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DEGREE SEQUENCE OF THE GENERALIZED SIERPIŃSKI GRAPH

MAHSA KHATIBI, ALI BEHTOEI, AND FATEMEH ATTARZADEH

Abstract. Sierpiński graphs are studied in fractal theory and have applications in diverse areas including dynamic systems, chemistry, psychology, probability, and computer science. Polymer networks and WK-recursive networks can be modeled by generalized Sierpiński graphs. The degree sequence of (ordinary) Sierpiński graphs and Hanoi graphs (and some of their topological indices) are determined in the literature. The number of leaves (vertices of degree one) of the generalized Sierpiński graph $S(T,t)$ of any tree T was determined in 2017 and in terms of t, $|V(T)|$, and the number of leaves of the base graph T. In this paper, we generalize these results. More precisely, for every simple graph G of order n, we completely determine the degree sequence of the generalized Sierpiński graph $S(G,t)$ of G in terms of n, t and the degree sequence of G. By using it, we determine the exact value of the general first Zagreb index of $S(G,t)$ in terms of the same parameters of G.

1. Introduction

All graphs considered in this paper are assumed to be simple and finite. Throughout the paper, $G=(V,E)$ will denote a graph of order $n=|V|$ with the vertex set V and the edge set E. The degree of a vertex v of G is denoted by $\deg_G(v)$ which is the size of the set of its neighborhood $N_G(v)$. Decomposition into special substructures inheriting significant properties is an important method for the investigation of some mathematical structures, especially when the considered structures have self-similarity properties. In these cases we typically only need to study the substructures and the way that they are related together. For example polymer networks can be modeled by generalized Sierpiński graphs, see [9]. In [9] the general Randić index of polymeric networks modeled by generalized Sierpiński graphs is studied and in [31] the Wiener index of Sierpiński-like graphs is investigated (they have applications in chemistry). Sierpiński and Sierpiński-type

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graphs are studied in fractal theory [39] and appear naturally in diverse areas of mathematics and in several scientific fields. This family of graphs was studied for the first time in [23] and [33], independently, and constitutes an extensively studied class of graphs of fractal nature with applications in computer science (as a model for interconnection networks which are known as WK-recursive networks), topology, and the mathematics of the Tower of Hanoi, see [6] and [37] for more details. One of the most important families of such graphs is formed by the Sierpiński gasket graphs introduced by Scorer, Grundy, and Smith in [38] which play an important role in psychology, dynamic systems, and probability, see [14], [22], and [28]. Sierpiński, Sierpiński-type, and generalized Sierpiński graphs have many interesting properties and were studied extensively in the literature.

Definition 1.1 ([13]). *Let $G = (V, E)$ be a graph of order $n \geq 2$, t be a positive integer and denote the set of words of length t on the alphabet V by V^t. The letters of a word $\mathbf{u} \in V^t$ (of length t) are denoted by $u_1u_2 \ldots u_t$. The generalized Sierpiński graph of G of dimension t, denoted by $S(G, t)$, is the graph with vertex set V^t and $\{\mathbf{u}, \mathbf{v}\}$ is an edge in it if and only if there exists $i \in \{1, \ldots, t\}$ such that:*

(i) $u_j = v_j$ *if* $j < i$,
(ii) $u_i \neq v_i$ *and* $\{u_i, v_i\} \in E(G)$,
(iii) $u_j = v_i$ *and* $v_j = u_i$ *if* $j > i$.

For example $S(C_4, 3)$ is depicted in Figure 1 in which C_4 is assumed to be a cycle with the vertex set $\{1, 2, 3, 4\}$ and the edge set

$$\{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}\}.$$

Note that $S(G, 1)$ is (isomorphic to) the base graph G and $S(G, 2)$ can be constructed by copying n times $S(G, 1)$ and adding an edge between the ith vertex of the jth copy and the jth vertex of the ith copy of $S(G, 1)$ whenever $\{i, j\}$ is an edge in G. In fact $S(G, t)$ is a fractal-like graph that uses G as a building block. When G is the complete graph K_n, the (ordinary) Sierpiński graph $S_n^t = S(K_n, t)$ is obtained. Klavžar et al. introduced the graph $S(K_n, t)$ for the first time and they show that $S(K_3, t)$ is isomorphic to the graph of the Tower of Hanoi H_3^t, see [23] and [24]. It is well-known that S_n^t contains n (extreme) vertices of degree $n - 1$ and all the other vertices are of degree n, see [16]. Hence, S_n^t is almost regular. The degree sequence of the Hanoi graph H_n^t (the state graph of the Tower of Hanoi game with n pegs and t discs) is more complex and is completely determined in [18]. In this paper, we want to determine the degree sequence of the generalized Sierpiński graph $S(G, t)$ of an arbitrary graph G. Sierpiński graphs are studied from numerous points of view. In [23] and [42] shortest paths in Sierpiński graphs are studied. In [15] an algorithm is proposed which makes use of three automata to determine all shortest paths in Sierpiński graphs. Metric properties of Sierpiński graphs are investigated in

[20] and [32]. For connections between the Sierpiński graphs and Stern's diatomic sequence see [17]. Identifying codes, locating-dominating codes, and total-dominating codes in Sierpiński graphs are studied in [12]. Sierpiński graphs contain (essentially) unique 1-perfect codes [24]. Covering codes in Sierpiński graphs is studied in [3] and equitable $L(2,1)$-labelings of them is considered in [10]. In [26] the canonical isometric representation of Sierpiński graphs is explicitly described. The crossing number of Sierpiński graphs is studied in [25], giving the first infinite families of graphs of fractal nature for which the crossing number is determined (up to the crossing number of complete graphs). Colorings and the chromatic number of these graphs are studied in [19] and their hub number is determined in [29]. Also, many

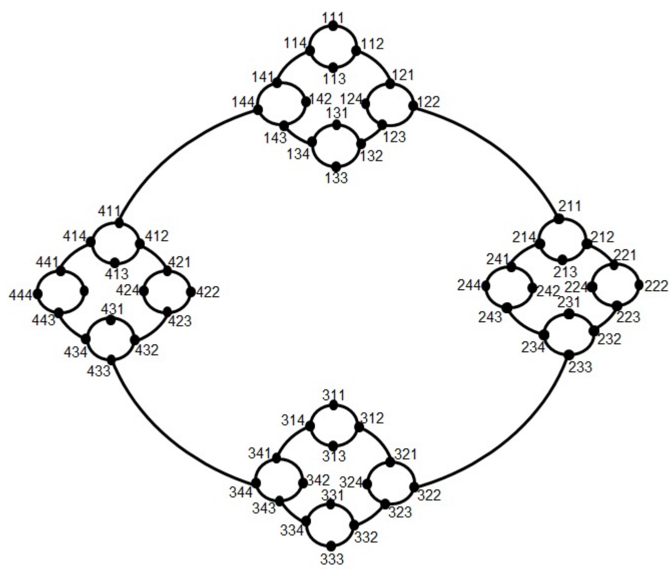

FIGURE 1. The generalized Sierpiński graph $S(C_4, 3)$.

papers studied the structure of the generalized Sierpiński graphs. In [13] some interesting results about the generalized Sierpiński graphs (concerning their automorphism groups, perfect codes, and distinguishing numbers) are obtained. The total chromatic number for some families of these graphs is determined by Geetha and Somasundaram in [11]. More precisely, the authors prove the tight bound of the Behzad and Vizing conjecture on total coloring for the generalized Sierpiński graphs of cycle graphs and hypercube graphs. They provide a total coloring for the WK-recursive topology, which also gives the tight bound. In [8] the distance between vertices of $S(G,t)$ is expressed in terms of the distance between vertices of the base graph G. In addition, the authors give an explicit formula for the diameter and the radius of $S(G,t)$ when the base graph G is a tree. In [35] their independence number, chromatic number, vertex cover number, clique number, and domination number are investigated in terms of the similar parameters of the base graph G. The strong metric dimension of these graphs is studied in [7]. Metric properties of generalized Sierpiński graphs over stars are considered in [2]. The Roman domination number of $S(G,t)$ is investigated in [34]. An explicit formula for the number of connected components of $S(G,t)$ is given in [27] and it is proved that the (edge-)connectivity of $S(G,t)$ equals the (edge-)connectivity of G. Also, it is demonstrated that $S(G,t)$ contains a perfect matching if and only if G contains a perfect matching. Moreover, the Hamiltonicity of these graphs is also discussed there. It is shown in [35] that for any tree T of order $n \geq 2$ and any positive integer t, $S(T,t)$ is a tree and the number of leaves of $S(T,t)$ is equal to

$$\frac{\varepsilon(T)\ (n^t - 2n^{t-1} + 1)}{n-1}$$

in which $\varepsilon(T)$ is the number of leaves of T. We generalize this result in Theorem 2.3. For more results in these subject and related subjects, see [13], [16], [18], [21], [35], and [36].

2. Main results

First, we determine the neighborhood of a vertex in $S(G,t)$ and hence, its degree in $S(G,t)$. Then, for each $0 \leq k \leq \Delta(S(G,t))$, we determine the number of vertices of degree k in $S(G,t)$ which leads to the degree sequence of the generalized Sierpiński graph $S(G,t)$. Finally, we show that the general first Zagreb index of $S(G,t)$ can be expressed as a linear combination of the general first Zagreb indices of the base graph G.

Lemma 2.1. *Let $G = (V,E)$ be a simple graph and $t \geq 1$ be an integer. Then, for each vertex $\mathbf{x} = x_1x_2\ldots x_t$ in the generalized Sierpiński graph $S(G,t)$ we have*

$$\deg_{S(G,t)}(x_1x_2\ldots x_t) = \begin{cases} 1 + \deg_G(x_t) & x_1x_2\ldots x_t \neq x_tx_t\ldots x_t \text{ and} \\ & x_i \in N_G(x_t) \text{ for} \\ & i = \max\{j : 1 \leq j \leq t-1, x_j \neq x_t\}, \\ \deg_G(x_t) & \text{otherwise.} \end{cases}$$

Proof. Obviously the result follows for $t = 1$ because $x_1\ldots x_t = x_t\ldots x_t = x_t$ and $\deg_{S(G,1)}(x_1) = \deg_G(x_1)$. Hence, we assume that $t \geq 2$. By the adjacency rule in $S(G,t)$, it is straightforward to see that each vertex $\mathbf{x}' = x'_1x'_2\ldots x'_t$ in $S(G,t)$ with $N_G(x'_t) = \emptyset$ is an isolated vertex in $S(G,t)$. Thus, if $\deg_G(x_t) = 0$, then $\mathbf{x} = x_1x_2\ldots x_t$ is an isolated vertex in $S(G,t)$ and the result directly follows. Assume that $\deg_G(x_t) = d \geq 1$ and $N_G(x_t) = \{y_1, y_2, \ldots, y_d\}$. For each $i \in \{1, 2, \ldots, d\}$, it is easy to check that two vertices $\mathbf{x} = x_1x_2\ldots x_{t-1}x_t$ and $\mathbf{y_i} = x_1x_2\ldots x_{t-1}y_i$ are adjacent in $S(G,t)$. Hence, $\deg_{S(G,t)}(\mathbf{x}) \geq d = \deg_G(x_t)$. Now let $\mathbf{z} = z_1z_2\ldots z_t$ be a neighbor of $\mathbf{x}$ in $S(G,t)$. Thus, by the adjacency rule in $S(G,t)$, there exists $i \in$

$\{1, 2, \ldots, t\}$ such that $x_j = z_j$ for each $j < i$, $\{x_i, z_i\} \in E(G)$ and for each $\ell > i$, we see that $x_\ell = z_i$ and $z_\ell = x_i$. If $i = t$, then $\mathbf{z} = x_1 x_2 \ldots x_{t-1} z_t$ and $\{x_t, z_t\} \in E(G)$. This implies that $z_t \in N_G(x_t) = \{y_1, y_2, \ldots, y_d\}$ and hence $\mathbf{z} \in \{\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_d}\}$. If $i < t$, then we must have

$$\mathbf{x} = x_1 x_2 \ldots x_{i-1} x_i z_i z_i \ldots z_i \ , \ \mathbf{z} = x_1 x_2 \ldots x_{i-1} z_i x_i x_i \ldots x_i.$$

Thus, $x_t = z_i$ and $x_i \in N_G(z_i) = N_G(x_t) = \{y_1, y_2, \ldots, y_d\}$. This implies that

$$\mathbf{x} = x_1 x_2 \ldots x_{i-1} x_i x_t x_t \ldots x_t \ , \ \mathbf{z} = x_1 x_2 \ldots x_{i-1} x_t x_i x_i \ldots x_i.$$

Note that $i = \max\{j : 1 \le j \le t-1, x_j \ne x_t\}$. Since the vertex $\mathbf{x}$ is given and its structure is specified, the index i is unique. Therefore, we have

$$N_{S(G,t)}(x_1 x_2 \ldots x_t) = \begin{cases} \{\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_d}, x_1 x_2 \ldots x_{i-1} x_t x_i x_i \ldots x_i\} & x_i \in N_G(x_t), \text{ and} \\ & x_1 x_2 \ldots x_t \ne x_t x_t \ldots x_t, \\ \{\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_d}\} & \text{otherwise}, \end{cases}$$

in which $i = \max\{j : \ 1 \le j \le t-1, \ x_j \ne x_t\}$. This completes the proof. □

Let G be a graph of order $n \ge 2$ with $E(G) \ne \emptyset$, and let x, y, z be vertices of G such that $\deg_G(x) = \delta(G)$, $\deg_G(y) = \Delta(G)$, and $z \in N_G(y)$. Lemma 2.1 implies that $\deg_{S(G,t)}(xx\ldots x) = \deg_G(x) = \delta(G)$ and for $t \ge 2$ we have $\deg_{S(G,t)}(zz\ldots zy) = 1+\deg_G(y) = 1+\Delta(G)$. Hence, the following corollary directly follows from Lemma 2.1, (also, see [35]).

Corollary 2.2. *Let G be a simple graph of order $n \ge 2$ and $t \ge 1$ be an integer. Then,*

(i) $\delta(S(G, t)) = \delta(G)$.
(ii) $\Delta(S(\overline{K_n}, t)) = \Delta(\overline{K_n}) = 0$ *and* $\Delta(S(G, 1)) = \Delta(G)$. *Also, we have* $\Delta(S(G, t)) = 1 + \Delta(G)$ *when* $E(G) \ne \emptyset$ *and* $t \ge 2$.

Theorem 2.3. *Let $G = (V, E)$ be a simple graph of order $n \ge 2$, $t \ge 1$ be an integer and for each k, $\delta(G) \le k \le \Delta(G)$, let $V_k = \{v \in V : \ \deg_G(v) = k\}$. Then, for each k, $\delta(S(G, t)) \le k \le \Delta(S(G, t))$, the number of vertices of degree k in the generalized Sierpiński graph $S(G, t)$ is*

$$|V_k| n^{t-1} - \frac{n^{t-1} - 1}{n - 1} \big(k|V_k| - (k-1)|V_{k-1}|\big).$$

Proof. It is easy to check that the result directly follows for the case $t = 1$. Hereafter we assume that $t \ge 2$. For each integer s, $\delta(G) \le s \le \Delta(G)$, and for each $x \in V_s$ define

$$\Omega_{s,x} = \{v_1 v_2 \ldots v_t \in V(S(G, t)) : v_t = x\}$$

and let $\Omega_s = \cup_{x \in V_s} \Omega_{s,x}$. Note that $|\Omega_{s,x}| = n^{t-1}$ and $|\Omega_s| = |V_s| n^{t-1}$. By Lemma 2.1, the degree of each vertex in Ω_s is s or $s + 1$. Now we want to determine the number of vertices in Ω_s which has degree s. Let $x \in V_s$ and $v_1v_2\dots v_{t-1}x \in \Omega_s$. If $\deg_{S(G,t)}(v_1v_2\dots v_{t-1}x) = s$, then by Lemma 2.1 we have $v_1v_2\dots v_{t-1}x = xx\dots x$ or there exists $1 \le i \le t-1$ such that $i = \max\{j:\ 1 \le j \le t-1,\ v_j \neq x\}$ and $v_i \notin N_G(x)$. For each i, $1 \le i \le t-1$, define

$$\Gamma_{s,x,i} = \{x_1x_2\dots x_t \in V(S(G,t)) : x_i \notin N_G(x) \cup \{x\}, x_j = x\ \forall j > i\}.$$

Note that $|\Gamma_{s,x,i}| = n^{i-1}\ (n-s-1)$. Thus, the degree of the vertex $v_1v_2\dots v_{t-1}x \in \Omega_s$ is equal to s if and only if

$$v_1v_2\dots v_{t-1}x \in \left(\cup_{i=1}^{t-1}\Gamma_{s,x,i}\right) \cup \{xx\dots x\}.$$

Since $x \in V_s$ and

$$\left|\left(\cup_{i=1}^{t-1}\Gamma_{s,x,i}\right) \cup \{xx\dots x\}\right| = 1 + (n-s-1)(1+n+n^2+\cdots+n^{t-2}),$$

the number of vertices in Ω_s of degree s is

$$|V_s|\big(1+(n-s-1)(1+n+n^2+\cdots+n^{t-2})\big) = |V_s|\ \big(n^{t-1}-s(1+n+\cdots+n^{t-2})\big).$$

Hence, the number of vertices in Ω_s of degree $s+1$ is given by

$$\begin{aligned}&|\Omega_s| - |V_s|\big(n^{t-1} - s(1+n+\cdots+n^{t-2})\big) = \\ &|V_s|n^{t-1} - |V_s|\big(n^{t-1} - s(1+n+\cdots+n^{t-2})\big) = s|V_s|(1+n+\cdots+n^{t-2}).\end{aligned}$$

Therefore, the number of vertices of degree k in the generalized Sierpiński graph $S(G,t)$ is equal to

$$|V_k|\big(n^{t-1} - k(1+n+\cdots+n^{t-2})\big) + (k-1)\ |V_{k-1}|(1+n+\cdots+n^{t-2})$$

Since $V_{1+\Delta(G)} = \emptyset = V_{-1}$ and $1+n+\cdots+n^{t-2} = (n^{t-1}-1)/(n-1)$, the proof is complete. $\square$

A large number of properties like chemical activity, biological activity, physicochemical properties, and thermodynamic properties are determined by the chemical applications of graph theory. These properties can be expressed by certain graph invariants (real numbers related to a structurally invariant graph) referred to as topological indices. Some of topological indices of Sierpiński networks and generalized Sierpiński graphs are determined, see [9], [21], and [36]. In [40] and [41] Li et al. considered the general first Zagreb index of a graph G as

$$Z_\alpha(G) = \sum_{\{u,v\}\in E(G)} \big((\deg_G(u))^{\alpha-1} + (\deg_G(v))^{\alpha-1}\big) = \sum_{u\in V(G)} (\deg_G(u))^\alpha$$

in which α is a real number. Specifically, we see that $Z_0(G) = n$, $Z_1(G) = \sum_{k=1}^{\Delta(G)} |V_k|\ k = 2|E(G)|$, $Z_2(G) = M_1(G)$ which is known as the first Zagreb index and $Z_3(G) = F(G)$ which is known as the forgotten topological index, see [1], [4], and [30] for more details.

Corollary 2.4. *For each simple graph G of order $n \geq 2$ and each integer $\alpha \geq 0$, the general first Zagreb index of the generalized Sierpiński graph $S(G,t)$, $t \geq 1$, is given by*

$$Z_\alpha(S(G,t)) = \frac{n^t - n^{t-1} + (n^{t-1}-1)\alpha}{n-1}\ Z_\alpha(G) + \frac{n^{t-1}-1}{n-1}\sum_{j=1}^{\alpha-1}\binom{\alpha}{j-1}\ Z_j(G).$$

Proof. By the definition of the general first Zagreb index, binomial expansion formula, and Theorem 2.3, $Z_\alpha(S(G,t))$ is equal to

$$\begin{aligned}
&\sum_{k=1}^{\Delta(G)+1}\left(|V_k|n^{t-1} - \frac{n^{t-1}-1}{n-1}\big(k\ |V_k| - (k-1)\ |V_{k-1}|\big)\right)k^\alpha \\
&= \sum_{k=1}^{\Delta(G)+1}\left(|V_k|\ n^{t-1}k^\alpha - \frac{n^{t-1}-1}{n-1}\big(k^{\alpha+1}\ |V_k| - k^\alpha(k-1)|V_{k-1}|\big)\right) \\
&= n^{t-1}Z_\alpha(G) - \frac{n^{t-1}-1}{n-1}Z_{\alpha+1}(G) + \frac{n^{t-1}-1}{n-1}\sum_{k=1}^{\Delta(G)+1}\left(k^\alpha(k-1)|V_{k-1}|\right) \\
&= n^{t-1}Z_\alpha(G) - \frac{n^{t-1}-1}{n-1}Z_{\alpha+1}(G) + \frac{n^{t-1}-1}{n-1}\sum_{\ell=1}^{\Delta(G)}\left((\ell+1)^\alpha\ell|V_\ell|\right) \\
&= n^{t-1}Z_\alpha(G) - \frac{n^{t-1}-1}{n-1}Z_{\alpha+1}(G) + \frac{n^{t-1}-1}{n-1}\sum_{\ell=1}^{\Delta(G)}\left(\left(\sum_{i=0}^{\alpha}\binom{\alpha}{i}\ell^i\right)\ell|V_\ell|\right) \\
&= n^{t-1}Z_\alpha(G) - \frac{n^{t-1}-1}{n-1}Z_{\alpha+1}(G) + \frac{n^{t-1}-1}{n-1}\sum_{i=0}^{\alpha}\binom{\alpha}{i}Z_{i+1}(G) \\
&= \frac{n^t - n^{t-1} + (n^{t-1}-1)\alpha}{n-1}Z_\alpha(G) + \frac{n^{t-1}-1}{n-1}\sum_{j=1}^{\alpha-1}\binom{\alpha}{j-1}Z_j(G).
\end{aligned}$$

□

Note that by using the generalized form of the binomial theorem, Corollary 2.4 can be motivated in such a way that α be a real number but then the finite sum should be replaced by an infinite series. By using Corollary 2.4, it can be easily seen that

$$|V(S(G,t))| = Z_0(S(G,t)) = n^t,$$

and

$$|E(S(G,t))| = \frac{1}{2}Z_1(S(G,t)) = \frac{n^t-1}{n-1}|E(G)|$$

which coincides with the previously obtained results, see [35]. Also, we have

$$M_1(S(G,t)) = Z_2(S(G,t)) = \frac{n^t + n^{t-1} - 2}{n-1} Z_2(G) + \frac{n^{t-1} - 1}{n-1} 2|E(G)|$$

and

$$\begin{aligned} F(S(G,t)) &= Z_3(S(G,t)) \\ &= \frac{n^t + 2n^{t-1} - 3}{n-1} F(G) + \frac{n^{t-1} - 1}{n-1} (2|E(G)| + 3M_1(G)). \end{aligned}$$

In [5], by using the Stirling numbers of the first kind, it is shown that for each integer $\alpha \geq \Delta(G)$, the general first Zagreb index $Z_\alpha(G)$ can be expressed as a linear combination of $Z_0(G), Z_1(G), \ldots, Z_{\Delta(G)-1}(G)$. This result using Corollary 2.4 implies that $Z_\alpha(S(G,t))$ can also be expressed as a linear combination of $Z_0(G), Z_1(G), \ldots, Z_{\Delta(G)-1}(G)$ for each $\alpha \geq \Delta(G)$.

The authors declare that they have no competing interests.

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Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin, Iran, PO Box: 34148 - 96818.
E-mail address: m.khatibi@edu.ikiu.ac.ir

Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin, Iran, PO Box: 34148 - 96818.
(Corresponding author)
E-mail address: a.behtoei@sci.ikiu.ac.ir

Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin, Iran, PO Box: 34148 - 96818.
E-mail address: prs.attarzadeh@gmail.com