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k-FORCING NUMBER FOR THE CARTESIAN PRODUCT OF SOME GRAPHS

ZEINAB MONTAZERI AND NASRIN SOLTANKHAH

Abstract. k-Forcing is an iterative graph coloring process based on a color change rule that describes how to color the vertices. k-Forcing is a generalization of zero forcing that is useful in multiple scientific branches, such as quantum control. In this paper, we investigate the k-forcing number of the Cartesian product of some graphs. The main contribution of this paper is to determine the k-forcing number of the Cartesian product of two complete bipartite graphs using a new representation of this graph.

1. Introduction

Let $G = (V_G, E_G)$ be a graph with vertex set V_G and edge set E_G. The order of G, denoted $|G|$, is the number of vertices of G. Throughout this paper, all graphs are simple, undirected, and have finite nonempty vertex sets. The *Cartesian product* of two graphs G and H, denoted $G\Box H$, is the graph with the vertex set $V_G \times V_H$ and two vertices (g_1, h_1) and (g_2, h_2) are adjacent if and only if either $g_1 = g_2$ and $h_1h_2 \in E_H$, or $h_1 = h_2$ and $g_1g_2 \in E_G$. For other graph-theoretic terminology, we refer to [10]. The k-forcing number of a graph G was introduced in [1] to generalize the zero forcing number of G. The *k-forcing number* is the minimum number of vertices that need to be initially black so that all vertices eventually become black during the discrete dynamical process described by the following color change rule. Let k be a positive integer and G be a graph with each vertex colored either white or black. If a black vertex of G has at most k white neighbors, then each of its white neighbors becomes black. The initial set of black vertices is called a *k-forcing set* if by iterating this process, all of the vertices in G become black. The *k-forcing number*, denoted $F_k(G)$, is the minimum of $|B|$ over all k-forcing sets $B \subseteq V_G$. We will call the discrete dynamical process of applying the color change rule to B and G the k-forcing process.

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When $k = 1$, the k-forcing number is equivalent to the definition of the zero forcing number. Zero forcing was introduced in [7] to bound the maximum nullity of the family of symmetric matrices associated with a graph. This concept was also studied in quantum physics [4], theoretical computer science [11], and other scientific branches [2, 3, 5, 8, 12]. The definition of k-forcing number is more than just a generalization of the zero forcing number. It is related to some other graph parameters such as the k-domination number and the k-power domination number [1, 6].

Checker Pattern. A graph can be formed from an $m \times n$ chessboard if taking the squares as the vertices and two vertices (e.g., v_i and v_j) are adjacent if a chess piece situated on one square (v_i), can be transferred to the other square (v_j) using the chess rules. Rook's graph is an example of this kind of graph. The rook's graph R_{mn} has mn squares as vertices, and two vertices are adjacent if they are on the same row or column. In other words, this graph describes all possible movements of a rook in an $m \times n$ chessboard. We can see that rook's graph is a $K_n \Box K_m$ graph, which is the Cartesian product of two complete graphs. The square (i, j) indicates the vertex located on the ith copy of K_n and the jth copy of K_m at the same time.

We introduced a similar representation for the Cartesian product of two complete bipartite graphs in [9]. Actually, we introduced generalized rook's graph as a generalization of rook's graph, corresponding to the graph $K_{m,n} \Box K_{m',n'}$. Consider the graph $K_{m,n} \Box K_{m',n'}$. Form an $(m' + n') \times (m + n)$ generalized chessboard from four smaller chessboards and denote them by C_1 to C_4. We denote the $m' \times m$ chessboard in the left top corner by C_1, the $m' \times n$ chessboard in the right top corner by C_2, the $n' \times n$ chessboard in the right bottom by C_3, and the $n' \times m$ chessboard in the left bottom by C_4 (see Figure 1). Hence, C_2 and C_4 indicate even chessboards and C_1 and C_3 odd chessboards. The square (i, j) indicates the vertex that is in the ith copy of $K_{m,n}$ and in the jth copy of $K_{m',n'}$ at the same time. The neighbors of the vertex (i, j) where $1 \leq i \leq m'$ and $1 \leq j \leq m$ are the vertices (i, j') and (i', j) with $m+1 \leq j' \leq m+n$ and $m'+1 \leq i' \leq m'+n'$. Furthermore, the neighbors of the vertex (i, j) where $m'+1 \leq i \leq m'+n'$ and $m+1 \leq j \leq m+n$ are the vertices (i, j') and (i', j) with $1 \leq j' \leq m$ and $1 \leq i' \leq m'$. Thus, the neighbors of each vertex in an odd (even) chessboard are in the even (odd) chessboard. Obviously, each smaller chessboard corresponds to an independent set in the graph $K_{m,n} \Box K_{m',n'}$.

In [9], we computed the zero forcing number of $K_{m,n} \Box K_{m',n'}$ and obtained some bounds about this graph using checker pattern. The main contribution of this paper is computing the k-forcing number of this graph.

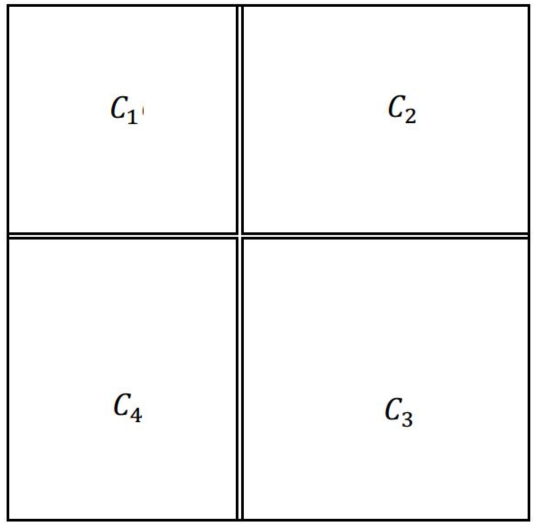

FIGURE 1. The location of chessboards in the generalized chessboard corresponding to the Cartesian product of two complete bipartite graphs.

2. Main results

In this section, we introduce two new concepts related to $F_k(G)$, denoted $mF_k(G)$ and $F_k^*(G)$. Then, we determine $F_k(G)$ and $F_k^*(G)$, where G is $K_n \square K_m$ or $K_{m,n} \square K_{m',n'}$.

For a graph G with k-forcing number $F_k(G)$ (for $k \geq 1$), define $mF_k(G) = |G| - F_k(G)$. Notice that, $mF_k(G)$ is the number of white vertices in a black-white coloring of G, where the set of black vertices is a minimum k-forcing set of G.

We also introduce another graph parameter similar to $F_k(G)$. Consider the definition of $F_k(G)$, we modify its color-change rule: every black vertex with exactly k white neighbors will cause each white neighbor to become black. We denote this parameter by $F_k^*(G)$ and the set of initial black vertices by k^**-forcing set*. Obviously, each k^*-forcing set is a k-forcing set. Thus for each graph G, $F_k(G) \leq F_k^*(G)$.

Now, we use the checker pattern to compute the k-forcing number of some graphs. In the corresponding chessboard of the rook's graph, the color change rule of the k-forcing process is as follows. If a square (i, j) is black and there exist at most k white squares in both the row i and the column j, then (i, j) will force all these white squares at the same time.

Theorem 2.1. *Suppose that m, n, k are integers with $m, n > k \geq 1$. Then,*

$$F_k(K_n \square K_m) = mn - k(m + n) + k^2 + k.$$

Proof. Let B be a k-forcing set for $K_m \Box K_n$ that is demonstrated with black squares in Figure 2. By applying the color change rule to B, all the white squares will be colored black after two steps.

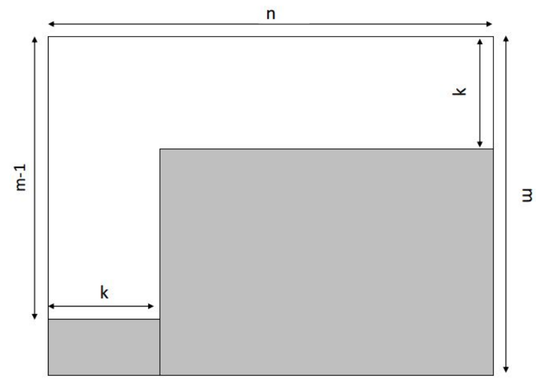

FIGURE 2. The corresponding $m \times n$ chessboard of graph $K_n \Box K_m$. The black squares representing the k-forcing set B.

Then we have

$$F_k(K_n \Box K_m) \leq (m-k)(n-k) + k$$
$$= mn - k(m+n) + k^2 + k.$$

On the other hand, we claim that $mf_k(K_n \Box K_m) \leq k(m+n-k-1)$. For a contradiction, assume that $mf_k(K_n \Box K_m) \geq k(m+n-k-1)+1$. Consider a black-white coloring of the $m \times n$ chessboard corresponding to $K_n \Box K_m$ with a set of black squares representing a minimum k-forcing set. Assume that a black square (i,j) can force its white neighbors. Therefore, in row i and column j, there exist at most k white squares altogether. Then, the other white squares are out of these two lines. This means that there exist at least $k(m+n-k-1)+1-k$ white squares in a smaller chessboard, in which the sum of its dimensions is $m+n-2$. The k-forcing process continues if there exists at least one line with at most k white squares or two lines (one row and one column) with at most k white squares together in this chessboard. Hence, there exist at least $k(m+n-k-1)+1-2k$ white squares in a smaller chessboard, in which the sum of its dimensions is at most $m+n-3$. By continuing this procedure for t steps (where $1 \leq t \leq m+n-2k-3$), we reach a chessboard with the sum of its dimensions is at most $m+n-1-t$ and has at least $k(m+n-k-1)+1-tk$ white squares. Thus, after $m+n-2k-3$ steps, we reach a chessboard in which the sum of its dimensions is at most $2(k+1)$ and has at least $(k+1)^2$ white squares. An $i \times (2k+2-i)$ chessboard, for $1 \leq i \leq k$, does not have $(k+1)^2$ squares so we reach a $(k+1) \times (k+1)$ white chessboard. It means the initial chessboard has $k+1$ rows and $k+1$ columns that have white squares in their crossing. Now, the color change rule cannot change the color of these squares, which is a contradiction. Then,

$$\begin{aligned} mf_k(K_n \square K_m) &\leq k(m+n-k-1) \\ &= k(m+n) - k^2 - k. \end{aligned}$$

Therefore, $F_k(K_n \square K_m) \geq mn - k(m+n) + k^2 + k$, and the equality holds. □

Corollary 2.2. *Let m, n, k be integers that $m, n > k \geq 1$. Then,*

$$F_k^*(K_n \square K_m) = F_k(K_n \square K_m) = mn - k(m+n) + k^2 + k.$$

Proof. In each step of the k-forcing process on the k-forcing set B that is shown in Figure 2, we have exactly k white squares that become black. Thus, $F_k^*(K_n \square K_m) \leq mn - k(m+n) + k^2 + k$, and since $F_k(G) \leq F_k^*(G)$ for every graph G, the equality holds. □

Here, we use the checker pattern of $G = K_{m,n} \square K_{m',n'}$ and determine the k-forcing number of this graph. Consider the corresponding $(m'+n') \times (m+n)$ generalized chessboard and every C_i for $i \in \{1, 2, 3, 4\}$. Let all squares be initially colored black or white. Then, in a k-forcing process of G, a black square (i, j) of C_1 (resp. C_3) can force its white neighbors if there are at most k white squares in the ith row of C_2 (resp. C_4) and jth column of C_4) (resp. C_2). Moreover, a black square (i, j) of C_2 (resp. C_4) can force its white neighbors if there are at most k white squares in the jth column of C_3 (resp. C_1) and ith row of C_1 (resp. C_3).

Now, we present some lemmas needed to compute the k-forcing number of $K_{m,n} \square K_{m',n'}$.

Lemma 2.3. *Consider the generalized chessboard, which demonstrates the graph $G = K_{m,n} \square K_{m',n'}$. In every minimum k-forcing set of G, there exist four lines in odd chessboards C_1 and C_3 (also in even chessboards C_2 and C_4) that have at most $2k$ white squares together.*

Proof. Consider a black-white coloring of the generalized chessboard corresponding to graph $K_{m,n} \square K_{m',n'}$ with the black squares representing a minimum k-forcing set. Recall that the white squares in the odd chessboards can be forced by the black squares in the even chessboards. Without loss of generality, assume that the first color change in the odd chessboards done by the vertex (i, j) of C_2. So the ith row of C_1 and the jth column of C_3 have at most k white squares together. The number of white squares in odd chessboards will decrease if all the color changes in these chessboards occur only on the black squares of C_2. Since the black squares form a minimum k-forcing set of G, some color changes in the odd chessboards must be done by some black squares of C_4. Assume that the first black square of C_4 that forces some squares is the square (i', j'). So the j'-th row of C_3 and the i'-th column of C_1 have at most k white squares together. Hence, C_1 and C_3 have four lines that have at most $2k$ white squares together. By the same argument, the even chessboards have four lines that have at most $2k$ white squares altogether. $\square$

Lemma 2.4. *Let m, n, m', and n' be integers greater than integer $k \geq 1$. Then,*

$$mf_k(K_{m,n}\square K_{m',n'}) \leq 2k(m+n+m'+n') - 4k^2 - 4k.$$

Proof. Consider a black-white coloring of the generalized chessboard correspondent to $K_{m,n}\square K_{m',n'}$ with the black squares representing a minimum k-forcing set. According to Lemma 2.3, eight lines (four rows and four columns) have at most $4k$ white squares together. So the other white squares are in four smaller chessboards ($(m'-1)\times(m-1)$, $(m'-1)\times(n-1)$, $(n'-1)\times(n-1)$, $(n'-1)\times(m-1)$ chessboards). At most k white squares in one line (a row or a column) or two lines (a row and a column) will be black in each application of the color change rule. If at most one line (a row or a column) of each C_i becomes black in each application of the color change rule, then the number of white squares in the chessboard can be maximized. Therefore, after the first step, the k-forcing process continues for each C_i, $i \in \{1, 2, 3, 4\}$, independently. Hence, using Theorem 2.1 for each C_i, there are at most $2k(m+n+m'+n') - 4k^2 - 8k$ white squares in the chessboard after the first step. Thus,

$$mf_k(K_{m,n}\square K_{m',n'}) \leq 2k(m+n+m'+n') - 4k^2 - 4k.$$

$\square$

Theorem 2.5. *Let m, n, m', and n' be integers greater than the integer $k \geq 1$. Then,*

$$F_k(K_{m,n}\square K_{m',n'}) = (m+n)(m'+n') - 2k(m+n+m'+n') + 4k^2 + 4k.$$

Proof. By Lemma 2.4, we have

$$\begin{aligned}(m+n)(m'+n') - F_k(K_{m,n}\square K_{m',n'}) &= mf_k(K_{m,n}\square K_{m',n'})\\ &\leq 2k(m+n+m'+n') - 4k^2 - 4k.\end{aligned}$$

Thus, it suffices to prove that $F(K_{m,n}\square K_{m',n'}) \leq (m+n)(m'+n') - 2k(m+n+m'+n') + 4k^2 + 4k$.

Without loss of generality, we assume that $m \geq n$, $m' \geq n'$, and $m \geq m'$. We color the generalized chessboard of $K_{m,n}\square K_{m',n'}$ (where $m \geq n \geq m' \geq n'$) by using the pattern, which is given in Figure 3, and the other cases are similar. It is easy to see that the black squares in this coloring demonstrate a k-forcing set for the graph $K_{m,n}\square K_{m',n'}$.

The number of black squares in C_1, C_2, C_3, and C_4 are $k^2 - k(m+m') + mm'$, $k^2 - k(m'+n) + nm'$, $(k+1)^2 + nn' - k(n+n') - 1$, and $(k+1)^2 + mn' - k(m+n') - 1$, respectively. Thus,

$$F_k(K_{m,n}\square K_{m',n'}) \leq (m+n)(m'+n') - 2k(m+n+m'+n') + 4k^2 + 4k,$$

and the equality holds. $\square$

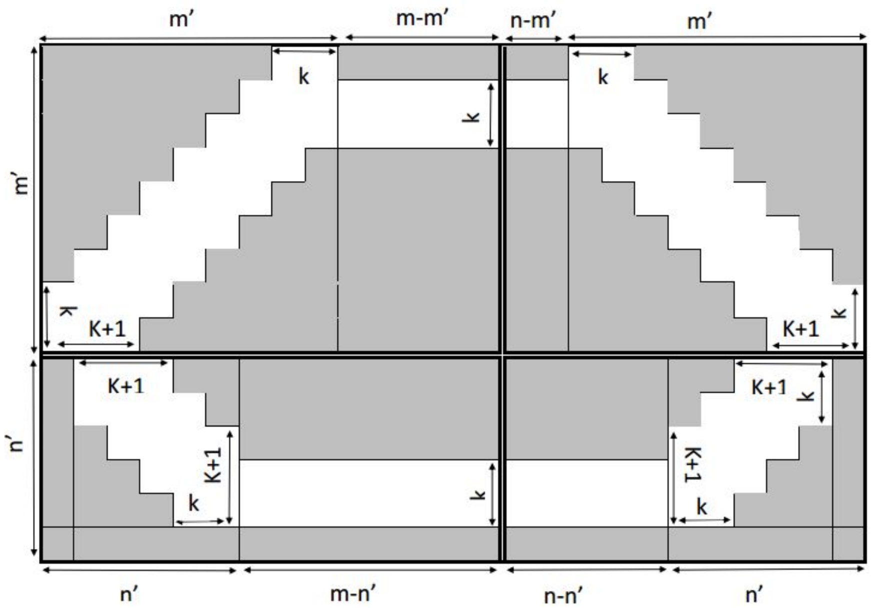

FIGURE 3. The corresponding generalized chessboard of graph $K_{m,n}\Box K_{m',n'}$ (where $m \geq n \geq m' \geq n'$). The black squares representing a k-forcing set.

To clarify how the k-forcing process starts and continues, we write two steps of this process for the coloring pattern in Figure 3. We consider the $(m'+n')\times(m+n)$ generalized chessboard and $x \longrightarrow (x_1, x_2, \ldots, x_k)$ means the vertex x forces the vertices $x_1, x_2, \ldots, x_k$ at the same time.

Step 1:

$$\begin{aligned}
&x_{(m'+n'),1} \longrightarrow (x_{(m'-k+1),1}, \ldots, x_{m',1})\\
&x_{(m'+n'),(m+n)} \longrightarrow (x_{(m'-k+1),(m+n)}, \ldots, x_{m',(m+n)})\\
&x_{1,1} \longrightarrow (x_{1,(m+n-m'+1)}, \ldots, x_{1,(m+n-m'+k)}\\
&x_{1,(m+n)} \longrightarrow (x_{1,(m'-k+1)}, \ldots, x_{1,m'})\\
&x_{(m'+n'),(m'+1)} \longrightarrow (x_{2,(m'+1)}, \ldots, x_{(k+1),(m'+1)})\\
&x_{(m'+n'),(m'+2)} \longrightarrow (x_{2,(m'+2)}, \ldots, x_{(k+1),(m'+2)}\\
&\qquad\qquad\vdots\\
&x_{(m'+n'),(m+n-m')} \longrightarrow (x_{2,(m+n-m')}, \ldots, x_{(k+1),(m+n-m')})
\end{aligned}$$

Step 2:

$$\begin{aligned}
&x_{1,2} \longrightarrow (x_{(m',2)}, \ldots, x_{(m'+k),2})\\
&x_{1,(m+n-1)} \longrightarrow (x_{(m'+1),(m+n-1)}, \ldots, x_{(m'+k),(m+n-1)})\\
&x_{m',1} \longrightarrow (x_{m',(m+n-k)}, \ldots, x_{m',(m+n-1)})\\
&x_{m',(m+n)} \longrightarrow (x_{m',2}, \ldots, x_{m',(k+1)})\\
&x_{(m'+n'),m'} \longrightarrow (x_{2,m'}, \ldots, x_{(k+1),m'})\\
&x_{(m'+n'),(m+n-m'+1)} \longrightarrow (x_{2,(m+n-m'+1)}, \ldots, x_{(k+1),(m+n-m'+1)})\\
&x_{1,(n'+1)} \longrightarrow (x_{(m'+n'-k),(n'+1)}, \ldots, x_{(m'+n'-1),(n'+1)})
\end{aligned}$$

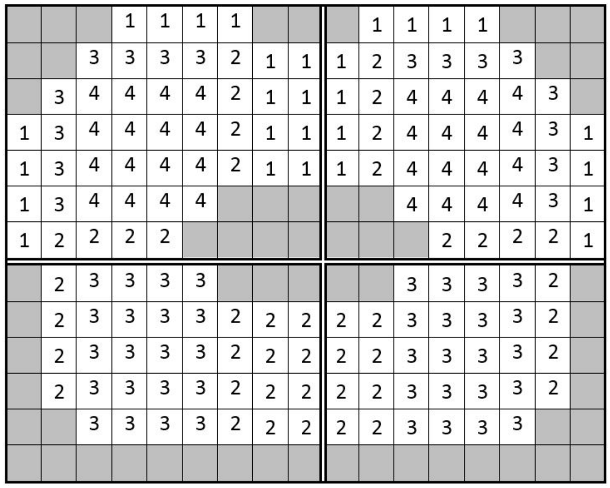

FIGURE 4. The corresponding chessboard of $K_{9,8}\Box K_{7,6}$.

$$x_{1,(n'+2)} \longrightarrow (x_{(m'+n'-k),(n'+2)},\dots,x_{(m'+n'-1),(n'+2)})$$
$$\vdots$$
$$x_{1,(m+n-n')} \longrightarrow (x_{(m'+n'-k),(m+n-n')},\dots,x_{(m'+n'-1),(m+n-n')})$$

Corollary 2.6. *Let m, n, m', and n' be integers greater than integer $k \geq 1$. Then,*

$$\begin{aligned} F_k^*(K_{m,n}\Box K_{m',n'}) &= F_k(K_{m,n}\Box K_{m',n'}) \\ &= (m+n)(m'+n') - 2k(m+n+m'+n') + 4k^2 + 4k. \end{aligned}$$

Proof. Since for every graph G, $F(G) \leq F^*(G)$, it is suffices to show

$$F^*(K_{m,n}\Box K_{m',n'}) \leq F(K_{m,n}\Box K_{m',n'}).$$

As seen in the above steps, for each application of the k-forcing process for the coloring pattern of Theorem 2.5, we have exactly k white squares that become black. Hence, $F_k^*(K_{m,n}\Box K_{m',n'}) \leq F_k(K_{m,n}\Box K_{m',n'})$ and the equality holds. $\square$

Example 2.7. *Figure 4 shows the corresponding generalized chessboard of $K_{9,8}\Box K_{7,6}$. The black squares demonstrate a minimum 4-forcing set (also a minimum 4^*-forcing set) which is obtained by Theorem 2.5 and the number in each of the white squares indicate the step that it will become black.*

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Department of Mathematics, Faculty of Mathematical Sciences, Alzahra University, Tehran, Iran
E-mail address: `z.montazeri@alzahra.ac.ir`

Department of Mathematics, Faculty of Mathematical Sciences, Alzahra University, Tehran, Iran
E-mail address: `soltan@alzahra.ac.ir, corresponding author`