By Proposition 4.2 (ii), if π_i contains exactly three points of B for some $i \in \{1, 2, 3\}$, then B_{π_i} is disjoint from $Q^-(3,3)$. Since $|B \setminus Q^-(3,3)| \leq 12$ and the points of B_{π_0} are outside $Q^-(3,3)$, it follows that we must have $|B_{\pi_0}| = 3$.

Without loss of generality, we may assume that $|B_{\pi_1}| = 4$, $|B_{\pi_2}| = 3$, and $|B_{\pi_3}| = 3$. Since $B_{\pi_1} \setminus \mathcal{C}_{\pi_1}$ is an $\mathcal{E}_{\pi_1}$-blocking set in π_1, we have $|B_{\pi_1} \setminus \mathcal{C}_{\pi_1}| \geq 2$ by Proposition 4.3. Since $|B \setminus Q^-(3,3)| \leq 12$, $B \cap Q^-(3,3) = B \cap \mathcal{C}_{\pi_1} = B_{\pi_1} \cap \mathcal{C}_{\pi_1}$ is nonempty. So $|B_{\pi_1} \setminus \mathcal{C}_{\pi_1}| \leq 3$. Thus $|B_{\pi_1} \setminus \mathcal{C}_{\pi_1}| \in \{2, 3\}$.

If $|B_{\pi_1} \setminus \mathcal{C}_{\pi_1}| = 3$, then $|B \setminus Q^-(3,3)| = 12$ and $|B \cap Q^-(3,3)| = 1$. In this case, Lemma 4.11 implies that B is a tangent plane which is not possible as $L \cap B = \emptyset$. So, $|B_{\pi_1} \setminus \mathcal{C}_{\pi_1}| = 2$. Then $|B \setminus Q^-(3,3)| = 11$ and $|B \cap Q^-(3,3)| = 2$. Put $B \cap Q^-(3,3) = \{x_1, x_2\}$.

Since L is disjoint from B, Lemma 4.12 implies that L meets $x_1x_2 \setminus \{x_1, x_2\}$ in a singleton. Denote by α the tangency point of L in $Q^-(3,3)$. Note that B_{π_0} is an $\mathcal{E}_{\pi_0}$-blocking set of size 3 in π_0. By Proposition 4.1, we then know that $B_{\pi_0} = U \setminus \{\alpha\}$ for some line U of π_0 through α distinct from L. If we denote by K a line of π_0 through α distinct from L and U, then K is another tangent line disjoint from B. By Lemma 4.12 again, we then know that K meets $x_1x_2 \setminus \{x_1, x_2\}$. But that is impossible: as $\pi_0 \cap x_1x_2 = L \cap x_1x_2$, L is the unique line through α in π_0 that meets x_1x_2. □

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Department of Mathematics: Algebra and Geometry, Ghent University, Krijgslaan 281 (S25), B-9000 Gent, Belgium
E-mail address: Bart.DeBruyn@Ugent.be

School of Mathematical Sciences, National Institute of Science Education and Research (NISER), Bhubaneswar, P.O.- Jatni, District- Khurda, Odisha-752050, India

Homi Bhabha National Institute (HBNI), Training School Complex, Anushakti Nagar, Mumbai-400094, India
E-mail address: puspendu.pradhan@niser.ac.in

School of Mathematical Sciences, National Institute of Science Education and Research (NISER), Bhubaneswar, P.O.- Jatni, District- Khurda, Odisha-752050, India

Homi Bhabha National Institute (HBNI), Training School Complex, Anushakti Nagar, Mumbai-400094, India
E-mail address: bksahoo@niser.ac.in